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CONTINUOUS EXTENSIONS OF REAL FUNCTIONS ON ARBITRARY SUBLOCALES AND C-, C^*- AND z-EMBEDDINGS

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ABSTRACT: This paper extends the extension theorem for localic real functions of [J. Gutiérrez García and T. Kubiak, General insertion and extension theorems for localic real functions, J. Pure Appl. Algebra 215 (2011) 1198-1204] from complemented sublocales to arbitrary sublocales. As an application, the theory of point-free C-, C^*- and z-embeddings is revisited.

KEYWORDS: Frame, locale, sublocale, cozero sublocale, completely separated sublocales, C- and C^*-embedded sublocales, z-embedded sublocale, localic real function, localic extension theorem.

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Introduction

One of the main differences between classical topology and point-free topology is that every subspace of a space has a complement (i.e. the lattice of subspaces of a space is a Boolean algebra) whereas most sublocales of a locale are not complemented (making the lattice of sublocales of a locale more complicated than its classical counterpart [15]).

The following point-free counterpart of the extension theorem of Mrówka ([13, 5]) was proved in [8] for a *complemented* sublocale S of a locale L:

Theorem. *The following statements about a bounded continuous real-valued function f in S are equivalent.*

(i) *There exists a continuous extension of f to L.*
(ii) *For every pair $r < s$ in $\mathbb{Q}$, the sublocales $f(r, —)$ and $f(—, s)$ are completely separated in L.*

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The main purpose of this paper is to present a proof of this theorem for *arbitrary* sublocales S. We take the opportunity to revisit the theory of C- and C^*-quotients of [2] from the point of view of sublocale embeddings (in the vein of [9, 12]). In addition, we also treat the parallel class of z-embedded sublocales [12] (*coz*-onto frame quotients in [2, 6]). Our purpose is to illustrate what the sublocale formulation adds to the theory of C-, C^*- and z-quotients.

The outline of the paper is as follows. The first two sections concern notation, terminology and background on real-valued functions in locales. Section 3 discusses the main concept of the paper: complete separation of sublocales. The main result, the general extension theorem, is proved at the fourth section. In Section 5 we present, in the language of sublocales, some needed facts about the product of two functions and the existence of the corresponding multiplicative inverse. Section 6 contains characterizations of C- and C^*-embedded sublocales and Section 7 discusses z-embedded sublocales.

1. Preliminaries and notation

1.1. The categories of frames and locales. In point-free topology, topological spaces are replaced by locales, seen as generalised spaces where points are not explicitly mentioned. The relevant categories are the category **Frm** of frames and frame homomorphisms and its dual category **Loc** of locales and localic maps. Our notation and terminology for frames and locales is that of [14] (we refer in particular to Appendix 1 for our notation on posets and lattices). We recall here some of the basic notions involved.

A *frame* (or *locale*) L is a complete lattice in which

$$a \wedge \bigvee S = \bigvee \{a \wedge b \mid b \in S\} \quad \text{for any } a \in L \text{ and } S \subseteq L. \tag{1.1.1}$$

A *frame homomorphism* preserves all joins (in particular, the bottom element 0 of the lattice) and all finite meets (in particular, the top element 1).

In a frame L the mappings $(x \mapsto (a \wedge x))\colon L \to L$ preserve suprema and hence they have right Galois adjoints $(y \mapsto (a \to y))\colon L \to L$, satisfying

$$a \wedge x \leq y \quad \text{iff} \quad x \leq a \to y$$

and making L a complete Heyting algebra. The *pseudocomplement* of $a \in L$ is the element $a^* = a \to 0 = \bigvee \{x \mid x \wedge a = 0\}$.

Let L be a frame, $a \in L$ and $r_a \colon L \twoheadrightarrow {\downarrow}a$ the surjective frame homomorphism defined by $x \mapsto x \wedge a$. We shall need the following basic property of r_a.

Lemma. *Let $h \colon L \to M$ be a frame homomorphism. If $h(a) = 1$ then there is a (unique) frame homomorphism $\overline{h} \colon {\downarrow}a \to M$ such that the diagram*

$$\begin{tikzcd} L \arrow[r, "h"] \arrow[d, two heads, "r_a"'] & M \\ {\downarrow}a \arrow[ur, "\overline{h}"'] & \end{tikzcd}$$

commutes.

The *rather below relation* $\prec_L$ in a frame L (briefly $\prec$ when there is no danger of confusion) is defined by $b \prec a$ iff $b^* \vee a = 1$. The *completely below relation* $\prec\!\!\prec_L$ (or just $\prec\!\!\prec$) is the interpolative modification of the rather below relation. Elements $a, b \in L$ satisfy $b \prec\!\!\prec a$ if and only if there exists a subset $\{a_q \mid q \in [0,1] \cap \mathbb{Q}\} \subseteq L$ with $a_0 = b$ and $a_1 = a$ such that $a_p \prec a_q$ whenever $p < q$ in $[0,1] \cap \mathbb{Q}$. A frame L is *completely regular* if $a = \bigvee\{b \in L \mid b \prec\!\!\prec a\}$ for any $a \in L$.

A frame is *normal* if for any $a, b \in L$ such that $a \vee b = 1$ there are $u, v \in L$ such that $a \vee u = 1 = b \vee v$ and $u \wedge v = 0$. In a normal frame, $\prec$ interpolates hence it coincides with $\prec\!\!\prec$.

It is of advantage to represent the category of locales as a concrete category as follows. Since frame homomorphisms $h \colon M \to L$ preserve all joins they have uniquely defined right adjoints $f = h_* \colon L \to M$. We will represent **Loc** as the category with frames for objects (in this context we often speak of frames as of locales) and meet preserving maps $f \colon L \to M$ such that f^* are frame homomorphisms (the *localic maps*) for morphisms. They are characterized by the following conditions:

> *a meet preserving $f \colon L \to M$ is a localic map iff $f(a) = 1 \Rightarrow a = 1$ and $f(f^*(a) \to b) = a \to f(b)$.*

1.2. The coframe of sublocales. A *sublocale* of a locale L is a subset $S \subseteq L$ closed under arbitrary meets such that

$$\forall x \in L \ \ \forall s \in S \ \ (x \to s \in S).$$

These are precisely the subsets of L for which the embedding $S \hookrightarrow L$ is a localic map. For alternative representations of sublocales in the literature (namely, *frame quotients* or *nuclei*) see [14, III.5].

The system $\mathsf{S}(L)$ of all sublocales of L, partially ordered by inclusion, is a *coframe*, that is, its dual lattice is a frame (see [14, Theorem III.3.2.1] for a proof). Infima and suprema are given by

$$\bigwedge_{i\in J} S_i = \bigcap_{i\in J} S_i, \quad \bigvee_{i\in J} S_i = \{\bigwedge M \mid M \subseteq \bigcup_{i\in J} S_i\}.$$

The least element is the *void sublocale* $\mathsf{O} = \{1\}$ and the greatest element is the entire locale L. Being a co-Heyting algebra, $\mathsf{S}(L)$ has co-pseudocomplements (usually called *supplements*), that we denote by $S^{\#}$.

We shall need the following property (that is valid in any coframe):

$$S \cap T = \mathsf{O} \;\Rightarrow\; S \subseteq T^{\#} \tag{1.2.1}$$

(see [7] for more information on supplements in $\mathsf{S}(L)$).

For any $a \in L$, the sublocales

$$\mathfrak{c}_L(a) = {\uparrow}a = \{x \in L \mid x \geq a\} \quad \text{and} \quad \mathfrak{o}_L(a) = \{a \to b \mid b \in L\}$$

are the *closed* and *open* sublocales of L, respectively (that we shall denote simply by $\mathfrak{c}(a)$ and $\mathfrak{o}(a)$ when there is no danger of confusion). For each $a \in L$, $\mathfrak{c}(a)$ and $\mathfrak{o}(a)$ are complements of each other in $\mathsf{S}(L)$ and satisfy the identities

$$\bigcap_i \mathfrak{c}(a_i) = \mathfrak{c}(\bigvee_i a_i), \quad \mathfrak{c}(a) \vee \mathfrak{c}(b) = \mathfrak{c}(a \wedge b), \tag{1.2.2}$$

$$\bigvee_i \mathfrak{o}(a_i) = \mathfrak{o}(\bigvee_i a_i) \quad \text{and} \quad \mathfrak{o}(a) \cap \mathfrak{o}(b) = \mathfrak{o}(a \wedge b). \tag{1.2.3}$$

Let S be a sublocale of L and let j_S^* be the left adjoint of the localic embedding $j_S\colon S \hookrightarrow L$, which is given by $j_S^*(a) = \bigwedge\{s \in S \mid s \geq a\}$. The closed (resp. open) sublocales $\mathfrak{c}_S(a)$ (resp. $\mathfrak{o}_S(a)$) of S ($a \in S$) are precisely the intersections $\mathfrak{c}(a) \cap S$ (resp. $\mathfrak{o}(a) \cap S$) and we have, for any $a \in L$,

$$\mathfrak{c}(a) \cap S = \mathfrak{c}_S(j_S^*(a)) \quad \text{and} \quad \mathfrak{o}(a) \cap S = \mathfrak{o}_S(j_S^*(a)). \tag{1.2.4}$$

The *closure* $\overline{S}$ of a sublocale S is the smallest closed sublocale containing S, and the *interior* $\operatorname{int} S$ is the largest open sublocale contained in S. There is a particularly simple formula for the closure, namely $\overline{S} = \mathfrak{c}(\bigwedge S)$. Hence $\overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*)$ and, consequently, $\operatorname{int} \mathfrak{c}(a) = \mathfrak{o}(a^*)$. Note that $a \prec b$ iff $\overline{\mathfrak{o}(a)} \subseteq \mathfrak{o}(b)$: indeed, $a^* \vee b = 1$ iff $\mathfrak{c}(a^*) \cap \mathfrak{c}(b) = \mathsf{O}$ iff $\overline{\mathfrak{o}(a)} \cap \mathfrak{c}(b) = \mathsf{O}$ iff $\overline{\mathfrak{o}(a)} \subseteq \mathfrak{o}(b)$.

1.3. Images and preimages. For any localic map $f\colon L \to M$ and any sublocale $S \subseteq L$ the standard set-theoretical image $f[S]$ is a sublocale of M. However the localic preimage $f_{-1}[T]$ of a sublocale $T \subseteq M$ does not coincide in general with the set-theoretical preimage $f^{-1}[T]$. It is given by

$$f_{-1}[T] = \bigvee\{S \mid S \in \mathsf{S}(L),\ S \subseteq f^{-1}[T]\}.$$

In particular, for a localic embedding $j\colon S \hookrightarrow L$, $j_{-1}[T] = T \cap S$.

One has the adjunction

$$\begin{tikzcd} \mathsf{S}(L) \arrow[r, bend left, "{f[-]}"] \arrow[r, phantom, "\perp"] & \mathsf{S}(M) \arrow[l, bend left, "{f_{-1}[-]}"] \end{tikzcd}$$

(since $f[S] \subseteq T$ iff $S \subseteq f_{-1}[T]$). The right adjoint $f_{-1}[-]$ is a coframe homomorphism (that is, $f_{-1}[-]\colon \mathsf{S}(M)^{op} \to \mathsf{S}(L)^{op}$ is a frame homomorphism) while $f[-]$ is a colocalic map.

(Localic) preimages of open resp. closed sublocales are open resp. closed and one has

$$f_{-1}[\mathfrak{o}(a)] = \mathfrak{o}(f^*(a)) \quad \text{and} \quad f_{-1}[\mathfrak{c}(a)] = f^{-1}[\mathfrak{c}(a)] = \mathfrak{c}(f^*(a)). \qquad (1.3.1)$$

(where f^* denotes the left adjoint of the localic map f).

1.4. The frame of reals. Recall the frame of reals $\mathfrak{L}(\mathbb{R})$ from [3]. Here we define it, equivalently, as the frame presented by generators $(r, \text{—})$ and $(\text{—}, r)$ for all rationals r, and relations

(r1) $(p, \text{—}) \wedge (\text{—}, q) = 0$ if $q \le p$,
(r2) $(p, \text{—}) \vee (\text{—}, q) = 1$ if $p < q$,
(r3) $(p, \text{—}) = \bigvee_{r>p}(r, \text{—})$,
(r4) $(\text{—}, q) = \bigvee_{s<q}(\text{—}, s)$,
(r5) $\bigvee_{p\in\mathbb{Q}}(p, \text{—}) = 1$,
(r6) $\bigvee_{q\in\mathbb{Q}}(\text{—}, q) = 1$.

Note that $(\text{—}, q)^* = (q, \text{—})$ and $(p, \text{—})^* = (\text{—}, p)$.

For each $p < q$ in $\mathbb{Q}$, the element $(p, \text{—}) \wedge (\text{—}, q)$ in $\mathfrak{L}(\mathbb{R})$ is denoted by (p, q). The *open interval frame* $\mathfrak{L}(p, q)$ is the frame

$$\mathfrak{L}(p, q) = {\downarrow}(p, q) = \{a \in \mathfrak{L}(\mathbb{R}) \mid a \le (p, q)\}.$$

Remark. It should be remarked that

$$\mathfrak{L}(p, q) \cong \mathfrak{L}(\mathbb{R}).$$

For the proof, consider any order isomorphism ψ (with inverse φ) from $\langle p,q\rangle$ into $\mathbb{Q}$ (where $\langle\cdot,\cdot\rangle$ stands for open interval in $\mathbb{Q}$) and define

$$\Phi\colon \mathfrak{L}(\mathbb{R}) \to \mathfrak{L}(p,q)$$

on generators by $\Phi(r,—) = (\varphi(r),q)$ and $\Phi(—,r) = (p,\varphi(r))$. It is straightforward to check that Φ turns defining relations (r1)–(r6) into identities in $\mathfrak{L}(p,q)$, hence it is a frame homomorphism, clearly surjective. Finally, define $\Psi_0\colon \mathfrak{L}(\mathbb{R}) \to \mathfrak{L}(\mathbb{R})$ by

$$\Psi_0(r,s) = \begin{cases} 1 & \text{if } r \le p < q \le s, \\ (—,\psi(s)) & \text{if } r \le p < s < q, \\ (\psi(r),\psi(s)) & \text{if } p < r < s < q, \\ (\psi(r),—) & \text{if } p < r < q \le s \\ 0 & \text{if } s \le p \text{ or } q \le r. \end{cases}$$

Again, it is straightforward to check that this is a frame homomorphism. Since $\Psi_0(p,q) = 1$, the restriction of Ψ_0 to $\mathfrak{L}(p,q)$ is, by Lemma 1.1, a frame homomorphism $\Psi\colon \mathfrak{L}(p,q) \to \mathfrak{L}(\mathbb{R})$, inverse to Φ.

1.5. Continuous real functions and scales. The ℓ-ring $\mathcal{R}(L)$ of *continuous real-valued functions* ([3]) on a frame L is the set of all frame homomorphism $f\colon \mathfrak{L}(\mathbb{R}) \to L$. Each element of $\mathcal{R}(L)$ is uniquely determined by a map defined on the generators of $\mathfrak{L}(\mathbb{R})$ that turns relations (r1)-(r6) into identities in L.

A *descending scale* (resp. *ascending scale*) in L is a family $(a_p)_{p\in\mathbb{Q}} \subseteq L$ such that

(S1) $p < q \;\Rightarrow\; a_q \prec a_p$ (resp. $a_p \prec a_q$).
(S2) $\bigvee_{p\in\mathbb{Q}} a_p = 1 = \bigvee_{p\in\mathbb{Q}} a_p^*$.

Remark. If all the a_p's are complemented then $a_p \prec a_p$ for any p and thus condition (S1) amounts only to $p < q \;\Rightarrow\; a_q \le a_p$ (resp. $a_p \le a_q$).

Proposition 1.5.1. *Let $f\colon \mathfrak{L}(\mathbb{R}) \to L$ be a continuous real function on L. Then:*

(1) *The family $(f(p,—))_{p\in\mathbb{Q}}$ is a descending scale in L.*
(2) *The family $(f(—,q))_{q\in\mathbb{Q}}$ is an ascending scale in L.*

Proof: Let $p < q$. Then, by (r1), $f(q,\text{—})^* \vee f(p,\text{—}) \geq f(\text{—},q) \vee f(p,\text{—}) = f(1) = 1$. Clearly, $\bigvee_{p\in\mathbb{Q}} f(p,\text{—}) = f(1) = 1$ and

$$\bigvee_{p\in\mathbb{Q}} f(p,\text{—})^* \geq \bigvee_{p\in\mathbb{Q}} f(\text{—},p) = f(1) = 1. \quad \blacksquare$$

Conversely, we have:[1]

Proposition 1.5.2. (1) *Let $(a_p)_{p\in\mathbb{Q}}$ be a descending scale in L. Then the formulas*

$$f(p,\text{—}) = \bigvee_{r>p} a_r \quad \text{and} \quad f(\text{—},q) = \bigvee_{s<q} a_s^*$$

define a frame homomorphism $f\colon \mathfrak{L}(\mathbb{R}) \to L$.

(2) *Let $(a_p)_{p\in\mathbb{Q}}$ be an ascending scale in L. Then the formulas*

$$g(p,\text{—}) = \bigvee_{r>p} a_r^* \quad \text{and} \quad g(\text{—},q) = \bigvee_{s<q} a_s$$

define a frame homomorphism $g\colon \mathfrak{L}(\mathbb{R}) \to L$.

1.6. Product of two functions. The *product* $f \cdot g$ of two continuous real functions $f, g\colon \mathfrak{L}(\mathbb{R}) \to L$ is given by the formula (see [3, Chapter 4])

$$(f \cdot g)(p,q) = \bigvee\{f(r,s) \wedge g(t,u) \mid \langle r,s\rangle \cdot \langle t,u\rangle \subseteq \langle p,q\rangle\} \tag{1.6.1}$$

where $\langle\cdot,\cdot\rangle$ denotes open intervals in $\mathbb{Q}$ and the inclusion on the right means that $x \cdot y \in \langle p,q\rangle$ whenever $x \in \langle r,s\rangle$ and $y \in \langle t,u\rangle$.

1.7. Cozero and zero sublocales. The σ-frame[2] $\operatorname{Coz} L \subseteq L$ of cozero elements plays an important role in the theory of continuous real functions ([3]). Recall that a *cozero element* of L is an element of the form $f((\text{—},0) \vee (0,\text{—}))$ for some frame homomorphism $f\colon \mathfrak{L}(\mathbb{R}) \to L$ (usually denoted as $\operatorname{coz}(f)$). Cozero elements can be described without reference to the frame of reals as follows: $a \in L$ is a cozero element if and only if $a = \bigvee_{n=1}^{\infty} a_n$ for some $a_n \prec\!\prec a$, $n = 1, 2, \ldots$.

$\operatorname{Coz} L$ is a *normal σ-frame*, that is, $a \vee b = 1$ ($a, b \in \operatorname{Coz} L$) implies there exist c and d in $\operatorname{Coz} L$ such that $a \vee c = 1 = b \vee d$ and $c \wedge d = 0$ ([4]).

[1] It is easy to check that the given formulas turn the defining relations (r1)–(r6) of $\mathfrak{L}(\mathbb{R})$ into identities in L.

[2] A *σ-frame* is a lattice L in which all *countable* subsets have a join such that the distribution law (1.1.1) holds for any $a \in L$ and any countable $S \subseteq L$.

The *cozero sublocales* (resp. *zero sublocales* [12]) are the $\mathfrak{c}(a)$ (resp. $\mathfrak{o}(a)$) with $a \in \operatorname{Coz} L$. We denote by

$$\mathsf{CoZS}(L) \quad \text{and} \quad \mathsf{ZS}(L)$$

the classes of cozero and zero sublocales respectively. The intersection

$$\mathsf{CoZS}(L) \cap \mathsf{ZS}(L)$$

is the class of clopen sublocales.

Any cozero sublocale is a G_δ-*sublocale*, that is, a countable intersection (in $\mathsf{S}(L)$) of open sublocales, while every zero sublocale is an F_σ-*sublocale*, that is, a countable join of closed sublocales (see [12, 5.3.1]).

2. Background on general localic real functions

2.1. General real functions and scales. Let L be a frame and consider its *assembly frame*, that is, the dual frame $\mathsf{S}(L)^{op}$ of the coframe of sublocales of L. Meets and joins in $\mathsf{S}(L)^{op}$ are given by respectively

$$\bigsqcap_{i\in J} S_i = \bigvee_{i\in J} S_i \quad \textit{and} \quad \bigsqcup_{i\in J} S_i = \bigcap_{i\in J} S_i.$$

By the identities in (1.2.2), the set of all closed sublocales of L form a subframe of $\mathsf{S}(L)^{op}$ isomorphic to the given L. Hence the ℓ-ring $\mathcal{R}(\mathsf{S}(L)^{op})$ is an extension of $\mathcal{R}(L)$, regarded as the ring of general real functions on L and denoted simply by $\mathsf{F}(L)$ (see [9, 11] for motivation and more information). It is partially ordered by

$$f \le g \;\text{ iff }\; f(\text{—}, r) \subseteq g(\text{—}, r) \;\text{ iff }\; g(r, \text{—}) \subseteq f(r, \text{—})$$

for all $r \in \mathbb{Q}$.

Note that a descending scale in the frame $\mathsf{S}(L)^{op}$ is a family $(S_p)_{p\in\mathbb{Q}}$ of sublocales of L satisfying

(S1) $p < q \;\Rightarrow\; S_q \prec S_p$ (i.e. $S_q^{\#} \cap S_p = \mathsf{O}$), and
(S2) $\bigcap_{p\in\mathbb{Q}} S_p = \mathsf{O} = \bigcap_{p\in\mathbb{Q}} S_p^{\#}$.

We will need the following two facts from [9, 4.4]:

Proposition. *Let $f_1, f_2 \in \mathsf{F}(L)$ be generated by descending scales $(S_r)_{r\in\mathbb{Q}}$ and $(T_r)_{r\in\mathbb{Q}}$ respectively. Then:*

(1) *$f_1(\text{—}, r)^{\#} \subseteq S_r \subseteq f_1(r, \text{—})$ for every $r \in \mathbb{Q}$.*
(2) *$f_2 \le f_1$ iff $S_r \subseteq T_s$ for every $r < s$.*

2.2. Semicontinuous functions. The extension $\mathsf{F}(L)$ of $\mathcal{R}(L)$ allows to deal with more general types of real functions. In particular, an $f \in \mathsf{F}(L)$ is *lower* (resp. *upper*) *semicontinuous* if $f(r,\text{—})$ (resp. $f(\text{—},r)$) is a closed sublocale for any $r \in \mathbb{Q}$. It is *continuous* if it is both lower and upper semicontinuous (that is, if $f(p,q)$ is a closed sublocale for every p,q). Of course, the subring of all continuous members of $\mathsf{F}(L)$, denoted by $\mathsf{C}(L)$, is an isomorphic copy of $\mathcal{R}(L)$ inside $\mathsf{F}(L)$. Along this paper, we will work always in $\mathsf{F}(L)$ and regard $\mathcal{R}(L)$ as the subring $\mathsf{C}(L)$ of $\mathsf{F}(L)$.

For any $f \in \mathsf{C}(L)$ and $r \in \mathbb{Q}$, both $f(\text{—},r)$ and $f(r,\text{—})$ are cozero sublocales ([12, 5.3.1]).

2.3. Constant functions. For each $r \in \mathbb{Q}$, $(S_p^r \mid p \in \mathbb{Q})$ defined by $S_p^r = \mathsf{O}$ if $p < r$ and $S_p^r = L$ if $p \geq r$ is a descending scale in $\mathsf{S}(L)^{op}$. The corresponding function in $\mathsf{C}(L)$, the *constant function* $\boldsymbol{r}$, is given by

$$\boldsymbol{r}(p,\text{—}) = \begin{cases} \mathsf{O} & \text{if } p < r \\ L & \text{if } p \geq r \end{cases} \quad \text{and} \quad \boldsymbol{r}(\text{—},q) = \begin{cases} L & \text{if } q \leq r \\ \mathsf{O} & \text{if } q > r. \end{cases}$$

2.4. Bounded functions. The *bounded part* $\mathsf{C}^*(L)$ of $\mathsf{C}(L)$ consists of all $f \in \mathsf{C}(L)$ such that $\boldsymbol{p} \leq f \leq \boldsymbol{q}$, that is, $f(\text{—},p) \cap f(q,\text{—}) = L$, for some pair $p < q$ in $\mathbb{Q}$.

By the isomorphism between $\mathcal{R}(L)$ and $\mathsf{C}(L)$ every cozero sublocale is of the form $\mathfrak{c}(a) = f(0,\text{—}) \cap f(\text{—},0)$ for some $f \in \mathsf{C}(L)$ (which, furthermore, can always be considered to be bounded); so we can always assume that a cozero sublocale is of the form $f(0,\text{—})$ for some continuous f satisfying $\mathbf{0} \leq f \leq \mathbf{1}$.

2.5. More examples. (1) By Remark 1.5, a family $(\mathfrak{c}(a_p))_{p\in\mathbb{Q}}$ of closed sublocales is a descending scale iff

(C1) $p < q \;\Rightarrow\; a_q \leq a_p$, and

(C2) $\bigcap_{p\in\mathbb{Q}} \mathfrak{c}(a_p) = \mathsf{O} = \bigcap_{p\in\mathbb{Q}} \mathfrak{o}(a_p)$.

In this case the formulas

$$f(p,\text{—}) = \bigcap_{r>p} \mathfrak{c}(a_r) \quad \text{and} \quad f(\text{—},q) = \bigcap_{s<q} \mathfrak{o}(a_s)$$

given by Proposition 1.5.2 induce a lower semicontinuous function $f \in \mathsf{F}(L)$.

(2) Similarly, a family $(\mathfrak{o}(a_p))_{p\in\mathbb{Q}}$ of open sublocales is a descending scale iff

(O1) $p < q \;\Rightarrow\; a_p \leq a_q$, and

(O2) $\bigcap_{p\in\mathbb{Q}} \mathfrak{o}(a_p) = \mathsf{O} = \bigcap_{p\in\mathbb{Q}} \mathfrak{c}(a_p)$.

In this case the formulas

$$f(p,\text{—}) = \bigcap_{r>p} \mathfrak{o}(a_r) \quad \text{and} \quad f(\text{—},q) = \bigcap_{s<q} \mathfrak{c}(a_s)$$

given by 1.5.2 induce an upper semicontinuous function $f \in \mathsf{F}(L)$.

(3) The condition $\bigcap_{p\in\mathbb{Q}} \mathfrak{c}(a_p) = \mathsf{O}$ means that $\bigvee_{p\in\mathbb{Q}} a_p = 1$ but the condition $\bigcap_{p\in\mathbb{Q}} \mathfrak{o}(a_p) = \mathsf{O}$ is generally weaker than $\bigvee_{p\in\mathbb{Q}} a_p^* = 1$. However, in case $(a_p)_{p\in\mathbb{Q}} \subseteq L$ satisfies 1.5(S1), then it is easy to see that

$$\bigcap_{p\in\mathbb{Q}} \mathfrak{o}(a_p) = \mathfrak{c}(\bigvee_{p\in\mathbb{Q}} a_p^*), \quad \bigcap_{s<q} \mathfrak{o}(a_s) = \mathfrak{c}(\bigvee_{s<q} a_s^*) \quad \text{and} \quad \bigcap_{r>p} \mathfrak{o}(a_r) = \mathfrak{c}(\bigvee_{r>p} a_r^*),$$

and, consequently, the conditions $\bigcap_{p\in\mathbb{Q}} \mathfrak{o}(a_p) = \mathsf{O}$ and $\bigvee_{p\in\mathbb{Q}} a_p^* = 1$ do coincide. Hence we have:

Proposition. (1) *Let $(a_p)_{p\in\mathbb{Q}}$ be a descending scale in L. Then the family $(\mathfrak{c}(a_p))_{p\in\mathbb{Q}}$ is a descending scale in $\mathsf{S}(L)^{\mathrm{op}}$ and the induced function $f \in \mathsf{F}(L)$ given by*

$$f(p,\text{—}) = \bigcap_{r>p} \mathfrak{c}(a_r) = \mathfrak{c}(\bigvee_{r>p} a_r) \quad \textit{and} \quad f(\text{—},q) = \bigcap_{s<q} \mathfrak{o}(a_s) = \mathfrak{c}(\bigvee_{s<q} a_s^*)$$

is continuous.

(2) *Let $(a_p)_{p\in\mathbb{Q}}$ be an ascending scale in L. Then the family $(\mathfrak{o}(a_p))_{p\in\mathbb{Q}}$ is a descending scale in $\mathsf{S}(L)^{\mathrm{op}}$ and the induced function $g \in \mathsf{F}(L)$ given by*

$$g(p,\text{—}) = \bigcap_{r>p} \mathfrak{o}(a_r) = \mathfrak{c}(\bigvee_{r>p} a_r^*) \quad \textit{and} \quad g(\text{—},q) = \bigcap_{s<q} \mathfrak{c}(a_s) = \mathfrak{c}(\bigvee_{s<q} a_s)$$

is continuous. ■

These are precisely (up to the isomorphism between L and the subframe of $\mathsf{S}(L)^{\mathrm{op}}$ of all closed sublocales) the functions f and g of 1.5.2.

2.6. Continuous extensions. Let S be a sublocale of L with localic embedding $j\colon S \hookrightarrow L$. A function $f \in \mathsf{C}(S)$ is said to have a *continuous extension* to L if there is an $\bar{f} \in \mathsf{C}(L)$ such that the diagram

$$\begin{tikzcd} \mathfrak{L}(\mathbb{R}) \arrow[r, "\bar{f}"] \arrow[dr, "f"'] & \mathsf{S}(L)^{op} \arrow[d, "j_{-1}[-]\colon T\mapsto T\cap S"] \\ & \mathsf{S}(S)^{op} \end{tikzcd}$$

commutes (that is, $\bar{f}(a) \cap S = f(a)$ for every $a \in \mathfrak{L}(\mathbb{R})$).

3. Completely separated sublocales

Two sublocales S and T of L are said to be *completely separated in* L if

$$S \subseteq f(0, -) \quad \text{and} \quad T \subseteq f(-, 1)$$

for some $f \in \mathsf{C}(L)$ such that $\mathbf{0} \leq f \leq \mathbf{1}$.

This notion was first studied in [2] in terms of quotient maps and cozero elements and equivalently reformulated in [8] in terms of sublocales and continuous real functions. We refer to [12] for several results about completely separated sublocales. E.g., sublocales $\mathfrak{c}(a)$ and $\mathfrak{o}(b)$ are completely separated iff $b \prec\!\!\prec a$ ([12, Lemma 5.4.2]).

Proposition 3.1. *Two sublocales S and T of L are completely separated in L if and only if they are contained in disjoint cozero sublocales of L.*

Proof: The implication '$\Rightarrow$' is obvious. Conversely, let $\mathfrak{c}(a), \mathfrak{c}(b) \in \mathsf{CozS}(L)$ such that $S \subseteq \mathfrak{c}(a)$, $T \subseteq \mathfrak{c}(b)$, and $\mathfrak{c}(a) \cap \mathfrak{c}(b) = \mathsf{O}$. The last identity means that $a \vee b = 1$. Since a and b are cozero elements of L and $\mathrm{Coz}\,(L)$ is a normal sub-σ-frame of L (recall 1.7), there exist $u, v \in \mathrm{Coz}\,(L)$ such that $a \vee u = 1 = b \vee v$ and $u \wedge v = 0$. This implies that $u \prec b$ and, again by normality, $\prec$ interpolates and thus $u \prec\!\!\prec b$. Then (see e.g. [12, 5.4.3]) there is an $f \in \mathsf{C}^*(L)$ such that $\mathfrak{o}(u) \subseteq f(0, -)$ and $\mathfrak{c}(b) \subseteq f(-, 1)$. Since $\mathfrak{c}(a) \subseteq \mathfrak{o}(u)$ (because $a \vee u = 1$) we may then conclude that $S \subseteq \mathfrak{c}(a) \subseteq f(0, -)$ and $T \subseteq \mathfrak{c}(b) \subseteq f(-, 1)$. ■

Remark 3.2. By the well-known Urysohn's separation lemma for locales ([3, Proposition 5]), in a normal locale any two disjoint closed sublocales are completely separated.

Let U and V be sublocales of a sublocale S of L (hence, sublocales of L). If U and V are completely separated in L with $f \in \mathsf{C}(L)$ satisfying $U \subseteq f(0, -)$ and $V \subseteq f(-, 1)$, consider the composite

$$\mathfrak{L}(\mathbb{R}) \xrightarrow{\ f\ } \mathsf{S}(L)^{op} \xrightarrow{j_{-1}[-]} \mathsf{S}(S)^{op}$$

where $j \colon S \hookrightarrow L$. Since $j_{-1}[-]$ is given by intersection with S, it is easy to see that $j_{-1}[-]f \in \mathsf{C}(S)$. Further, $U \subseteq f(0, -) \cap S = j_{-1}[f(0, -)]$ and $V \subseteq f(-, 1) \cap S = j_{-1}[f(-, 1)]$. Hence U and V are completely separated in S.

Hence, if two sublocales of S are completely separated in L, then they are completely separated in S. Of course, the converse does not hold in general:

two sublocales that are completely separated in S may not be completely separated in L as some easy examples show.

More specifically, we have:

Proposition 3.3. *The following statements are equivalent for a sublocale S of L.*

(i) *Any two completely separated sublocales of S are completely separated in L.*

(ii) *For every $f \in \mathsf{C}^*(S)$ and every pair $r < s$ in $\mathbb{Q}$, $f(r, \text{—})$ and $f(\text{—}, s)$ are completely separated in L.*

Proof: (i)⇒(ii): For any $f \in \mathsf{C}^*(S)$ and $r < s$, $f(r, \text{—})$ and $f(\text{—}, s)$ are disjoint cozero sublocales of S:

$$f(r, \text{—}) \cap f(\text{—}, s) = f((r, \text{—}) \vee (\text{—}, s)) = f(1) = \mathsf{O}.$$

Therefore, by the previous proposition, they are completely separated in S hence in L (by assumption).

(ii)⇒(i): Let U, V be sublocales of S, completely separated in S. Then $U \subseteq f(0, \text{—})$ and $V \subseteq f(\text{—}, 1)$ for some $f \in \mathsf{C}^*(S)$ and by assumption, $f(0, \text{—})$ and $f(\text{—}, 1)$ are completely separated in L. Hence U and V are also completely separated in L. ■

Remark 3.4. It is an easy exercise to check that each one of the following conditions is also equivalent to the assertions in Proposition 3.3:

(iii) *Any two closed (resp. open) sublocales of S that are completely separated in S, are completely separated in L.*

(iv) *Any two cozero (resp. zero) sublocales of S that are completely separated in S, are completely separated in L.*

(v) *Any open and any closed (resp. any zero and any cozero) sublocales of S that are completely separated in S, are completely separated in L.*

They are the localic formulations of some of the equivalent conditions in [2, Theorem 7.1.1] presented in terms of frame quotients. We believe that the language of sublocales helps to clarify the notions and statements. For example, it is now clear that the terminology "m-completely separated" (with respect to a quotient m of a frame L, that is, a sublocale M of L) introduced in [2, pp. 123] is superfluous. In fact, it is precisely "complete separation": for any sublocale M of L, two sublocales of M are M-completely separated if and only if they are completely separated in L.

Another equivalent condition in [2, Theorem 7.1.1] is formulated in terms of certain order relation

$$\prec\!\prec_m$$

in L. The condition $b \prec\!\prec_m a$ translates into the present language as the condition that the sublocales $\mathfrak{o}_S(b)$ and $\mathfrak{c}_S(a)$ of S are completely separated in L. As we observed earlier, this is stronger than saying that $\mathfrak{o}_S(b)$ and $\mathfrak{c}_S(a)$ are completely separated in S, that is, that $b \prec\!\prec_S a$ (by the result recalled above that in any locale L, $\mathfrak{o}_L(b)$ and $\mathfrak{c}_L(a)$ are completely separated in L iff $b \prec\!\prec_L a$).

4. Extension theorem for arbitrary sublocales

In this section we generalize the general extension theorem of [8] to arbitrary sublocales. To prove it we will need the following insertion theorem from [8]:

Theorem 4.1. ([8, Theorem 4.2]) *Let L be a frame and $f_1, f_2 \in \mathsf{F}(L)$. The following statements are equivalent.*

(i) *There exists $h \in \mathsf{C}(L)$ such that $f_2 \leq h \leq f_1$.*
(ii) *The sublocales $f_2(\text{—}, s)$ and $f_1(r, \text{—})$ are completely separated in L for every $r < s$ in $\mathbb{Q}$.*

Theorem 4.2. *Let S be a sublocale of L. The following statements about an $f \in \mathsf{C}^*(S)$ are equivalent.*

(i) *There exists a bounded continuous extension of f to L.*
(ii) *The sublocales $f(r, \text{—})$ and $f(\text{—}, s)$ are completely separated in L for every $r < s$ in $\mathbb{Q}$.*

Proof: (i)⇒(ii): If f has a bounded continuous extension to L, say $\bar{f}$, then, for every $r < s$,

$$f(r, \text{—}) = \bar{f}(r, \text{—}) \cap S \subseteq \bar{f}(r, \text{—}) \quad \text{and} \quad f(\text{—}, s) = \bar{f}(\text{—}, s) \cap S \subseteq \bar{f}(\text{—}, s).$$

Moreover,

$$\bar{f}(r, \text{—}) \cap \bar{f}(\text{—}, s) = \bar{f}((r, \text{—}) \vee (\text{—}, s)) = \bar{f}(1) = \mathsf{O}.$$

Hence $f(r, \text{—})$ and $f(\text{—}, s)$ are contained in the disjoint cozero sublocales $\overline{f}(r, \text{—})$ and $\overline{f}(\text{—}, s)$, and thus they are completely separated.

(ii)⇒(i): Let $f \in \mathsf{C}^*(S)$. We may assume that $\mathbf{0} \leq f \leq \mathbf{1}$ with no loss of generality. For each $r \in \mathbb{Q}$ set

$$S_r = \begin{cases} \mathsf{O} & \text{if } r < 0 \\ \bigcap \{C \in \mathsf{CoZS}(L) \mid f(r,\text{—}) \subseteq C\} & \text{if } 0 \leq r < 1 \\ L & \text{if } r \geq 1 \end{cases}$$

and

$$T_r = \begin{cases} \mathsf{O} & \text{if } r \leq 0 \\ \bigvee \{C \in \mathsf{ZS}(L) \mid f(\text{—},r) \subseteq C^{\#}\} & \text{if } 0 < r \leq 1 \\ L & \text{if } r > 1. \end{cases}$$

Each S_r is a closed sublocale of L whereas T_r is open. For any $r < s$, we have $S_r \subseteq S_s$, since $f(r,\text{—}) \subseteq f(s,\text{—})$, and $T_r \subseteq T_s$, since $f(\text{—},s) \subseteq f(\text{—},r)$, hence $T_s^{\#} \subseteq T_r^{\#}$. Note that for any $0 < r \leq 1$,

$$T_r^{\#} = \bigcap\{C \in \mathsf{CoZS}(L) \mid f(\text{—},r) \subseteq C\}$$

because T_r is a join of open sublocales, that is, an open sublocale, hence its complement is simply the intersection of the complements of those open sublocales.

Further $\bigcap_{r\in\mathbb{Q}} S_r = \mathsf{O} = \bigcap_{r\in\mathbb{Q}} T_r$, hence $(S_r)_{r\in\mathbb{Q}}$ and $(T_r)_{r\in\mathbb{Q}}$ are descending scales, with corresponding functions $f_1, f_2 \in F(L)$ defined by (recall 1.5.2)

$$\begin{array}{ll} f_1\colon \mathfrak{L}(\mathbb{R}) \to \mathsf{S}(L)^{op} & f_2\colon \mathfrak{L}(\mathbb{R}) \to \mathsf{S}(L)^{op} \\ f_1(r,\text{—}) = \bigcap_{p>r} S_p & f_2(r,\text{—}) = \bigcap_{p>r} T_p \\ f_1(\text{—},s) = \bigcap_{q<s} S_q^{\#} & f_2(\text{—},s) = \bigcap_{q<s} T_q^{\#}. \end{array}$$

In particular,

$$f_1(r,\text{—}) = \begin{cases} \mathsf{O} & \text{if } r < 0 \\ \bigcap \{C \in \mathsf{CoZS}(L) \mid f(p,\text{—}) \subseteq C \text{ for some } p > r\} & \text{if } 0 \leq r < 1 \\ L & \text{if } r \geq 1 \end{cases}$$

and

$$f_2(\text{—},s) = \begin{cases} L & \text{if } s \leq 0 \\ \bigcap\{C \in \mathsf{CoZS}(L) \mid f(\text{—},q) \subseteq C \text{ for some } q < s\} & \text{if } 0 < s \leq 1 \\ \mathsf{O} & \text{if } s > 1. \end{cases}$$

Claim 1: $f_2 \leq f_1$.

We will show this using Proposition 2.1, by proving that $S_r \subseteq T_s$ for every $r < s$. If $r < 0$ then $S_r = \mathsf{O} \subseteq T_s$. If $s > 1$, then $S_r \subseteq L = T_s$. Finally, if $0 \leq r < s \leq 1$, then $f(r,\text{—})$ and $f(\text{—},s)$ are completely separated in L thus there exist disjoint $C_1, C_2 \in \mathsf{CoZS}(L)$ such that $f(r,\text{—}) \subseteq C_1$ and $f(\text{—},s) \subseteq C_2$ and then, by (1.2.1),

$$S_r \subseteq C_1 \subseteq C_2^{\#} \subseteq T_s.$$

Claim 2: *There exists $h \in \mathsf{C}^*(L)$ such that $f_2 \leq h \leq f_1$.*

By Theorem 4.1 it suffices to show that $f_1(r,\text{—})$ and $f_2(\text{—},s)$ are completely separated for any $r < s$. Again, the cases $r < 0$ and $s > 1$ are trivial. If $0 \leq r < s \leq 1$ consider $p, q \in \mathbb{Q}$ such that $0 \leq r < p < q < s \leq 1$. By the assumption, there are disjoint $C_1, C_2 \in \mathsf{CoZS}(L)$ such that $f(p,-) \subseteq C_1$ and $f(-,q) \subseteq C_2$. Then

$$f_1(r,\text{—}) \subseteq S_p \subseteq C_1 \quad \text{ and } \quad f_2(\text{—},s) \subseteq T_q^{\#} \subseteq C_2.$$

Hence $f_1(r,\text{—})$ and $f_2(\text{—},s)$ are completely separated.

Claim 3: *h is a continuous bounded extension of f to L.*

(1) We need to show that $h(r,\text{—}) \cap S = f(r,\text{—})$ for every $r \in \mathbb{Q}$. We have the following three cases:

- $r < 0$: We have $h(r,\text{—}) \subseteq f_2(r,\text{—}) = \bigcap_{p>r} T_p = \mathsf{O}$ and $f(r,\text{—}) = \mathsf{O}$ (because $\mathbf{0} \leq f$). Hence $h(r,\text{—}) \cap S = \mathsf{O} = f(r,\text{—})$.

- $r \geq 1$: In this case, $L = f_1(r,\text{—}) \subseteq h(r,\text{—})$ and $f(r,\text{—}) = S$ since $f \leq \mathbf{1}$. Hence $h(r,\text{—}) \cap S = L \cap S = S = f(r,\text{—})$.

- $0 \leq r < 1$: For every $p > r$, $f(r,\text{—}) \subseteq f(p,\text{—}) \subseteq S_p$, hence $f(r,\text{—}) \subseteq f_1(r,\text{—}) \subseteq h(r,\text{—})$ (since $h \leq f_1$). On the other hand, since $f_2 \leq h$, then

$$S \cap h(r,\text{—}) \subseteq S \cap f_2(r,\text{—}) = S \cap \bigcap_{p>r} T_p = \bigcap_{p>r} (S \cap T_p).$$

  Moreover, since $f(\text{—},p) \subseteq T_p^{\#}$, we have $f(\text{—},p) \cap T_p = \mathsf{O}$. In particular, $f(\text{—},p) \cap S \cap T_p = \mathsf{O}$. Hence, by (1.2.1), $S \cap T_p \subseteq f(\text{—},p)^{\#_S} \subseteq f(p,\text{—})$

(where $(-)^{\#_S}$ denotes supplementation in $\mathsf{S}(S)$). Note that $S \cap T_p \subseteq f(p, -)$ also holds for $p > 1$ since $T_p = L$ and $f(p, -) = S$. Hence,

$$S \cap h(r, -) \subseteq \bigcap_{p>r} (S \cap T_p) \subseteq \bigcap_{p>r} f(p, -) = f(r, -).$$

(2) Finally, we need to show that $h(-, s) \cap S = f(-, s)$ for every $s \in \mathbb{Q}$. Again it suffices to analyse the following three cases:

- $s \leq 0$: In this case, $L = f_2(-, s) \subseteq h(-, s)$ and $f(-, s) = S$ (because $\mathbf{0} \leq f$). Then $h(-, s) \cap S = L \cap S = S = f(-, s)$.

- $1 < s$: Then $h(-, s) \subseteq f_1(-, s) = \bigcap_{q<s} S_q^{\#} = \mathsf{O}$ and $f(-, s) = \mathsf{O}$ since $f \leq \mathbf{1}$. Then $h(-, s) \cap S = \mathsf{O} = f(-, s) \cap S$.

- $0 < s \leq 1$: First, we have (since $\mathbf{0} \leq f$ and $f_2 \leq h$):

$$\begin{aligned} f(-, s) &= \bigcap_{0<t<s} f(-, t) \subseteq \bigcap \{C \in \mathsf{CoZS}(L) \mid f(-, t) \subseteq C \text{ for some } 0 < t < s\} \\ &= \bigcap_{0<t<s} T_t^{\#} = f_2(-, s) \subseteq h(-, s). \end{aligned}$$

On the other hand, since $f(t, -) \subseteq S_t$ for every $t \geq 0$, we have $f(t, -) \cap S \cap S_t^{\#} = \mathsf{O}$, that is (by (1.2.1)), $S \cap S_t^{\#} \subseteq f(t, -)^{\#_S} \subseteq f(-, t)$. Finally, from $h \leq f_1$ it follows that

$$S \cap h(-, s) \subseteq S \cap f_1(-, s) \subseteq \bigcap_{t<s} (S \cap S_t^{\#}) \subseteq \bigcap_{t<s} f(-, t) = f(-, s).$$

In conclusion, h is a continuous extension of f to L. ∎

This theorem is the extension to arbitrary sublocales of the main theorem of [8] (proved only for complemented sublocales). It was originally stated as part of [2, Theorem 7.1.1], but the proof there requires some background results on the localic Yosida representation, complete separation in archimedean f-rings and uniformities. The proof above uses only basic facts about localic real functions and sublocale lattices.

5. The multiplicative inverse of a function

Let $f \cdot g$ denote the product of two real functions $f, g \in \mathsf{F}(L)$. It may be computed with formula (1.6.1) applied on frame $\mathsf{S}(L)^{op}$. Alternative formulas for the computation of $f \cdot g$ may be consulted in [12, 4.4] or [11, 4.3]. Here we only need to recall the particular case $f, g \geq \mathbf{0}$.

Proposition 5.1. *Let* $\mathbf{0} \leq f, g \in \mathsf{F}(L)$. *Then:*

(1) $$(f \cdot g)(p, \text{–}) = \begin{cases} \bigcap_{r>0} \big(f(r, \text{–}) \vee g(\tfrac{p}{r}, \text{–})\big) & \textit{if } p \geq 0 \\ \mathsf{0} & \textit{if } p < 0. \end{cases}$$

(2) $$(f \cdot g)(\text{–}, q) = \begin{cases} \bigcap_{s>0} \big(f(\text{–}, s) \vee g(\text{–}, \tfrac{q}{s})\big) & \textit{if } q > 0 \\ L & \textit{if } q \leq 0. \end{cases}$$

We will also need the familiar fact ([12, 5.3]) that

$$\operatorname{coz}(f \cdot g) = \operatorname{coz}(f) \wedge \operatorname{coz}(g). \tag{5.1.1}$$

The following proposition can be found in [2].

Proposition 5.2. *A frame homomorphism* $f \colon \mathfrak{L}(\mathbb{R}) \to L$ *has a multiplicative inverse if and only if* $\operatorname{coz}(f) = 1$.

This result can be also proved with the point-free description of the reals as a frame presented by generators and relations. The idea for the proof is to mimick the classical proof that constructs the multiplicative inverse of a function $f \colon X \to \mathbb{R}$ by composing it with $g \colon \mathbb{R} \smallsetminus \{0\} \to \mathbb{R}$ $(x \mapsto \frac{1}{x})$ provided the image of f is contained in $\mathbb{R} \smallsetminus \{0\}$.

Indeed, if $\operatorname{coz}(f) = 1$ there is by Lemma 1.1 a frame homomorphism $\overline{f}$ such that the diagram

$$\begin{tikzcd} \mathfrak{L}(\mathbb{R}) \arrow[r, "f"] \arrow[d, two heads] & L \\ {\downarrow}((0,\text{–}) \vee (\text{–},0)) \arrow[ur, "\overline{f}"] & \end{tikzcd}$$

commutes. We can compose $\overline{f}$ with

$$g \colon \mathfrak{L}(\mathbb{R}) \to \mathfrak{L}(\mathbb{R} \smallsetminus \{0\}) = {\downarrow}((0,\text{–}) \vee (\text{–},0)),$$

the point-free version of the mapping $x \mapsto \frac{1}{x}$ above, given by

$$g(p, \text{–}) = \begin{cases} (0, \frac{1}{p}) & \text{if } p > 0 \\ (0, \text{–}) & \text{if } p = 0 \\ (\text{–}, \frac{1}{p}) \vee (0, \text{–}) & \text{if } p < 0 \end{cases}$$

and

$$g(\text{—},q)=\begin{cases}(\frac{1}{q},0) & \text{if } q<0\\ (\text{—},0) & \text{if } q=0\\ (\text{—},0)\vee(\frac{1}{q},\text{—}) & \text{if } q>0.\end{cases}$$

The composite $\overline{f}g$ is the multiplicative inverse of f. The verification details are left to the reader.

Note. Classically, when a function does not have a multiplicative inverse, one restricts it to its cozero set in order to compose it with $x\mapsto\frac{1}{x}$. Similarly, if $f\colon\mathfrak{L}(\mathbb{R})\to L$ is a frame homomorphism, by 1.1 there exists $\overline{f}$ such that the diagram

$$\begin{tikzcd}
\mathfrak{L}(\mathbb{R}) \arrow[r, "f"] \arrow[d, two heads] & L \arrow[r, two heads, "p"] & {\downarrow}\mathrm{coz}(f) \\
\mathfrak{L}(\mathbb{R}\smallsetminus\{0\}) \arrow[urr, "\overline{f}"'] & &
\end{tikzcd}$$

commutes. Then $\mathfrak{L}(\mathbb{R})\xrightarrow{g}\mathfrak{L}(\mathbb{R}\smallsetminus\{0\})\xrightarrow{\overline{f}}L$ is the multiplicative inverse of pf.

6. C- and C^*-embedded sublocales

Recall 2.6. A sublocale S of L is said to be *C-embedded* (resp. *C^*-embedded*) if every f in $\mathsf{C}(S)$ (resp. in $\mathsf{C}^*(S)$) has a continuous extension (resp. bounded continuous extension) to L.

As an immediate consequence of the Extension Theorem 4.2 and Proposition 3.3, we have:[3]

Theorem 6.1. *The following statements about a sublocale S of L are equivalent.*

(i) *S is C^*-embedded in L.*
(ii) *For every $f\in\mathsf{C}^*(S)$ and every pair $r<s$ in $\mathbb{Q}$, $f(r,\text{—})$ and $f(\text{—},s)$ are completely separated in L.*
(iii) *Any two completely separated sublocales of S are completely separated in L.* ∎

[3] The counterpart to equivalence (i)⇔(iii) in the classical setting is known as the Uryshon's Extension Theorem (cf. [1, 6.6]).

Next result identifies C-embedded sublocales in among C^*-embedded sublocales.

Theorem 6.2. *The following statements about a sublocale S of L are equivalent.*

(i) *S is C-embedded in L.*
(ii) *S is C^*-embedded and every cozero sublocale of L disjoint from S is completely separated from S.*

Proof: Let S be a C-embedded sublocale of L. Of course, S is C^*-embedded. Consider a cozero sublocale C of L disjoint from S, say $C = f(0,\text{—})$, for some f such that $\mathbf{0} \le f \le \mathbf{1}$ (recall 2.4). Then consider the composite

$$h\colon \mathfrak{L}(\mathbb{R}) \xrightarrow{f} \mathsf{S}(L)^{op} \xrightarrow{j_{-1}[-]} \mathsf{S}(S)^{op}$$

where j is the localic embedding $S \hookrightarrow L$. The cozero sublocale defined by h is

$$\mathrm{coz}(h) = h(0,\text{—}) \cap h(\text{—},0) = j_{-1}[C] \cap j_{-1}[L] = \mathsf{O} \cap S = \mathsf{O}.$$

By Proposition 5.2, h has a multiplicative inverse $g\colon \mathfrak{L}(\mathbb{R}) \to \mathsf{S}(S)^{op}$ and since S is C-embedded, there is an extension

$$\bar{g}\colon \mathfrak{L}(\mathbb{R}) \to \mathsf{S}(L)^{op}$$

such that $j_{-1}[-]\bar{g} = g$. We may assume, without loss of generality, that $\bar{g} \ge \mathbf{0}$ (because $\bar{g} \vee 0$ will be also an extension of $g \ge \mathbf{0}$). Then we can use the formula for the product in 5.1 and conclude that $\bar{g} \cdot f$ completely separates the sublocales S and C. Indeed,

$$(\bar{g} \cdot f)(0,\text{—}) = \bigcap_{s>0} \bar{g}(s,\text{—}) \vee f(0,\text{—}) = C \vee \bigcap_{s>0} \bar{g}(s,\text{—}) = C \vee \bar{g}(0,\text{—}) \supseteq C$$

and

$$(\bar{g} \cdot f)(\text{—},1) = \bigcap_{s>0} \bar{g}(\text{—},s) \vee f(\text{—},\tfrac{1}{s})$$

from which it follows that

$$\begin{aligned} j_{-1}[(\bar{g} \cdot f)(\text{—},1)] &= \bigcap_{s>0} j_{-1}[\bar{g}(\text{—},s)] \vee j_{-1}[f(\text{—},\tfrac{1}{s})] = \\ &= \bigcap_{s>0} g(\text{—},s) \vee h(\text{—},\tfrac{1}{s}) = (h \cdot g)(\text{—},1) = \mathbf{1}(\text{—},1) = S \end{aligned}$$

and hence $S \cap (\bar{g} \cdot f)(\text{—},1) = S$, that is, $S \subseteq (\bar{g} \cdot f)(\text{—},1)$.

Conversely, assume that S is a C^*-embedded sublocale of L, with localic embedding $j\colon S \hookrightarrow L$ such that every cozero sublocale disjoint from S is completely separated from S. In order to show that S is also C-embedded consider an $f \in \mathsf{C}(S)$. Now recall Remark 1.4 and consider an order isomorphism ψ from $\langle -1, 1\rangle$ into $\mathbb{Q}$. Using the notation from 1.4, we have the commutative diagram

$$\begin{tikzcd}
\mathfrak{L}(\mathbb{R}) \arrow[rr, "\Psi_0"] \arrow[dr, two heads, "r_{(-1,1)}"'] & & \mathfrak{L}(\mathbb{R}) \arrow[dl, bend right, "\Phi"'] \\
& \mathfrak{L}(-1,1) \arrow[ur, bend right, "\Psi=\Psi_0|_{\mathfrak{L}(-1,1)}"'] \arrow[ur, phantom, "\cong"] &
\end{tikzcd}$$

The composite $f\Psi_0$ is a bounded frame homomorphism (since $\Psi_0(-1,1) = 1$). Hence, since S is C^*-embedded, there is an $\overline{f} \in \mathsf{C}^*(L)$ (with $\mathbf{p} < \overline{f} < \mathbf{q}$) such that the diagram

$$\begin{tikzcd}
\mathfrak{L}(\mathbb{R}) \arrow[rr, dotted, "\exists\overline{f}"] \arrow[d, two heads, "r_{(-1,1)}"'] \arrow[dr, "\Psi_0"] & & \mathsf{S}(L)^{op} \arrow[d, two heads, "j_{-1}[-]"] \\
\mathfrak{L}(-1,1) \arrow[r, bend right, "\Psi"'] \arrow[r, phantom, "\cong"] & \mathfrak{L}(\mathbb{R}) \arrow[l, bend right, "\Phi"'] \arrow[r, "f"] & \mathsf{S}(S)^{op}
\end{tikzcd}$$

commutes. To show that S is C-embedded it suffices to find a

$$\overline{g} \colon \mathfrak{L}(-1,1) \to \mathsf{S}(L)^{op}$$

such that $\overline{g}\, r_{(-1,1)} = \overline{f}$, because then $\overline{g}\,\Phi$ will be a continuous extension of f to L:

$$f\,\Psi\, r_{(-1,1)} = j_{-1}[-]\,\overline{f} = j_{-1}[-]\,\overline{g}\, r_{(-1,1)} \;\Rightarrow\; f\,\Psi = j_{-1}[-]\,\overline{g} \;\Leftrightarrow\; f = j_{-1}[-]\,\overline{g}\,\Phi.$$

For that, by Lemma 1.1, it suffices to find a $g \colon \mathfrak{L}(\mathbb{R}) \to \mathsf{S}(L)^{op}$ such that $g(-1,1) = \mathsf{O}$ and $j_{-1}[-]g = f\Psi_0$:

$$\begin{tikzcd}
\mathfrak{L}(\mathbb{R}) \arrow[rr, "\overline{f}"] \arrow[rr, dotted, bend left, "\exists g"] \arrow[d, two heads, "r_{(-1,1)}"'] \arrow[dr, "\Psi_0"] & & \mathsf{S}(L)^{op} \arrow[d, two heads, "j_{-1}[-]"] \\
\mathfrak{L}(-1,1) \arrow[urr, dotted, "\exists\overline{g}"] \arrow[r, bend right, "\Psi"'] \arrow[r, phantom, "\cong"] & \mathfrak{L}(\mathbb{R}) \arrow[l, bend right, "\Phi"'] \arrow[r, "f"] & \mathsf{S}(S)^{op}
\end{tikzcd}$$

We conclude the proof by showing how to get such map g.

Let $\overline{f}(-1,1) = \mathfrak{c}_L(a) \in \mathsf{CozS}(L)$. Then $j_{-1}[\mathfrak{c}_L(a)] = f\Psi_0(-1,1) = f(1) = \mathsf{O}$. On the other hand (recall 1.3 and (1.2.4)), $j_{-1}[\mathfrak{c}_L(a)] = \mathfrak{c}_S(j^*(a)) = \mathfrak{c}_L(a) \cap S$. Hence, $S \cap \mathfrak{c}_L(a) = \mathsf{O}$. Then, by assumption, S and $\mathfrak{c}_L(a)$ are completely separated, that is, there is an

$$h\colon \mathfrak{L}(\mathbb{R}) \to \mathsf{S}(L)^{op} \quad (\mathbf{0} \le h \le \mathbf{1})$$

such that $\mathfrak{c}_L(a) \subseteq h(0,\text{—})$ and $S \subseteq h(\text{—},1)$. We claim that $h\cdot\overline{f}$ is the function g we are searching for. We only need to check that $(h\cdot\overline{f})(-1,1) = \mathsf{O}$ and $j_{-1}[-](h\cdot\overline{f}) = f\Psi_0$.

By 1.6.1 we have

$$\begin{aligned}(h\cdot\overline{f})(-1,1) &= \bigcap\big\{h(r,s)\vee\overline{f}(t,u) \mid \langle r,s\rangle\cdot\langle t,u\rangle \subseteq \langle -1,1\rangle\big\}\\ &\subseteq \bigcap\big\{h(-y,y)\vee\overline{f}(-\tfrac{1}{y},\tfrac{1}{y}) \mid 1<y\big\}\\ &\overset{(*)}{=} \bigcap\big\{\overline{f}(-\tfrac{1}{y},\tfrac{1}{y}) \mid 1<y\big\} = \overline{f}(-1,1)\end{aligned}$$

(the equality $(*)$ follows from the fact that $\mathbf{0}\le h\le\mathbf{1}$). Consequently,

$$\begin{aligned}(h\cdot\overline{f})(-1,1) &= (h\cdot\overline{f})(-1,1)\cap\overline{f}(-1,1)\\ &= \bigcap\big\{h(r,s)\vee\overline{f}(u,v) \mid \langle r,s\rangle\cdot\langle u,v\rangle \subseteq \langle -1,1\rangle\big\}\cap\overline{f}(-1,1)\\ &\subseteq \bigcap\big\{h(-\tfrac{1}{y},\tfrac{1}{y})\vee\overline{f}(-y,y) \mid y>\max\{|p|,|q|,1\}\big\}\cap\overline{f}(-1,1)\\ &\overset{(*)}{=} \bigcap\big\{h(-\tfrac{1}{y},\tfrac{1}{y}) \mid y>\max\{|p|,|q|,1\}\big\}\cap\overline{f}(-1,1)\\ &\subseteq \bigcap\big\{h(-\tfrac{1}{y},\tfrac{1}{y}) \mid y>\max\{|p|,|q|,1\}\big\}\cap h(0,\text{—})\\ &= \bigcap\big\{h(-\tfrac{1}{y},\tfrac{1}{y})\cap h(0,\text{—}) \mid y>\max\{|p|,|q|,1\}\big\}\\ &\overset{(**)}{=} \bigcap\big\{h(-\tfrac{1}{y},\text{—}) \mid y>\max\{|p|,|q|,1\}\big\} = \mathsf{O}\end{aligned}$$

(where $(*)$ follows from $\mathbf{p}<\overline{f}<\mathbf{q}$ and $(**)$ from $h\ge\mathbf{0}$).

Finally, in order to show that $j_{-1}[-](h\cdot\overline{f}) = f\Psi_0$ note first that, for any $(u,v)\in\mathfrak{L}(\mathbb{R})$, if $1\notin\langle u,v\rangle$, then

$$\begin{aligned}j_{-1}[-]h(u,v) &\supseteq j_{-1}[-]h\left((1,\text{—})\vee(\text{—},1)\right) = j_{-1}[-]\left(h(1,\text{—})\vee h(\text{—},1)\right)\\ &\supseteq j_{-1}[-]\left(h(1,\text{—})\cap S\right) = j_{-1}[-](S) = S,\end{aligned}$$

otherwise,

$$\begin{aligned} \mathsf{O} = j_{-1}[-]h(1) &= j_{-1}[-]h\left((-,1) \vee (1,-) \vee (u,v)\right) \\ &= j_{-1}[-]h(u,v) \cap S = j_{-1}[-]h(u,v). \end{aligned}$$

Hence,

$$\begin{aligned} &j_{-1}[-](h{\cdot}\overline{f})(r,s) = \\ &\qquad = \bigcap \{ j_{-1}[h(u,v)] \vee f\Psi_0(z,w) \mid 1 \in \langle u,v\rangle, \langle u,v\rangle \cdot \langle z,w\rangle \subseteq \langle r,s\rangle \} \\ &\qquad = \bigcap \{ f\Psi_0(z,w) \mid 1 \in \langle u,v\rangle, \langle u,v\rangle \cdot \langle z,w\rangle \subseteq \langle r,s\rangle \} \\ &\qquad = \bigcap \{ f\Psi_0(z,w) \mid r < z < w < s \} = f\Psi_0(r,s). \end{aligned}$$ ■

7. z-embedded sublocales

As is well known (see e.g. [2]), normality can be characterized in terms of C-embedded and C^*-embedded sublocales:

Theorem 7.1. *The following are equivalent for a locale L:*

(i) *L is normal.*
(ii) *Any two disjoint closed sublocales are completely separated.*
(iii) *Every closed sublocale is C-embedded.*
(iv) *Every closed sublocale is C^*-embedded.*

Our purpose in this final section is to study another class of sublocales, the z-embedded sublocales, inspired by the classical results in [1, Section 7]. In particular, we will present a characterization of normality, similar to the one above, in terms of z-embeddings.

Recall from [12] that a sublocale S of L is *z-embedded* if for every zero sublocale Z of S there is a zero sublocale W of L such that $W \cap S = Z$; in other words, S is z-embedded if for every cozero sublocale C of S there is a cozero sublocale D of L such that $D \cap S = C$.

Remark 7.2. The motivation for this notion is the following. Let $f\colon L \to M$ be a localic map and recall the adjunction $f[-] \dashv f_{-1}[-]$ from 1.3. Recall also (1.3.1). Since frame homomorphisms preserve cozero elements, $f_{-1}[-]$ restricts to maps

$$f^{\mathrm{z}}_{-1}[-]\colon \mathsf{ZS}(M) \to \mathsf{ZS}(L) \quad \text{and} \quad f^{\mathrm{coz}}_{-1}[-]\colon \mathsf{CozS}(M) \to \mathsf{CozS}(L).$$

The former (the *zero map*) is a σ-frame homomorphism and the latter (the *cozero map*) is a σ-coframe homomorphism. Clearly, $f^{\mathrm{z}}_{-1}[-]$ is surjective if and only if $f_{-1}^{\mathrm{coz}}[-]$ is surjective. In this case, we say that f is a *z-map*. For the particular case of a localic embedding $j\colon S \hookrightarrow L$, j is a z-map iff S is *z-embedded* in L.

We do not pursue this approach in the present paper. The study of general z-maps is left to a subsequent article.

We start with a result that can be found in [6, Proposition 3.5] formulated in terms of frame quotients.

Proposition 7.3. *The following statements about a sublocale S of L are equivalent.*

(i) *S is z-embedded in L.*

(ii) *For any two completely separated sublocales U, V of S there is a $C \in \mathsf{CozS}(L)$, such that $U \subseteq C$ and $V \subseteq C^{\#}$.*

(iii) *If U and V are completely separated sublocales of S, then they are S-separated; i.e., there exist cozero sublocales of L, say C_1 and C_2, such that*

$$U \subseteq C_1 \quad V \subseteq C_2 \quad \textit{and} \quad C_1 \cap C_2 \cap S = \mathsf{O}.$$

Proof: (i)⇒(ii): Let U and V be completely separated in S. There exist cozero sublocales C_1 and C_2 in S such that $U \subseteq C_1$, $V \subseteq C_2$ and $C_1 \cap C_2 = \mathsf{O}$. Since S is z-embedded, there exists also a cozero sublocale C of L such that $C_1 = S \cap C$. Clearly, $U \subseteq C_1 \subseteq C$. Since $C_1 \cap V \subseteq C_1 \cap C_2 = \mathsf{O}$ we have

$$V \subseteq C_1^{\#} = (S \cap C)^{\#} = \mathsf{O} \vee C^{\#} = C^{\#}.$$

(ii)⇒(iii): By 3.1 it suffices to show statement (iii) for disjoint cozero sublocales of S. Take $C_1, C_2 \in \mathsf{CozS}(S)$ such that $C_1 \cap C_2 = \mathsf{O}$. From the assumption there is a cozero sublocale C of L such that $C_1 \subseteq C$ and $C_2 \subseteq C^{\#}$. Now, $S \cap C$ is a cozero sublocale in S disjoint from C_2. Applying the assumption again we get a cozero sublocale D of L such that $C_2 \subseteq D$ and $S \cap C \subseteq D^{\#}$. Hence, $C \cap S \cap D = \mathsf{O}$ as required.

(iii)⇒(i): Let C be a cozero sublocale of S. Then $C = f(0, —)$ for some $f \in \mathsf{C}(S)$ with $\mathbf{0} \leq f \leq \mathbf{1}$. Therefore, C is a closed sublocale $\mathfrak{c}_S(a)$ with $a \in \mathrm{Coz}\,(S)$. Furthermore, consider the cozero sublocales

$$B_n = f(—, \tfrac{1}{n}) = \mathfrak{c}_S(b_n) \quad (b_n \in \mathrm{Coz}\,(S), n = 1, 2, \ldots).$$

Note that B_n and C are completely separated in S. Indeed,

$$B_n \cap C = f(—, \tfrac{1}{n}) \cap f(0, —) = f((—, \tfrac{1}{n}) \vee (0, —)) = f(1) = \mathsf{O}.$$

By assumption, there are $\mathfrak{c}_L(a_n)$ and $\mathfrak{c}_L(d_n)$ with $a_n, d_n \in \mathrm{Coz}\,(L)$ such that

$$\mathfrak{c}_S(a) \subseteq \mathfrak{c}_L(a_n), \quad \mathfrak{c}_S(b_n) \subseteq \mathfrak{c}_L(d_n) \quad \text{and} \quad \mathfrak{c}_L(a_n) \cap \mathfrak{c}_L(d_n) \cap S = \mathsf{O}$$

for every $n \in \mathbb{N}$. Consider now the cozero sublocale $\bigcap_{n=1}^{\infty} \mathfrak{c}_L(a_n) = \mathfrak{c}_L(v)$ where $v \in \mathrm{Coz}\,(L)$ and recall (1.2.4). Then

$$\mathfrak{c}_L(v) \cap S = \mathfrak{c}_S(j_S^*(v)) \quad \text{and} \quad \mathfrak{c}_L(d_n) \cap S = \mathfrak{c}_S(j_S^*(d_n)).$$

Clearly, $\mathfrak{c}_S(a) \subseteq \mathfrak{c}_S(j_S^*(v))$. In order to show that this is indeed an equality, note first that $\mathfrak{c}_S(j_S^*(v)) \cap \mathfrak{c}_S(j_S^*(d_n)) = \mathsf{O}$. Then

$$\mathfrak{c}_S(j_S^*(v)) \subseteq \mathfrak{c}_S(j_S^*(d_n))^{\#_S} = \mathfrak{o}_S(j_S^*(d_n))$$

and $f(\text{—}, \frac{1}{n}) = \mathfrak{c}_S(b_n) \subseteq \mathfrak{c}_S(j_S^*(d_n))$. Therefore, taking supplements in S,

$$\mathfrak{c}_S(j_S^*(v)) \subseteq \mathfrak{o}_S(b_n) \subseteq f(\text{—}, \tfrac{1}{n})^{\#_S} \subseteq f((\text{—}, \tfrac{1}{n})^*) \subseteq f(\tfrac{1}{n}, \text{—})$$

for $n = 1, 2, \ldots$. Hence,

$$\mathfrak{c}_L(v) \cap S = \mathfrak{c}_S(j_S^*(v)) \subseteq \bigcap_{n=1}^{\infty} f(\tfrac{1}{n}, \text{—}) = f\left(\bigvee_{n=1}^{\infty} (\tfrac{1}{n}, \text{—})\right) = f(0, \text{—}) = c_S(a). \ \blacksquare$$

Clearly, if T is a sublocale of S, z-embedded in L, then it is z-embedded in S. It is also easy to see that any C^*-embedded sublocale in L is z-embedded in L. Indeed, if $C \in \mathsf{CozS}(S)$ then $C = f(0, \text{—})$ for some $f \in \mathsf{C}^*(S)$ and $\mathbf{0} \leq f \leq \mathbf{1}$; then there exists an $\overline{f} \in \mathsf{C}^*(L)$ such that $j_{-1}[-]\overline{f} = f$, which means that

$$S \cap \overline{f}(0, \text{—}) \cap \overline{f}(\text{—}, 0) = j_{-1}[\overline{f}((0, \text{—}) \vee (\text{—}, 0))] = f(0, \text{—}) = C$$

(and $\overline{f}(0, \text{—}) \cap \overline{f}(\text{—}, 0) = \overline{f}((\text{—}, 0) \vee (0, \text{—})) \in \mathrm{Coz}\, L$).

In conclusion,

$$C\text{-embedded} \;\Rightarrow\; C^*\text{-embedded} \;\Rightarrow\; z\text{-embedded}.$$

Next result is a consequence of our extension theorem (via Theorem 6.1).

Proposition 7.4. *A sublocale S of a locale L is C^*-embedded in L if and only if it is z-embedded in L and for any sublocale T of S and $C \in \mathsf{CozS}(L)$, if T and $S \cap C$ are S-separated in L, then T and $S \cap C$ are completely separated.*

Proof: If S is a C^*-embedded sublocale of L then it is z-embedded. Consider a sublocale T of S and $C \in \mathsf{CozS}(L)$ such that T and $S \cap C$ are S-separated in L. This means there are $C_1, C_2 \in \mathsf{CozS}(L)$ such that

$$T \subseteq C_1, \quad S \cap C \subseteq C_2 \quad \text{and} \quad C_1 \cap C_2 \cap S = \mathsf{O}.$$

In particular, $T \subseteq C_1 \cap S$ and $S \cap C \subseteq C_2 \cap S$. Thus, T and $S \cap C$ are completely separated in S because $C_1 \cap S, C_2 \cap S \in \mathsf{CozS}(S)$. Finally, by Theorem 6.1, T and $S \cap C$ are completely separated in L.

Conversely, we will prove that S is C^*-embedded using the characterization in 6.1. Let T and V be completely separated in S. There exist $C_1, C_2 \in \mathsf{CozSub}(S)$ such that $T \subseteq C_1$, $V \subseteq C_2$ and $C_1 \cap C_2 = \mathsf{O}$. Since S is z-embedded, then $C_1 = S \cap U_1$ and $C_2 = S \cap U_2$ for some $U_1, U_2 \in \mathsf{CozS}(L)$. Thus, $S \cap U_1$ and $S \cap U_2$ are S-separated, and by assumption they must be completely separated in L. Hence, T and V are also completely separated in L. ■

Remark 7.5. This result is another good example of the advantages of sublocale language in terms of conciseness and clarity. Indeed, the result is stated in [6, Proposition 4.3] in terms of frame quotients; a closer inspection to assertions (2) and (3) reveals, when formulated in terms of sublocales, that they express precisely the same fact.

A further consequence of the extension theorem is the following:

Proposition 7.6. *A sublocale S of a locale L is C-embedded in L if and only if it is z-embedded in L and it is completely separated from every cozero sublocale disjoint from it.*

Proof: If S is C-embedded then it is z-embedded. The other conclusion follows from Theorem 6.2. Conversely, assume that S is z-embedded in L and it is completely separated from every cozero sublocale disjoint from it. By 6.2, it suffices to show that S is C^*-embedded. We will do this using Theorem 6.1. Consider completely separated sublocales T and M in S. They are S-separated by 7.3, meaning that there are $C_1, C_2 \in \mathsf{CozS}(L)$ such that

$$T \subseteq C_1,\ M \subseteq C_2 \quad \text{and} \quad C_1 \cap C_2 \cap S = \mathsf{O}.$$

Take $C = C_1 \cap C_2 \in \mathsf{CozS}(L)$. Then $C \cap S = \mathsf{O}$ and by assumption, there is a $D \in \mathsf{CozS}(L)$ such that $S \subseteq D$ and $C \cap D = \mathsf{O}$. Hence,

$$T \subseteq D \cap C_1, \quad M \subseteq D \cap C_2 \quad \text{and} \quad D \cap C_1 \cap C_2 = D \cap C = \mathsf{O},$$

which means that S and T are completely separated in L. ■

Recall that a sublocale S of L is G_δ-*dense* in L if $S \cap A \neq \mathsf{O}$ for every G_δ-sublocale A of L. The preceding result characterizes C-embedded sublocales in among z-embedded sublocales. Next proposition shows that the class of z-embedded G_δ-dense sublocales is an example of such C-embedded sublocales.

Proposition 7.7. *Any z-embedded G_δ-dense sublocale of L is C-embedded in L.*

Proof: Let S be a z-embedded G_δ-dense sublocale of L. Recall from [12, Corollary 5.6.1] that every cozero sublocale is a G_δ-sublocale. Hence there is no cozero sublocale disjoint from S and S is C-embedded by the previous proposition. ∎

Lemma 7.8. *In a normal locale, every F_σ-sublocale is z-embedded.*

Proof: Let S be an F_σ-sublocale of L, say $S = \bigvee_{n=1}^{\infty} \mathfrak{c}(a_n)$. Consider a cozero sublocale $C = g(0,—)$ in S for some $g \in \mathsf{C}^*(S)$ with $\mathbf{0} \leq g \leq \mathbf{1}$. Let $\mathfrak{c}(b)$ be the closure of C in L. Furtermore, consider for each $n = 1, 2, \ldots$

$$T_n = \mathfrak{c}(a_n) \vee \mathfrak{c}(b) = \mathfrak{c}(a_n \wedge b)$$

and $g_n \colon \mathfrak{L}(\mathbb{R}) \to \mathsf{S}(T_n)^{op}$ defined by

$$g_n(p,—) = \begin{cases} \mathsf{O} & \text{if } p < 0 \\ \mathfrak{c}(b) \vee (g(p,—) \cap \mathfrak{c}(a_n)) & \text{if } 0 \leq p < 1 \\ \mathfrak{c}(b) \vee \mathfrak{c}(a_n) & \text{if } p \geq 1 \end{cases}$$

and

$$g_n(—,q) = \begin{cases} \mathfrak{c}(b) \vee \mathfrak{c}(a_n) & \text{if } q \leq 0 \\ g(—,q) \cap \mathfrak{c}(a_n) & \text{if } 0 < q \leq 1 \\ \mathsf{O} & \text{if } q > 1. \end{cases}$$

Let us confirm that this defines indeed a frame homomorphism by checking that it turns relations (r1)-(r6) into identities in the frame $\mathsf{S}(T_n)^{op}$:

(r1) $g_n(p,—) \vee g_n(—,q) = \mathfrak{c}(b) \vee \mathfrak{c}(a_n)$ whenever $p \geq q$:
The only nontrivial case is when $0 \leq p < 1$ and $0 < q \leq 1$ where we have

$$\begin{aligned} g_n(p,—) \vee g_n(—,q) &= (\mathfrak{c}(b) \vee (g(p,—) \cap \mathfrak{c}(a_n))) \vee (g(—,q) \cap \mathfrak{c}(a_n)) \\ &= \mathfrak{c}(b) \vee (\mathfrak{c}(a_n) \cap (g(p,—) \vee g(—,q)) = \mathfrak{c}(b) \vee (\mathfrak{c}(a_n) \cap S) \\ &= \mathfrak{c}(b) \vee \mathfrak{c}(a_n). \end{aligned}$$

(r2) $g_n(p,\text{—}) \cap g_n(\text{—},q) = \mathsf{O}$ whenever $p < q$:
The only nontrivial case is when $0 \le p < 1$ and $0 < q \le 1$ and we have

$$\begin{aligned} g_n(p,\text{—}) \cap g_n(\text{—},q) &= (\mathfrak{c}(b) \vee (g(p,\text{—}) \cap \mathfrak{c}(a_n))) \cap (g(\text{—},q) \cap \mathfrak{c}(a_n)) \\ &= (\mathfrak{c}(b) \cap g(\text{—},q) \cap \mathfrak{c}(a_n)) \vee (g(p,\text{—}) \cap \mathfrak{c}(a_n) \cap g(\text{—},q)) \\ &= (\mathfrak{c}(b) \cap g(\text{—},q) \cap \mathfrak{c}(a_n) \cap S) \vee \mathsf{O} \\ &= g(0,\text{—}) \cap g(\text{—},q) \cap \mathfrak{c}(a_n) = \mathsf{O}. \end{aligned}$$

(Note that we are using the fact that $\mathfrak{c}(b)$ is the closure in L of $g(0,\text{—})$; hence, $\mathfrak{c}(b) \cap S = g(0,\text{—})$.)

(r3) $\bigcap_{r>p} g_n(r,\text{—}) = g_n(p,\text{—})$:
For the only nontrivial case, when $0 \le p < 1$, we have

$$\begin{aligned} \bigcap_{r>p} g_n(r,\text{—}) &= \mathfrak{c}(b) \vee \bigcap_{1>r>p} (g(r,\text{—}) \cap \mathfrak{c}(a_n)) \\ &= \mathfrak{c}(b) \vee (g(p,\text{—}) \cap \mathfrak{c}(a_n)) = g_n(p,\text{—}). \end{aligned}$$

Note that the second equality holds since g is a frame homomorphism and $g \le \mathbf{1}$.

(r4) $\bigcap_{s<q} g_n(\text{—},s) = g_n(\text{—},q)$:
For the only nontrivial case $0 < q \le 1$ we have

$$\bigcap_{s<q} g_n(\text{—},s) = \bigcap_{0<s<q} (g(\text{—},s) \cap \mathfrak{c}(a_n)) = g(\text{—},q) \cap \mathfrak{c}(a_n) = g_n(\text{—},q)$$

(the second equality holds since g is a frame homomorphism and $\mathbf{0} \le g$).

(r5) $\bigcap_{p\in\mathbb{Q}} g_n(p,\text{—}) = \mathsf{O}$ is clear.

(r6) $\bigcap_{q\in\mathbb{Q}} g_n(\text{—},q) = \mathsf{O}$ is also obvious.

In order to see that g_n is continuous for every n it suffices to check that $g(p,\text{—}) \cap \mathfrak{c}(a_n)$ and $g(\text{—},q) \cap \mathfrak{c}(a_n)$ are closed sublocales in T_n for every $0 \le p < 1$ and $0 < q \le 1$. Regarding the former, since g is continuous, $g(p,\text{—})$ is closed in S and thus there is a $d \in L$ such that $\mathfrak{c}(d) \cap S = g(p,\text{—})$. Hence

$$\begin{aligned} (\mathfrak{c}(d) \cap \mathfrak{c}(a_n)) \cap (\mathfrak{c}(a_n) \vee \mathfrak{c}(b)) &= (\mathfrak{c}(d) \cap \mathfrak{c}(a_n)) \vee (\mathfrak{c}(d) \cap \mathfrak{c}(a_n) \cap \mathfrak{c}(b)) \\ &= (g(p,\text{—}) \cap \mathfrak{c}(a_n)) \vee (g(p,\text{—}) \cap \mathfrak{c}(a_n) \cap g(0,\text{—})) \\ &= (g(p,\text{—}) \cap \mathfrak{c}(a_n)) \vee (\mathfrak{c}(a_n) \cap g(0,\text{—})) \\ &= \mathfrak{c}(a_n) \cap (g(p,\text{—}) \vee g(0,\text{—})) = g(p,\text{—}) \cap \mathfrak{c}(a_n). \end{aligned}$$

Similarly, if $\mathfrak{c}(d) \cap S = g(—, q)$ we have

$$\begin{aligned}(\mathfrak{c}(d) \cap \mathfrak{c}(a_n)) \cap (\mathfrak{c}(a_n) \vee \mathfrak{c}(b)) &= (\mathfrak{c}(d) \cap \mathfrak{c}(a_n)) \vee (\mathfrak{c}(d) \cap \mathfrak{c}(a_n) \cap \mathfrak{c}(b)) \\ &= (g(—, q) \cap \mathfrak{c}(a_n)) \vee (g(—, q) \cap \mathfrak{c}(a_n) \cap g(0, —)) \\ &= (g(—, q) \cap \mathfrak{c}(a_n)) \vee \mathsf{O} = g(—, q) \cap \mathfrak{c}(a_n).\end{aligned}$$

By Theorem 7.1 we know that T_n is C-embedded in L. Consequently, there are $f_n \in \mathsf{C}(L)$ $(n = 1, 2, \ldots)$ such that $(j_n)_{-1}[-]f_n = g_n$ where j_n is the localic embedding of T_n in L. Take

$$F = \bigcap_{n=1}^{\infty} (f_n(0, —) \cap f_n(—, 0)) \in \mathsf{CozS}(L).$$

We claim that $F \cap S = C$. First note that

$$\begin{aligned}g_n(0, —) \cap g_n(—, 0) &= (\mathfrak{c}(b) \vee (g(0, —) \cap \mathfrak{c}(a_n))) \cap (\mathfrak{c}(b) \vee \mathfrak{c}(a_n)) \\ &= \mathfrak{c}(b) \vee ((\mathfrak{c}(b) \vee (g(0, —) \cap \mathfrak{c}(a_n))) \cap \mathfrak{c}(a_n)) \\ &= \mathfrak{c}(b) \vee (\mathfrak{c}(b) \cap \mathfrak{c}(a_n)) \vee (\mathfrak{c}(a_n) \cap g(0, —)) \\ &= \mathfrak{c}(b) \vee (\mathfrak{c}(a_n) \wedge g(0, —)) = (\mathfrak{c}(b) \vee \mathfrak{c}(a_n)) \cap (\mathfrak{c}(b) \vee g(0, -)) \\ &= (\mathfrak{c}(b) \vee \mathfrak{c}(a_n)) \cap \mathfrak{c}(b) = \mathfrak{c}(b),\end{aligned}$$

hence $g(0, —) \subseteq \mathfrak{c}(b) = g_n(0, —) \cap g_n(—, 0) \subseteq f_n(0, —) \cap f_n(—, 0)$ for every n. Therefore, $g(0, —) \subseteq F \cap S$. For the converse inclusion we have

$$\begin{aligned}F \cap S &= \bigvee F \cap \mathfrak{c}(a_n) \subseteq \bigvee f_n(0, —) \cap f_n(—, 0) \cap \mathfrak{c}(a_n) \\ &= \bigvee f_n(0, —) \cap f_n(—, 0) \cap \mathfrak{c}(a_n) \cap T_n = \bigvee g_n(0, —) \cap g_n(—, 0) \cap \mathfrak{c}(a_n) \\ &= \bigvee \mathfrak{c}(b) \cap \mathfrak{c}(a_n) = \mathfrak{c}(b) \cap S = g(0, —)\end{aligned}$$

where the first and the last equalities hold because F and $\mathfrak{c}(b)$ are closed sublocales in particular complemented sublocales ([14, VI.4.4.1]). We have shown that an arbitrary cozero sublocale of S is the intersection of S with a cozero sublocale in L. In conclusion, S is z-embedded in L. ■

Lemma 7.9. *If S is a sublocale of L with the property that whenever $S \subseteq \mathfrak{o}(a)$ there is a normal and z-embedded F such that $S \subseteq F \subseteq \mathfrak{o}(a)$, then S is z-embedded in L.*

Proof: Let $A = \mathfrak{c}_S(b)$ (with $b \in \mathrm{Coz}(S)$) be a cozero sublocale of S. Then A is a G_δ-sublocale of S, that is, $A = \bigcap_{n=1}^{\infty} \mathfrak{o}_S(b_n)$ for some $b_n \in S$. Consider the open sublocales $\mathfrak{o}_L(b \vee b_n)$ for $n = 1, 2, \ldots$. Since $\mathfrak{c}_S(b) \subseteq \mathfrak{o}_S(b_n)$ we have

$$S \cap \mathfrak{c}_L(b \vee b_n) = S \cap \mathfrak{c}_L(b) \cap \mathfrak{c}_L(b_n) = \mathfrak{c}_S(b) \cap \mathfrak{c}_S(b_n) = \mathsf{O}.$$

Hence $S \subseteq \mathfrak{o}_L(b \vee b_n)$. By assumption, there is for each n a normal and z-embedded sublocale T_n such that $S \subseteq T_n \subseteq \mathfrak{o}_L(b \vee b_n)$. Note that $\mathfrak{c}_L(b) \cap T_n$ and $\mathfrak{c}_L(b_n) \cap T_n$ are disjoint; indeed

$$\mathfrak{c}_L(b) \cap T_n \cap \mathfrak{c}_L(b_n) \subseteq \mathfrak{c}_L(b \vee b_n) \cap \mathfrak{o}_L(b \vee b_n) = \mathsf{O}.$$

Recall Remark 3.2. By the normality of T_n, $\mathfrak{c}_L(b) \cap T_n$ and $\mathfrak{c}_L(b_n) \cap T_n$ are then completely separated in T_n. Consequently, there is a $C_n \in \mathsf{CozS}(T_n)$ such that

$$T_n \cap \mathfrak{c}_L(b) \subseteq C_n \quad \text{and} \quad T_n \cap \mathfrak{c}_L(b_n) \cap C_n = \mathsf{O}.$$

On the other hand, by z-embeddedness of T_n there is a $C'_n \in \mathsf{CozS}(L)$ such that $T_n \cap C'_n = C_n$. Finally, consider the cozero sublocale $\bigcap_{n=1}^{\infty} C'_n$. We claim $A = S \cap \bigcap_{n=1}^{\infty} C'_n$. The inclusion '$\subseteq$' is clear because

$$A \subseteq S \quad \text{and} \quad A \subseteq \mathfrak{c}_L(b) \cap T_n \subseteq C_n \subseteq C'_n$$

for every n. Conversely,

$$S \cap \bigcap_{n=1}^{\infty} C'_n = S \cap \bigcap_{n=1}^{\infty} (C'_n \cap T_n) = S \cap \bigcap_{n=1}^{\infty} C_n \overset{(*)}{\subseteq} S \cap \bigcap_{n=1}^{\infty} \mathfrak{o}_L(b_n) = \bigcap_{n=1}^{\infty} \mathfrak{o}_S(b_n) = A.$$

where $(*)$ holds because $T_n \cap \mathfrak{c}_L(b_n) \cap C_n = \mathsf{O}$ hence $C_n = T_n \cap C_n \subseteq \mathfrak{o}_L(b_n)$. ∎

Finally, we say that a sublocale S of L is *F_σ-generalized* if for every open sublocale T of L with $S \subseteq T$, there exists an F_σ-sublocale F such that $S \subseteq F \subseteq T$. Obviously, any F_σ-sublocale is F_σ-generalized.

Theorem 7.10. *The following statements about a locale L are equivalent.*

(i) *L is normal.*
(ii) *Every closed sublocale of L is z-embedded in L.*
(iii) *Every F_σ-sublocale of L is z-embedded in L.*
(iv) *Every generalized F_σ-sublocale of L is z-embedded in L.*
(v) *For any closed sublocale F of L and any cozero sublocale C of L, $F \vee C$ is z-embedded in L.*
(vi) *For any closed sublocale F of L and any cozero sublocale C of L such that $F \cap C = \mathsf{O}$, $F \vee C$ is z-embedded in L.*

Proof: (i)⇒(iv): Let S be a generalized F_σ-sublocale of L. If $S \subseteq \mathfrak{o}(a)$ then there is an F_σ-sublocale F such that $S \subseteq F \subseteq \mathfrak{o}(a)$. By Lemma 7.8, F is z-embedded. On the other hand, being an F_σ-sublocale of a normal locale, F is also normal (see [10, Proposition 6.4]). Hence, the conclusion follows from Lemma 7.9.

(iv)⇒(iii)⇒(ii) is obvious.

(ii)⇒(i): To prove that L is normal it suffices to show that every closed sublocale is C-embedded (by Theorem 7.1). Let $\mathfrak{c}(s)$ be a closed sublocale of L; by assumption, it is z-embedded. We will use Proposition 7.6 to show that it is also C-embedded. Let $\mathfrak{c}(a) \in \mathsf{CozS}(L)$ such that $\mathfrak{c}(s) \cap \mathfrak{c}(a) = \mathsf{O}$ and consider the closed sublocale $T = \mathfrak{c}(a) \vee \mathfrak{c}(s)$. We claim $\mathfrak{c}(s)$ is a cozero sublocale of T. Indeed,

$$\mathfrak{o}(a) \cap T = \mathfrak{o}(a) \cap (\mathfrak{c}(a) \vee \mathfrak{c}(s)) = \mathfrak{c}(s) \quad \text{and} \quad \mathfrak{c}(s) \cap T = \mathfrak{c}(s) \cap (\mathfrak{c}(a) \vee \mathfrak{c}(s)) = \mathfrak{c}(s)$$

assert that $\mathfrak{c}(s)$ is a clopen sublocale of T; therefore (recall 1.7) it is both a zero and a cozero sublocale of T. By assumption, T is z-embedded because it is closed; thus, $\mathfrak{c}(s) = T \cap C$ for some $C \in \mathsf{CozS}(L)$. Hence,

$$\mathfrak{c}(s) \subseteq C \quad \text{and} \quad C \cap \mathfrak{c}(a) = C \cap (\mathfrak{c}(a) \cap T) = \mathfrak{c}(s) \cap \mathfrak{c}(a) = \mathsf{O}$$

which means that $\mathfrak{c}(a)$ and $\mathfrak{c}(s)$ are completely separated, as required.

(iii)⇒(v) is clear (since $F \vee C$ is an F_σ-sublocale), (v)⇒(vi) is trivial and (vi)⇒(ii) follows by taking $C = \mathsf{O}$. ■

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