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INTRINSIC SCHREIER SPECIAL OBJECTS

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Abstract: Motivated by the categorical-algebraic analysis of split epimorphisms of monoids, we study the concept of a *special object* induced by the *intrinsic Schreier split epimorphisms* in the context of a regular unital category with binary coproducts, comonadic covers and a natural imaginary splitting in the sense of our article [20]. In this context, each object comes naturally equipped with an imaginary magma structure. We analyse the intrinsic Schreier split epimorphisms in this setting, showing that their properties improve when the imaginary magma structures happen to be associative. We compare the intrinsic Schreier special objects with the protomodular objects, and characterise them in terms of the imaginary magma structure. We furthermore relate them to the Engel property in the case of groups and Lie algebras.

Keywords: Imaginary morphism; approximate operation; regular, unital, protomodular category; monoid; 2-Engel group, Lie algebra; Jónsson–Tarski variety.
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1. Introduction

Recently, two different categorical approaches have been developed which aim to describe the homological properties of monoids, mainly in comparison with the properties groups have. The first one started with the observation that an important class of split epimorphisms of monoids, called *Schreier split epimorphisms*, satisfies the convenient properties of split epimorphisms of groups [8, 9]. The idea of considering Schreier split epimorphisms originated from the fact that these split epimorphisms correspond to monoid actions in the usual sense [21, 18]. Although the category of monoids is not protomodular, Schreier split epimorphisms satisfy the properties that are typical for split epimorphisms in a protomodular category. This led to the notion of an $\mathscr{S}$*-protomodular category*, with respect to a chosen class $\mathscr{S}$ of points—i.e., split epimorphisms with fixed section [10]. In an $\mathscr{S}$-protomodular category, it is always possible

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to identify a full subcategory which is protomodular [2], called in [9] the *protomodular core* with respect to the class $\mathscr{S}$. The objects of this subcategory are the *$\mathscr{S}$-special objects*, namely those objects X for which the split epimorphism $X \times X \leftrightarrows X$, given by the second product projection and the diagonal morphism, belongs to $\mathscr{S}$. The category of monoids is not protomodular but it is $\mathscr{S}$-protomodular with respect to the class of Schreier split epimorphisms, and its protomodular core is the category of groups.

The second approach consists in considering, in a pointed category with finite limits, a suitable class of objects, called *protomodular objects* [19]. These are the objects Y such that every split epimorphism with codomain Y is *stably strong*. A split epimorphism is *strongly split* if its kernel and section are jointly extremal-epimorphic. It is *stably strong* if every pullback of it along any morphism is a strongly split epimorphism. As proved in [19], in the category of monoids the protomodular objects are precisely the groups.

The notion of protomodular object makes sense in every (pointed) category with finite limits, while Schreier special objects can apparently be considered only in the context of a Jónsson–Tarski variety [15], because the notion of Schreier split epimorphism depends on the existence of a function, which is not a morphism in general, called the *Schreier retraction*. In order to study this from a categorical perspective, we introduced in [20] the concept of *intrinsic Schreier split epimorphism*, in the context of a regular unital category [3] equipped with a comonadic cover (in the sense we recall in Subsection 2.3). This approach is inspired by the notion of imaginary morphism [5]: indeed, the Schreier retraction we need is such an imaginary morphism. We showed in [20] that these categories are $\mathscr{S}$-protomodular with respect to the class of intrinsic Schreier split epimorphisms, and we obtained an intrinsic version of the so-called Schreier special objects. It is shown in [20] that the concepts of intrinsic Schreier special object and protomodular object are independent. Since, however, the two coincide in the category of monoids, the question of understanding when the two notions are related arises naturally.

One of the goals of the present paper is to give an answer to this question. An important ingredient here is the observation that, when considering the Kleisli category associated with the comonad involved in the definition of an intrinsic Schreier split epimorphism, the definition itself simplifies greatly (Section 5). Also, each object admits a canonical imaginary magma structure whose operation (called *imaginary addition* in the text) depends on a choice of a natural imaginary splitting, which is part of our initial setting (Section 4). It turns out

where $\phi(\underline{x}, \overline{y})$ is an expression in terms of Lie brackets of $\underline{x}$'s and $\overline{y}$'s. Now using essentially the same proof as in groups, we see that if $\mathfrak{y}$ is 2-Engel, all higher-order brackets vanish, and we deduce that

$$t_{\mathfrak{x},\mathfrak{y}}(x, y) = \underline{x} + \overline{y} + k[\underline{x}, \overline{y}]$$

for some $k \in \mathbb{K}$. As in Proposition 11.9, it follows that $x * y = x + y + k[x, y]$. It is then again easy to check that **(iSs1)** and **(iSs2)** hold.

Proposition 12.1. *Any 2-Engel $\mathbb{K}$-Lie algebra is intrinsic Schreier special, with respect to all natural imaginary splittings in* $\mathsf{Lie}_{\mathbb{K}}$. ∎

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Proof: The proof is similar to that of Proposition 11.5. We define the imaginary retraction by $q(x,y) = x * y^{-1} = [x, y^{-1}]^k xy^{-1}$ (Proposition 11.9). We use the properties in Definition 11.1 and Lemma 11.4 to prove that **(iSs1)** holds:

$$\begin{aligned}
(x * y^{-1}) * y &= ([x, y^{-1}]^k\, xy^{-1}) * y \\
&= \left[[x, y^{-1}]^k xy^{-1}, y\right]^k\, [x, y^{-1}]^k\, xy^{-1}y \\
&= \left[[x, y]^{-k} xy^{-1}, y\right]^k [x, y]^{-k} x \\
&= \left({}^{[x,y]^{-k}}[x, y][[x, y]^{-k}, y]\right)^k [x, y]^{-k} x \\
&= \left(\left({}^{[x,y]^{-k}}[x, y]\right)\left({}^{[x,y]^{-k}}[[x, y]^{-k}, y]\right)\right)^k [x, y]^{-k} x \\
&= [x, y]^k [x, y]^{-k} x \\
&= x.
\end{aligned}$$

As for **(iSs2)**, the equality $q(x * y, y) = (x * y) * y^{-1} = x$ follows from **(iSs1)** by replacing y with y^{-1}. ∎

It remains an open question whether or not the converse of Proposition 11.10 holds; we are currently working on this question. Essentially the same result holds for Lie algebras, as we shall explain now.

12. Lie algebras

Let $\mathbb{K}$ be a field, and consider the variety $\mathsf{Lie}_{\mathbb{K}}$ of $\mathbb{K}$-Lie algebras. Recall that a 2-**Engel** Lie algebra is a Lie algebra $\mathfrak{e}$ that satisfies the commutator identity $[[x, y], y] = 1$ for all $x,\ y \in \mathfrak{e}$. The aim of this section is to relate the variety $\mathsf{Eng}_2(\mathsf{Lie}_{\mathbb{K}})$ of 2-Engel Lie algebras over $\mathbb{K}$ to the Schreier special objects with respect to all natural imaginary splittings: Theorem 12.1. We can actually just follow the pattern of the previous section; since furthermore things are somewhat simpler here, we will only sketch the basic idea.

We may proceed as in Proposition 11.5, now taking the natural imaginary splitting in $\mathsf{Lie}_{\mathbb{K}}$ defined by

$$t_{\mathfrak{x},\mathfrak{y}} \colon \mathfrak{x} \times \mathfrak{y} \dashrightarrow \mathfrak{x} + \mathfrak{y} \colon (x, y) \mapsto \underline{x} + \overline{y} + [\underline{x}, \overline{y}].$$

Recall that the free Lie algebra over $\mathbb{K}$ on a single generator is $\mathbb{K}$ itself, equipped with the trivial bracket. Mimicking the proof of Lemma 11.8, we see that for any pair of $\mathbb{K}$-Lie algebras $\mathfrak{x}$ and $\mathfrak{y}$ and any $x \in \mathfrak{x}$, $y \in \mathfrak{y}$, necessarily

$$t_{\mathfrak{x},\mathfrak{y}}(x, y) = \underline{x} + \overline{y} + \phi(\underline{x}, \overline{y}),$$

that an object is intrinsic Schreier special precisely when its imaginary magma structure is a one-sided loop structure (Theorem 8.2). Under the assumption that the imaginary addition is associative (Section 6) we are able to extend several stability properties and homological lemmas which hold for Schreier extensions of monoids [8] to our intrinsic context (Section 7). Moreover, we prove that every intrinsic Schreier special object is a protomodular object (Corollary 10.4).

It was shown in [20] that there are only two possible choices for the natural imaginary splitting in the category of monoids, which leads to only two possible imaginary additions. This is no longer true for the category of groups or Lie algebras, where many options are available. Therefore, we focus on studying intrinsic Schreier special objects with respect to *all* natural imaginary additions in these categories. We prove that 2-Engel groups are intrinsic Schreier special with respect to all possible imaginary additions (Proposition 11.10). The similar result also holds for Lie algebras (Proposition 12.1).

2. Imaginary morphisms

In this section we recall the notions of imaginary morphism needed throughout the text. We fix the particular setting for which we consider imaginary morphisms in this work.

2.1. Imaginary morphisms [7]**.** We take $\mathbb{X}$ to be the functor category $\mathsf{Set}^{\mathbb{C}^{\mathrm{op}} \times \mathbb{C}}$, where $\mathbb{C}$ is an arbitrary (small) category. Consider functors $\hom_{\mathbb{C}}$ and $A \colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathsf{Set}$ and a natural transformation $\alpha \colon \hom_{\mathbb{C}} \Rightarrow A$. If all the components $\alpha_{X,Y} \colon \hom_{\mathbb{C}}(X, Y) \to A(X, Y)$ are injective, then all sets $A(X, Y)$ contain (an isomorphic copy of) $\hom_{\mathbb{C}}(X, Y)$. So, we may think of $A(X, Y)$ as an *extension* of $\hom_{\mathbb{C}}(X, Y)$, and indeed in [7] the triple $(\mathbb{C}, A, \alpha)$ was called an *extended category*. The elements of $A(X, Y) \setminus \hom_{\mathbb{C}}(X, Y)$ will be called **imaginary morphisms**. Sometimes it will be convenient to call a morphism in $\hom_{\mathbb{C}}(X, Y)$ a **real morphism** to emphasise that it is an actual morphism in $\mathbb{C}$.

We use arrows of the type

$$X \dashrightarrow Y$$

to represent an element of $A(X, Y)$, which could be an imaginary morphism or not. To distinguish those which are not, i.e., the elements of $A(X, Y)$ corresponding to a real morphism, say $f \colon X \to Y$, we tag the dashed arrow with

the name of that real morphism overlined (instead of $\alpha_{X,Y}(f)$):

$$X \overset{\overline{f}}{\dashrightarrow} Y.$$

It is possible to define an extended composition, denoted by $\circ$, between real and imaginary morphisms as follows:

$$X \overset{a}{\dashrightarrow} Y \xrightarrow{v} V, \quad \text{where } v \circ a = A(1_X, v)(a)$$

(with $v \circ a$ the composite $X \dashrightarrow V$)

and

$$U \xrightarrow{u} X \overset{a}{\dashrightarrow} Y, \quad \text{where } a \circ u = A(u, 1_Y)(a).$$

(with $a \circ u$ the composite $U \dashrightarrow Y$)

If a corresponds to a real morphism, i.e., $a = \overline{f} = \alpha_{X,Y}(f\colon X \to Y)$, then the same is true for $v \circ a$ and $a \circ u$. Indeed, by the naturality of α we have

$$A(1_X, v)(\alpha_{X,Y}(f)) = \alpha_{X,V}(vf),$$

so that $v \circ \overline{f} = \overline{vf} (= \alpha_{X,V}(vf))$ corresponds to the real morphism vf. Similarly, $\overline{f} \circ u = \overline{fu} (= \alpha_{U,Y}(fu))$ corresponds to the real morphism fu. In particular, we obtain identity axioms $v \circ \overline{1_Y} = \overline{v}$ and $\overline{1_X} \circ u = \overline{u}$. There is also an associativity axiom, which follows from the fact that A is a functor

$$\begin{aligned}(v \circ a) \circ u &= A(u, 1_V)(A(1_X, v)(a)) = A(u, v)(a) = A(1_U, v)(A(u, 1_Y)(a)) \\ &= v \circ (a \circ u).\end{aligned}$$

Definition 2.2. We say that a real morphism $f\colon X \to Y$ admits an **imaginary splitting** when there exists an imaginary morphism s such that the following diagram commutes

$$Y \overset{s}{\dashrightarrow} X \xrightarrow{f} Y.$$

(with $f \circ s = \overline{1_Y}$)

2.3. Comonadic covers. We assume that $\mathbb{C}$ is a regular category equipped with a comonad (P, δ, ε) whose counit ε is a regular epimorphism. We write $\varepsilon_X\colon P(X) \twoheadrightarrow X$ for the chosen cover of some object X in $\mathbb{C}$: ε_X is a regular epimorphism. Note that for any morphism $f\colon X \to Y$ in $\mathbb{C}$

$$f\varepsilon_X = \varepsilon_Y P(f) \tag{1}$$

and

$$P^2(f)\delta_X = \delta_Y P(f), \tag{2}$$

Proof: If $X = Y = \mathbb{Z}$, then $t_{\mathbb{Z},\mathbb{Z}}(k, l) \in \mathbb{Z} + \mathbb{Z}$ must be of the form $\underline{k_1}\overline{l_1} \cdots \underline{k_m}\overline{l_m}$, for some $m \in \mathbb{N}$ and $k_1, \ldots, k_m, l_1, \ldots, l_m \in \mathbb{Z}$ such that $\sum_{1 \leqslant i \leqslant m} k_i = k$ and $\sum_{1 \leqslant i \leqslant m} l_i = l$, for all $(k, l) \in \mathbb{Z} \times \mathbb{Z}$ (see Subsection 4.3). Consider the group homomorphisms $f\colon \mathbb{Z} \to X\colon 1 \mapsto x$ and $g\colon \mathbb{Z} \to Y\colon 1 \mapsto y$. The naturality of t gives the commutative diagram (see (6))

$$\begin{array}{ccc} \mathbb{Z} \times \mathbb{Z} & \overset{t_{\mathbb{Z},\mathbb{Z}}}{\dashrightarrow} & \mathbb{Z} + \mathbb{Z} \\ {\scriptstyle f \times g}\downarrow & & \downarrow{\scriptstyle f+g} \\ X \times Y & \underset{t_{X,Y}}{\dashrightarrow} & X + Y, \end{array}$$

from which we conclude that $t_{X,Y}(x, y) = t_{X,Y}(f \times g)(1, 1) = (f + g)t_{\mathbb{Z},\mathbb{Z}}(1, 1)$. Suppose that $t_{\mathbb{Z},\mathbb{Z}}(1, 1) = \underline{k_1}\overline{l_1} \cdots \underline{k_m}\overline{l_m}$, where $\sum_{1 \leqslant i \leqslant m} k_i = 1 = \sum_{1 \leqslant i \leqslant m} l_i$. We get $t_{X,Y}(x, y) = \underline{x}^{k_1}\overline{y}^{l_1} \cdots \underline{x}^{k_m}\overline{y}^{l_m}$, as desired. ∎

Proposition 11.9. *If Y is a 2-Engel group and t is a natural imaginary splitting in* Gp*, then the induced operation $x * y = \mu^Y(x, y) = \nabla_Y(t_{Y,Y}(x, y))$ may be written as*

$$x * y = [x, y]^k xy$$

for some $k \in \mathbb{Z}$.

Proof: Lemma 11.8 tells us that

$$x * y = x^{k_1} y^{l_1} \cdots x^{k_m} y^{l_m},$$

for some $m \in \mathbb{N}$ and $k_1, \ldots, k_m, l_1, \ldots, l_m \in \mathbb{Z}$ such that $\sum_{1 \leqslant i \leqslant m} k_i = 1 = \sum_{1 \leqslant i \leqslant m} l_i$. We rewrite the expression above as

$$x * y = (x^{k_1} y^{l_1} \cdots x^{k_m} y^{(l_m - 1)} x^{-1} y^0)xy,$$

where the product in brackets is such that the sums of the exponents of the x's and y's are zero. Hence this expression is a commutator word in x and y: it is a product of (nested) commutators. By Lemma 11.4, all higher-order commutators in this product vanish; furthermore, the expression is equal to a product of commutators of the form $[x, y]$ or $[y, x] = [x, y]^{-1}$. Hence it is of the form $[x, y]^k$ for some integer k. ∎

Proposition 11.10. *If Y is a 2-Engel group, then Y is intrinsic Schreier special with respect to all natural imaginary splittings in* Gp*.*

— D_{10} is not an intrinsic Schreier special object with respect to the natural imaginary splitting which gives rise to $x *' y = [x, y]^2 xy$. For example, $q(1, b)$ should be the unique element of D_{10} such that $q(1, b) *' b = 1$. However, all of the elements $b, \dots, a^4 b$ satisfy this equality.

This example shows that the converse of Proposition 11.5 is false. However, we may claim the following:

Proposition 11.7. *If a group Y is an intrinsic Schreier special object with respect to the natural imaginary splitting* (27) *and such that $q(x, y) = x * y^{-1}$, then Y is a 2-Engel group.*

Proof: It suffices we use **(iSs1)**

$$(x * y^{-1}) * y = x$$

to see that Y is in $\mathsf{Eng}_2(\mathsf{Gp})$. Indeed, this is equivalent to

$$\begin{aligned} x = (x^{-1}y^{-1}x^2) * y &= x^{-2}yxyx^{-1}y^{-1}x^2x^{-1}y^{-1}x^2 \\ &= x^{-2}yxyx^{-1}y^{-1}xy^{-1}x^2, \end{aligned}$$

which may be rewritten as

$$1 = x^{-2}yxyx^{-1}y^{-1}xy^{-1}x,$$

so

$$1 = x^{-1}yxyx^{-1}y^{-1}xy^{-1}.$$

This gives

$$y^{-1}xyx^{-1} = xyx^{-1}y^{-1},$$

or, equivalently, $[y^{-1}, x] = [x, y] = [y, x]^{-1}$. ∎

Next we aim to prove the that a 2-Engel group Y is an intrinsic Schreier special object with respect to *all* natural imaginary splittings t. So, we need to extend Proposition 11.5 to all t.

Lemma 11.8. *If t is a natural imaginary splitting in Gp, then for each pair of groups X, Y and all $x \in X$, $y \in Y$ we have that $t_{X,Y}(x, y) \in X + Y$ may be written as a product*

$$\underline{x}^{k_1}\overline{y}^{l_1} \cdots \underline{x}^{k_m}\overline{y}^{l_m},$$

for some $m \in \mathbb{N}$ and $k_1, \dots, k_m, l_1, \dots, l_m \in \mathbb{Z}$ such that

$$\sum_{1 \leqslant i \leqslant m} k_i = 1 = \sum_{1 \leqslant i \leqslant m} l_i.$$

where $P^2 = PP$. Also

$$\varepsilon_{P(X)}\delta_X = 1_{P(X)} = P(\varepsilon_X)\delta_X \tag{3}$$

and

$$P(\delta_X)\delta_X = \delta_{P(X)}\delta_X, \tag{4}$$

for all objects X in $\mathbb{C}$.

Example 2.4. If $\mathbb{V}$ is a variety of universal algebras, then we may consider the free algebra comonad (P, δ, ε). For any algebra X, we have

$$\begin{array}{rcl} \varepsilon_X \colon P(X) & \twoheadrightarrow & X \\ [x] & \mapsto & x \end{array} \quad \text{and} \quad \begin{array}{rcl} \delta_X \colon P(X) & \hookrightarrow & P^2(X), \\ [x] & \mapsto & [[x]] \end{array}$$

where $[x]$ denotes the one letter word x; such words are the generators of $P(X)$. In this case, any function $f \colon X \to Y$ between algebras X and Y extends uniquely to a morphism

$$\begin{array}{rcl} P(X) & \to & Y \\ [x] & \mapsto & f(x) \end{array}$$

in $\mathbb{V}$.

2.5. Imaginary morphisms induced from comonadic covers. The idea behind functions extending to real morphisms in Example 2.4 can be captured through the notion of imaginary morphism: it is like a function (not a morphism) $X \dashrightarrow Y$ of algebras X and Y that extends to an actual morphism of algebras $P(X) \to Y$. More precisely, given a regular category $\mathbb{C}$ with comonadic covers we define the functor

$$\begin{array}{rcl} A : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} & \longrightarrow & \mathsf{Set}, \\ (X, Y) & \longmapsto & \hom_{\mathbb{C}}(P(X), Y) \\ u \uparrow \downarrow v & & \downarrow A(u, v) \\ (U, V) & \longmapsto & \hom_{\mathbb{C}}(P(U), V) \end{array}$$

where $A(u, v) = \hom_{\mathbb{C}}(P(u), v)$. So, A is just the functor $\hom_{\mathbb{C}}(P^{\mathrm{op}} \times 1_{\mathbb{C}})$.

The components of $\alpha \colon \hom_{\mathbb{C}} \Rightarrow A$ are defined, for all objects X, Y by

$$\begin{array}{rcl} \alpha_{X,Y} : \hom_{\mathbb{C}}(X, Y) & \longrightarrow & \hom_{\mathbb{C}}(P(X), Y). \\ X \xrightarrow{f} Y & \longmapsto & P(X) \overset{\varepsilon_X}{\twoheadrightarrow} X \xrightarrow{f} Y \end{array}$$

Note that α is indeed a natural transformation because ε is (see (1)). Also, the components $\alpha_{X,Y}$ are injective, for all objects X, Y, since ε_X is a regular epimorphism. Since the elements of $A(X, Y) = \hom_{\mathbb{C}}(P(X), Y)$ are actual

morphisms in $\mathbb{C}$, an arrow of the type $X \dashrightarrow Y$ corresponds to a morphism $P(X) \to Y$. According to Subsection 2.1:

— if $X \xrightarrow{f} Y$ is a real morphism, then $X \overset{\overline{f}}{\dashrightarrow} Y$ corresponds to the morphism $P(X) \xrightarrow{f\varepsilon_X} Y$, so $\overline{f} = f\varepsilon_X$;
— an imaginary morphism $X \overset{a}{\dashrightarrow} Y$ is a (real) morphism $P(X) \xrightarrow{a} Y$ which is *not* of the type $a = f\varepsilon_X$, for some real morphism $X \xrightarrow{f} Y$;
— the composition of a real morphism with an imaginary one is defined by

$$\begin{cases} X \overset{a}{\dashrightarrow} Y \xrightarrow{v} V\,, & \text{where } v \circ a = va\colon P(X) \longrightarrow V, \\ \qquad\quad {\scriptstyle v\circ a}\searrow \\ U \xrightarrow{u} X \overset{a}{\dashrightarrow} Y\,, & \text{where } a \circ u = aP(u)\colon P(U) \longrightarrow Y. \end{cases}$$

Convention 2.6. From now on, we only consider imaginary morphisms that are induced from comonadic covers.

Remark 2.7. It is clear that in this setting the existence of an imaginary splitting (Definition 2.2) for a morphism f implies that f is a regular epimorphism ($f \circ s = \overline{1_Y}$ implies that $fs = \varepsilon_Y$, which is a regular epimorphism). The converse holds when the values of P are projective objects in $\mathbb{C}$. If $f\colon X \twoheadrightarrow Y$ is a regular epimorphism, then f admits an imaginary splitting because $P(Y)$ is projective

$$\begin{array}{ccc} & & P(Y) \\ & {\scriptstyle \exists\, s}\swarrow & \downarrow{\scriptstyle \varepsilon_Y} \\ X & \xrightarrow[f]{} & Y; \end{array}$$

thus $fs = \varepsilon_Y$. So the existence of imaginary splittings characterises regular epimorphisms in this setting. Moreover, $P(f)$ is a split epimorphism since $P(f)P(s)\delta_Y = P(\varepsilon_Y)\delta_Y \overset{(3)}{=} 1_{P(Y)}$.

3. The Kleisli category

Let $\mathbb{C}$ be a regular category with comonadic covers. We denote by $\mathbb{K}$ the Kleisli category associated to the comonad (P, δ, ε). Its objects are those of $\mathbb{C}$ and $\hom_{\mathbb{K}}(X, Y) = \hom_{\mathbb{C}}(P(X), Y)$. The morphisms of $\mathbb{K}$ are the imaginary morphisms together with those of the type $\overline{f}\colon X \dashrightarrow Y$, for some real morphism $f\colon X \to Y$ (Subsection 2.5).

It is easy to check that $x * 1 = x = 1 * x$ and that $x * x^{-1} = 1 = x^{-1} * x$; however $*$ is not associative.

A group Y is an intrinsic Schreier special object with respect to (27) if there exists an imaginary retraction $q\colon Y \times Y \dashrightarrow Y$ such that **(iSs1)** and **(iSs2)** hold. In this case

(iSs1) means that $q(x, y) * y = x$, for all x, $y \in Y$, and
(iSs2) means that $q(x * y, y) = x$, for all x, $y \in Y$

—see Section 8.

Proposition 11.5. *If Y is a 2-Engel group, then Y is an intrinsic Schreier special object with respect to the natural imaginary splitting* (27).

Proof: We define the imaginary retraction by $q(x, y) = x * y^{-1}$. Then, for all x, $y \in Y$, **(iSs1)** holds:

$$\begin{array}{rcl} q(x,y) * y & = & (x * y^{-1}) * y \\ & = & (x^{-1}y^{-1}x^2) * y \\ & = & x^{-2}yxyx^{-1}y^{-1}x^2x^{-1}y^{-1}x^2 \\ & = & x^{-2}y[x,y]xy^{-1}x^2 \\ & \overset{11.1.4}{=} & x^{-2}y[y^{-1},x]xy^{-1}x^2 \\ & = & x^{-2}yy^{-1}xyx^{-1}xy^{-1}x^2 \\ & = & x. \end{array}$$

As for **(iSs2)**, the equality $q(x * y, y) = (x * y) * y^{-1} = x$ holds by swapping y and y^{-1} in **(iSs1)**. ∎

Example 11.6. The Dihedral group D_{10} is generated by elements a and b such that $a^5 = 1$, $b^2 = 1$ and $abab = 1$. We have

$$D_{10} = \{1, a, a^2, a^3, a^4, b, ab, a^2b, a^3b, a^4b\},$$

where the elements b, …, a^4b are all inverses to themselves. We have

— D_{10} is not a 2-Engel group: $[a, ab]ab = a^2$, while $ab[a, ab] = a^4b$.
— D_{10} is an intrinsic Schreier special object with respect to the natural imaginary splitting (27). It suffices to build the Cayley table for the product $*$ and observe that it gives a Latin square. The fact that it is a Latin square guarantees the existence of a unique element, which is equal to $q(x, y)$, satisfying the equality **(iSs1)** $q(x, y) * y = x$. The equality **(iSs2)** follows from the uniqueness of each $q(x, y)$.

2. The group of quaternions Q_8 is a 2-Engel group which is not abelian.
3. The smallest non 2-Engel (thus non-abelian) group is the symmetric group S_3 (which is isomorphic to the dihedral group D_6).
4. The dihedral group D_8 is 2-Engel, but the Dihedral group D_{10} is not (see Example 11.6).

Lemma 11.3. *Let E be a 2-Engel group. Then:*

1. $[xy, z] = {}^x([y, z][x, z])$, *for all* x, y, $z \in E$;
2. $[x, yz] = {}^y([x, y][x, z])$, *for all* x, y, $z \in E$.

Proof: 1. $[xy, z] = {}^x[y, z][x, z] = x[y, z]x^{-1}[x, z] \overset{11.1.1}{=} x[y, z][x, z]x^{-1}$.
2. $[x, yz] = [x, y]{}^y[x, z] = [x, y]y[x, z]y^{-1} \overset{11.1.2}{=} y[x, y][x, z]y^{-1}$. ∎

It is known (and in fact not hard to see) that the free object on two generators in the variety $\mathsf{Eng}_2(\mathsf{Gp})$ of 2-Engel groups is 2-nilpotent. This allows us to prove the following result.

Lemma 11.4. *In a 2-Engel group E, let a, b and c be products of given elements x, y of E, or their inverses. Then:*

1. $[[a, b], c] = 1$;
2. $[ab, c] = [b, c][a, c]$ *and* $[a, bc] = [a, b][a, c]$;
3. $[a^{-1}, b] = [a, b]^{-1} = [a, b^{-1}]$;
4. $[a^k, b] = [a, b]^k = [a, b^k]$, *for all* $k \in \mathbb{Z}$.

Proof: 1. Follows from the fact that the free 2-Engel group on two generators x and y is necessarily 2-nilpotent.
2. $[ab, c] = {}^a[b, c][a, c] = a[b, c]a^{-1}[a, c] \overset{1.}{=} [b, c][a, c]$. The proof of the second claim is similar.
3. $[a^{-1}, b][a, b] \overset{2.}{=} [aa^{-1}, b] = [1, b] = 1$. The proof of the second claim is similar.
4. Is just a particular case of 11.1.5 and 11.1.6. ∎

We now look at a specific natural imaginary splitting in Gp: the one defined by the function

$$t_{X,Y} \colon X \times Y \dashrightarrow X + Y \colon (x, y) \mapsto \underline{x}^{-1}\overline{y}\underline{x}^2, \tag{27}$$

for any pair of groups X and Y. It is easy to see that this t is indeed a natural imaginary splitting. When $X = Y$, we write

$$x * y = \mu^Y(x, y) = \nabla_Y(t_{Y,Y}(x, y)) = x^{-1}yx^2.$$

composition $\circ$	composition in $\mathbb{C}$
$X \overset{a}{\dashrightarrow} Y \overset{v}{\dashrightarrow} V$ $v \circ a$	$P(X) \xrightarrow{a} Y \xrightarrow{v} V$
$X \overset{a}{\dashrightarrow} Y \overset{\overline{v}}{\dashrightarrow} V$ $\overline{v} \circ a$	$P(X) \xrightarrow{\delta_X} P^2(X) \xrightarrow{P(a)} P(Y) \xrightarrow{\varepsilon_Y} V$, (3) $\varepsilon_{P(X)}$, (1), a, $P(X)$
$U \overset{u}{\dashrightarrow} X \overset{a}{\dashrightarrow} Y$ $a \circ u$	$P(U) \xrightarrow{P(u)} P(X) \xrightarrow{a} Y$
$U \overset{\overline{u}}{\dashrightarrow} X \overset{a}{\dashrightarrow} Y$ $a \circ \overline{u}$	$P(U) \xrightarrow{\delta_U} P^2(U) \xrightarrow{P(\varepsilon_U)} P(U) \xrightarrow{P(u)} P(X) \xrightarrow{a} Y$, (3)

TABLE 1. Composition in the Kleisli category

The composition in $\mathbb{K}$ will also be denoted by $\circ$ (as in Subsections 2.5 and 2.1)

$$A \overset{a}{\dashrightarrow} B \overset{b}{\dashrightarrow} C,$$
$$b \circ a$$

where $b \circ a$ corresponds to the morphism in $\mathbb{C}$

$$P(A) \xrightarrow{\delta_A} P^2(A) \xrightarrow{P(a)} P(B) \xrightarrow{b} C.$$

Remark 3.1. If any of the morphisms in a composite in $\mathbb{K}$ corresponds to a real morphism, then this composite coincides with the one defined in Subsections 2.5 and 2.1—See Table 1.

The comonad (P, δ, ε) gives rise to an adjunction $\mathbb{K} \rightleftarrows \mathbb{C}$, where the right adjoint is the embedding

$$I \colon \mathbb{C} \to \mathbb{K} \colon\ X \xrightarrow{f} Y \ \mapsto\ X \overset{\overline{f}}{\dashrightarrow} Y$$

Consequently, $\mathbb{K}$ has a limit for every finite diagram in $\mathbb{C}$, which is just the limit of that diagram in $\mathbb{C}$, embedded into $\mathbb{K}$.

4. Imaginary addition in unital categories

In this section we define an *imaginary addition* on each object X of a unital category with comonadic covers, i.e., an imaginary morphism $\mu^X \colon X \times X \dashrightarrow X$

such that $\mu^X \circ \langle 1_X, 0\rangle = \overline{1_X}$ and $\mu^X \circ \langle 0, 1_X\rangle = \overline{1_X}$. Such an imaginary addition provides one of the tools needed to define intrinsic Schreier split epimorphisms in Section 5.

4.1. Unital categories [3]**.** A pointed and finitely complete category is called **unital** when, for all objects A, B,

$$A \xrightarrow{\langle 1_A, 0\rangle} A \times B \xleftarrow{\langle 0, 1_B\rangle} B$$

is a jointly extremal-epimorphic pair.

Example 4.2. As shown in [1], a variety of universal algebras $\mathbb{V}$ is unital if and only if it is a **Jónsson-Tarski variety** [15]. Recall that a Jónsson–Tarski variety is such that its theory contains a unique constant 0 and a binary operation $+$ satisfying the identities $x+0 = x = 0+x$. So an algebra is a unitary magma, possibly equipped with additional operations.

A pointed finitely complete category $\mathbb{C}$ is unital if and only if for any **punctual span** in $\mathbb{C}$

$$A \underset{f}{\overset{s}{\rightleftarrows}} C \underset{g}{\overset{t}{\leftrightarrows}} B, \qquad\qquad fs = 1_A, gt = 1_B, ft = 0, gs = 0$$

the factorisation $\langle f, g\rangle \colon C \twoheadrightarrow A \times B$ is a strong epimorphism (Theorem 1.2.12 in [1]). Consequently, a pointed regular category with binary coproducts is unital if and only if for all objects A, B, the comparison morphism

$$r_{A,B} = \left(\begin{smallmatrix} 1_A & 0 \\ 0 & 1_B \end{smallmatrix}\right) \colon A + B \twoheadrightarrow A \times B$$

is a regular epimorphism.

4.3. Natural imaginary splittings [20]**.** If $\mathbb{C}$ is a regular unital category with binary coproducts and comonadic covers, then for all objects A, B, the comparison morphism $r_{A,B} = \left(\begin{smallmatrix} 1_A & 0 \\ 0 & 1_B \end{smallmatrix}\right) \colon A + B \twoheadrightarrow A \times B$ is a regular epimorphism. When $P(A \times B)$ is a projective object, as in the varietal case, there exists a (not necessarily unique) morphism $t_{A,B} \colon P(A \times B) \to A + B$ such that

$$r_{A,B} t_{A,B} = \varepsilon_{A \times B} \tag{5}$$

(see Remark 2.7).

products of $\underline{x}$ or $\underline{x}^{-1}$, and of $\overline{y}$ or $\overline{y}^{-1}$, for which the products of the $x's$ gives x and the products of the $y's$ gives y. For example $\underline{x}^{-1}\overline{y}\underline{x}^2$ or $\underline{x}\overline{y}\underline{x}^{-1}\overline{y}^{-1}\underline{x}\overline{y}$.

Although these notions are independent in general, as we have already observed, there are special properties of the category of groups that make the notions coincide. From Corollary 10.4, we know that the associativity of the group operation implies that all intrinsic Schreier special objects are protomodular objects. This associativity is not enough to guarantee that every protomodular object is intrinsic Schreier special (Remark 10.6). This leads us to the following question:

> *What property of* Gp *guarantees that all protomodular objects are intrinsic Schreier special ones?*

We cannot answer this question now, but we can see that groups lack a certain homogeneity, in the sense that the concept of an intrinsic Schreier special object strongly depends on the chosen natural imaginary splitting. We can eliminate this discrepancy by considering groups which satisfy the property with respect to *all* natural imaginary splittings. Then we find:

11. The variety of 2-Engel groups

Recall that a 2-*Engel group* is a group E that satisfies the commutator identity $[[x, y], y] = 1$ for all x, $y \in E$. The aim of this section is to show that 2-Engel groups are intrinsic Schreier special objects with respect to *all* natural imaginary splittings: Proposition 11.10.

We begin by recalling the definition and main properties of 2-Engel groups needed in the sequel, which can be found in [12, 13, 16].

Here we denote the conjugation of an element x by an element y as ${}^y x = yxy^{-1}$ and we write $[x, y]$ for $xyx^{-1}y^{-1}$, so that $[xy, z] = {}^x[y, z][x, z]$ and $[x, yz] = [x, y]^y[x, z]$. Note also that $[x, y]^{-1} = [y, x]$.

Definition 11.1. A group E is called a 2-**Engel group** if it satisfies any of the following equivalent conditions, for all elements x, $y \in E$,

1. $[[x, y], x] = 1$;
2. $[[x, y], y] = 1$;
3. $[x, y^{-1}] = [x, y]^{-1}$;
4. $[x^{-1}, y] = [x, y]^{-1}$;
5. $[x, y^k] = [x, y]^k$, for all $k \in \mathbb{Z}$;
6. $[x^k, y] = [x, y]^k$, for all $k \in \mathbb{Z}$.

Example 11.2. 1. Any abelian group is obviously a 2-Engel group.

Now we must check **(iS1)** for (16):

$$\begin{aligned}
\mu^X \circ \langle k \circ \rho, \overline{sf} \rangle &= \mu^X \circ \langle \mu^X \circ \langle \overline{1_X}, s \circ q \circ \langle 0, 1_Y \rangle f \rangle, \overline{sf} \rangle \\
&\overset{(17)}{=} \mu^X \circ \langle \overline{1_X}, \mu^X \circ \langle s \circ q \circ \langle 0, 1_Y \rangle f, \overline{sf} \rangle \rangle \\
&= \mu^X \circ \langle \overline{1_X}, \mu^X \circ (s \times s) \circ \langle q \circ \langle 0, 1_Y \rangle, \overline{1_Y} \rangle \circ f \rangle \\
&\overset{(14)}{=} \mu^X \circ \langle \overline{1_X}, s \circ \mu^Y \circ \langle q \circ \langle 0, 1_Y \rangle, \overline{1_Y} \rangle \circ f \rangle \\
&\overset{(25)}{=} \mu^X \circ \langle \overline{1_X}, \overline{0} \rangle \\
&= \mu^X \circ \langle 1_X, 0 \rangle \\
&\overset{(12)}{=} \overline{1_X}.
\end{aligned}$$

■

Corollary 10.4. *Suppose that the natural addition* $(\mu^X \colon X \times X \dashrightarrow X)_{X \in \mathbb{C}}$ *is associative. Then every intrinsic Schreier special object is a protomodular object.*

Proof: This follows from Proposition 10.3 and Proposition 5.8 in [20], which states that any split epimorphism satisfying **(iS1)** is stably strong. ■

Remark 10.5. Even if the natural addition $(\mu^X \colon X \times X \dashrightarrow X)_{X \in \mathbb{C}}$ is associative, the converse of Corollary 10.4 may be false. As mentioned above, the variety HSLat of Heyting semilattices provides an example of a category where all objects are protomodular, but not all are Schreier special objects. The natural addition for HSLat, given by the meet, is associative.

Remark 10.6. The variety Loop of (left and right) loops gives an example where the natural addition is non-associative. All loops are intrinsic Schreier special objects (see Section 8) and they are also all protomodular objects (because Loop is a semi-abelian category, thus a protomodular category). So, the fact that all intrinsic Schreier special objects are protomodular objects does not imply that the natural addition is associative.

From the remark above, in Gp all objects are intrinsic Schreier special with respect to its usual group operation. Also, all objects are protomodular since Gp is a protomodular category. So the two notions coincide in Gp, just as in the case of Mon. However, in Mon there are only two possible choices of imaginary splittings (see Subsection 4.3). In Gp there are countably many possibilities. Given groups X and Y, a natural imaginary splitting $t \colon P(X \times Y) \to X + Y$ may be defined by making $t([(x, y)])$ equal to any combination of alternating

Example 4.4. Let $\mathbb{V}$ be a Jónsson–Tarski variety. For any pair of algebras (A, B) in $\mathbb{V}$, we can make the following choices of an imaginary splitting for $r_{A,B}$: the *direct imaginary splitting* t^d

$$[(a, b)] \mapsto \underline{a} + \overline{b}$$

which sends a generator $[(a, b)] \in P(A \times B)$ to the sum of $\underline{a} = \iota_A(a)$ with $\overline{b} = \iota_B(b)$ in $A + B$ (where ι_A and ι_B are the coproduct inclusions); and the *twisted imaginary splitting* t^w

$$[(a, b)] \mapsto \overline{b} + \underline{a}$$

which does the same, but in the opposite order. Note that each of those choices determines a natural transformation

$$t \colon P((\cdot) \times (\cdot)) \Rightarrow (\cdot) + (\cdot)$$

such that $rt = \varepsilon_{(\cdot)\times(\cdot)}$, where $r \colon (\cdot) + (\cdot) \Rightarrow (\cdot) \times (\cdot)$ and

$$\varepsilon_{(\cdot)\times(\cdot)} \colon P((\cdot) \times (\cdot)) \Rightarrow (\cdot) \times (\cdot).$$

It was shown in [20] that when $\mathbb{V}$ is the category Mon of monoids, then the above choices of natural imaginary splittings (direct and twisted) are the only options. As we shall see below, this is far from being true in general.

We make the existence of a natural t into an axiom. Let $\mathbb{C}$ be a pointed regular (unital) category $\mathbb{C}$ with binary coproducts and comonadic covers. Suppose also that there exist $t_{A,B}$ such that (5) holds and that they are the components of a natural transformation

$$t \colon P((\cdot) \times (\cdot)) \Rightarrow (\cdot) + (\cdot)$$

where $rt = \varepsilon_{(\cdot)\times(\cdot)}$. Then all $r_{A,B}$ are necessarily regular epimorphisms (because the $\varepsilon_{A\times B}$ are) and, consequently, $\mathbb{C}$ is a unital category. In [20] such a natural transformation t was called a **natural imaginary splitting**.

Remark 4.5. Any natural imaginary splitting $t \colon P((\cdot) \times (\cdot)) \Rightarrow (\cdot) + (\cdot)$ has the following properties:

1. $t_{A,0}$ is isomorphic to ε_A

$$\begin{array}{ccccc} P(A) & \xrightarrow{\varepsilon_A} & A & \xrightarrow{1_A} & A \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ P(A\times 0) & \xrightarrow{t_{A,0}} & A+0 & \xrightarrow{r_{A,0}} & A\times 0, \end{array} \qquad P(A\times 0) \xrightarrow[\varepsilon_{A\times 0}]{} A\times 0,$$

   for all objects A in $\mathbb{C}$;
2. the naturality of t gives the commutative diagram

$$\begin{array}{ccc} P(A\times B) & \xrightarrow{t_{A,B}} & A+B \\ {\scriptstyle P(u\times v)}\downarrow & & \downarrow{\scriptstyle u+v} \\ P(C\times D) & \xrightarrow[t_{C,D}]{} & C+D \end{array} \tag{6}$$

   for all $u\colon A\to C$, $v\colon B\to D$ in $\mathbb{C}$;
3. from (5), we deduce

$$\begin{bmatrix} 1_A & 0 \end{bmatrix} t_{A,B} = \pi_A \varepsilon_{A\times B} \overset{(1)}{=} \varepsilon_A P(\pi_A) \tag{7}$$

   and

$$\begin{bmatrix} 0 & 1_B \end{bmatrix} t_{A,B} = \pi_B \varepsilon_{A\times B} \overset{(1)}{=} \varepsilon_B P(\pi_B) \tag{8}$$

   for all objects A and B in $\mathbb{C}$;
4. using properties 1. and 2. above, we obtain the (regular epimorphism, monomorphism) factorisations

$$\begin{array}{ccccc} P(A) & \xrightarrow{P(\langle 1_A,0\rangle)} & P(A\times B) & \xrightarrow{t_{A,B}} & A+B \\ & {\scriptstyle \varepsilon_A}\searrow & & \nearrow{\scriptstyle \iota_A} & \\ & & A & & \end{array} \tag{9}$$

   and

$$\begin{array}{ccccc} P(B) & \xrightarrow{P(\langle 0,1_B\rangle)} & P(A\times B) & \xrightarrow{t_{A,B}} & A+B, \\ & {\scriptstyle \varepsilon_B}\searrow & & \nearrow{\scriptstyle \iota_B} & \\ & & B & & \end{array} \tag{10}$$

   for all objects A and B in $\mathbb{C}$.

Proof: In Subsection 8.1 we saw that **(iL2)** follows from **(iSs2)**. If we precompose **(iL2)** with $\langle \mu^Y \circ \langle \overline{1_Y}, q\circ\langle 0,1_Y\rangle\rangle, \overline{1_Y}\rangle\colon Y \dashrightarrow Y\times Y$, we obtain

$$q\circ\langle \mu^Y, \overline{\pi_2}\rangle \circ \langle \mu^Y \circ \langle \overline{1_Y}, q\circ\langle 0,1_Y\rangle\rangle, \overline{1_Y}\rangle = \overline{\pi_1}\circ\langle \mu^Y \circ \langle \overline{1_Y}, q\circ\langle 0,1_Y\rangle\rangle, \overline{1_Y}\rangle,$$

which is equivalent to

$$\begin{aligned} & q\circ\langle \mu^Y \circ \langle \mu^Y \circ \langle \overline{1_Y}, q\circ\langle 0,1_Y\rangle\rangle, \overline{1_Y}\rangle, \overline{1_Y}\rangle = \mu^Y \circ \langle \overline{1_Y}, q\circ\langle 0,1_Y\rangle\rangle \\ \overset{(17)}{\Leftrightarrow}\ & q\circ\langle \mu^Y \circ \langle \overline{1_Y}, \mu^Y \circ \langle q\circ\langle 0,1_Y\rangle, \overline{1_Y}\rangle\rangle, \overline{1_Y}\rangle = \mu^Y \circ \langle \overline{1_Y}, q\circ\langle 0,1_Y\rangle\rangle \\ \overset{(25)}{\Leftrightarrow}\ & q\circ\langle \mu^Y \circ \langle \overline{1_Y}, \overline{0}\rangle, \overline{1_Y}\rangle = \mu^Y \circ \langle \overline{1_Y}, q\circ\langle 0,1_Y\rangle\rangle \\ \Leftrightarrow\ & q\circ\langle \mu^Y \circ \langle 1_Y, 0\rangle, \overline{1_Y}\rangle = \mu^Y \circ \langle \overline{1_Y}, q\circ\langle 0,1_Y\rangle\rangle \\ \overset{(12)}{\Leftrightarrow}\ & q\circ\langle \overline{1_Y}, \overline{1_Y}\rangle = \mu^Y \circ \langle \overline{1_Y}, q\circ\langle 0,1_Y\rangle\rangle \\ \Leftrightarrow\ & q\circ\langle 1_Y, 1_Y\rangle = \mu^Y \circ \langle \overline{1_Y}, q\circ\langle 0,1_Y\rangle\rangle \\ \overset{\textbf{(iSs4)}}{\Leftrightarrow}\ & \overline{0} = \mu^Y \circ \langle \overline{1_Y}, q\circ\langle 0,1_Y\rangle\rangle. \end{aligned}$$

■

Proposition 10.3. *Suppose that the natural addition $(\mu^X\colon X\times X \dashrightarrow X)_{X\in\mathbb{C}}$ is associative. If Y is an intrinsic Schreier special object, then any split epimorphism* (16) *satisfies* **(iS1)**.

Proof: We define an imaginary morphism $\rho\colon X \dashrightarrow K$ through the universal property of the kernel

$$\begin{array}{ccccc} K & \overset{k}{\rhd\!\!\longrightarrow} & X & \underset{f}{\overset{s}{\leftleftarrows\!\!\rightrightarrows}} & Y. \\ & {\scriptstyle \rho}\nwarrow & \uparrow{\scriptstyle \mu^X\circ(1_X\times s)\circ(1_X\times(q\circ\langle 0,1_Y\rangle))\circ\langle 1_X,f\rangle} & & \\ & & X & & \end{array}$$

Indeed,

$$\begin{aligned} & f\circ\mu^X\circ(1_X\times s)\circ(1_X\times(q\circ\langle 0,1_Y\rangle))\circ\langle 1_X,f\rangle \\ \overset{(14)}{=}\ & \mu^Y\circ(f\times f)\circ\langle \overline{1_X}, s\circ q\circ\langle 0,1_Y\rangle f\rangle \\ =\ & \mu^Y\circ\langle \overline{f}, q\circ\langle 0,1_Y\rangle f\rangle \\ =\ & \mu^Y\circ\langle \overline{1_Y}, q\circ\langle 0,1_Y\rangle\rangle\circ f \\ \overset{(26)}{=}\ & \overline{0}. \end{aligned}$$

an example of a category where all objects are protomodular, but not every object is Schreier special ([20], Example 7.7).

On the other hand, the cyclic group $C_2 = (\{0,1\},+)$ gives an example of a Schreier special object in the Jónsson-Tarski variety of unitary magmas Mag, because it is a right loop. However, we gave an example of a point $X \leftrightarrows C_2$ which is not strong. Consequently, C_2 is not a protomodular object ([20], Example 7.4). Of key importance here is that the unitary magma $(X,+)$ is *non-associative*. As we shall prove next in Corollary 10.4, the presence of the associativity axiom (Section 6) gives a link between intrinsic Schreier special objects and protomodular objects: then

Every intrinsic Schreier special object is a protomodular object.

The proof of this statement follows the same proof for monoids, i.e., that a Schreier special monoid Y is necessarily a group; the inverse of an element $y \in Y$ is given by $q(0,y)$, where q is the imaginary retraction for (23) (see Proposition 3.1.6 of [8]). Also, that all points over a group are necessarily Schreier split epimorphisms (see Corollary 3.1.7 of [8]).

Lemma 10.1. *If* (23) *satisfies* **(iSs1)**, *then* $\mu^Y \circ \langle q \circ \langle 0, 1_Y\rangle, \overline{1_Y}\rangle = \overline{0}$*:*

$$\begin{tikzcd} Y \arrow[r, dashed, "\langle q\circ\langle 0,1_Y\rangle,\overline{1_Y}\rangle"] \arrow[dr, dashed, "\overline{0}"'] & Y \times Y. \arrow[d, dashed, "\mu^Y"] \\ & Y \end{tikzcd} \tag{25}$$

Proof: In Subsection 8.1 we saw that **(iL1)** follows from **(iSs1)**. If we precompose **(iL1)** with $\langle 0, 1_Y\rangle \colon Y \to Y \times Y$, we get

$$\begin{aligned} & \mu^Y \circ \langle q, \overline{\pi_2}\rangle \circ \langle 0, 1_Y\rangle = \overline{\pi_1} \circ \langle 0, 1_Y\rangle \\ \Leftrightarrow\ & \mu^Y \circ \langle q \circ \langle 0, 1_Y\rangle, \overline{1_Y}\rangle = \overline{0} \end{aligned}$$

■

Lemma 10.2. *Suppose that the natural addition* $(\mu^X \colon X \times X \dashrightarrow X)_{X\in\mathbb{C}}$ *is associative. If* Y *is an intrinsic Schreier special object, then*

$$\begin{tikzcd} Y \arrow[r, dashed, "\langle\overline{1_Y},q\circ\langle 0,1_Y\rangle\rangle"] \arrow[dr, dashed, "\overline{0}"'] & Y \times Y \arrow[d, dashed, "\mu^Y"] \\ & Y \end{tikzcd} \tag{26}$$

$\mu^Y \circ \langle \overline{1_Y}, q \circ \langle 0, 1_Y\rangle\rangle = \overline{0}$.

4.6. Imaginary addition. Let $\mathbb{C}$ be a regular unital category with binary coproducts, comonadic covers and a natural imaginary splitting t. For every object X, we consider the imaginary morphism $\mu^X \colon X \times X \dashrightarrow X$ given by

$$\begin{tikzcd} P(X \times X) \arrow[r, "t_{X,X}"] \arrow[rr, dashed, bend right, "\mu^X"'] & X + X \arrow[r, "(1_X\ 1_X)"] & X. \end{tikzcd} \tag{11}$$

We call μ^X an **imaginary addition** on X since:

$$\begin{tikzcd} X \arrow[r, "\langle 1_X,0\rangle"] \arrow[rr, dashed, bend right, "\mu^X\circ\langle 1_X,0\rangle=\overline{1_X}"'] & X \times X \arrow[r, dashed, "\mu^X"] & X \end{tikzcd} \tag{12}$$

and

$$\begin{tikzcd} X \arrow[r, "\langle 0,1_X\rangle"] \arrow[rr, dashed, bend right, "\mu^X\circ\langle 0,1_X\rangle=\overline{1_X}"'] & X \times X \arrow[r, dashed, "\mu^X"] & X. \end{tikzcd} \tag{13}$$

Indeed,

$$\mu^X \circ \langle 1_X, 0\rangle = (1_X\ \ 1_X)t_{X,X}P(\langle 1_X, 0\rangle) \overset{(9)}{=} (1_X\ \ 1_X)\iota_1\varepsilon_X = \varepsilon_X = \overline{1_X}$$

and

$$\mu^X \circ \langle 0, 1_X\rangle = (1_X\ \ 1_X)t_{X,X}P(\langle 0, 1_X\rangle) \overset{(10)}{=} (1_X\ \ 1_X)\iota_2\varepsilon_X = \varepsilon_X = \overline{1_X}.$$

We adapt Definition 3.15 in [7] to the unital context and call the family $(\mu^X \colon X \times X \dashrightarrow X)_{X\in\mathbb{C}}$ a **natural addition**. Here natural means that for any morphism $f \colon X \to Y$ the diagram

$$\begin{tikzcd} X \times X \arrow[r, dashed, "\mu^X"] \arrow[d, "f\times f"'] & X \arrow[d, "f"] \\ Y \times Y \arrow[r, dashed, "\mu^Y"'] & Y \end{tikzcd} \tag{14}$$

commutes. In fact,

$$\begin{aligned} f \circ \mu^X &= f(1_X\ \ 1_X)t_{X,X} = (1_Y\ \ 1_Y)(f+f)t_{X,X} \\ &\overset{(6)}{=} (1_Y\ \ 1_Y)t_{Y,Y}P(f \times f) = \mu^Y \circ (f \times f). \end{aligned}$$

5. Intrinsic Schreier split extensions

In this section we recall the notion of a Schreier split epimorphism of monoids and its extended categorical version, the notion of an intrinsic Schreier split epimorphism. We actually give a simplified version of the intrinsic definition by using the direct composition of imaginary morphisms, which is simply the composition in the Kleisli category associated with the comonad of the comonadic covers.

5.1. Schreier split extensions of monoids [8, 9]**.** We recall the definition and the main properties concerning Schreier split epimorphisms.

A split epimorphism of monoids f with chosen section s and kernel K

$$K \underset{k}{\rightarrowtriangle} (X, \cdot, 1) \underset{f}{\overset{s}{\leftrightarrows}} Y \tag{15}$$

is called a **Schreier split epimorphism** if, for every $x \in X$, there exists a unique element $a \in K$ such that $x = k(a) \cdot sf(x)$. Equivalently, if there exists a unique function $q\colon X \dashrightarrow K$ such that $x = kq(x) \cdot sf(x)$ for all $x \in X$. We emphasise the fact that q is just a function (not necessarily a morphism of monoids) by using an arrow of type $\dashrightarrow$.

The uniqueness property may be replaced [9, Proposition 2.4] by an extra condition on q: the couple (f, s) is a Schreier split epimorphism if and only if

(S1) $x = kq(x) \cdot sf(x)$, for all $x \in X$;
(S2) $q(k(a) \cdot s(y)) = a$, for all $a \in K$, $y \in Y$.

Remark 5.2. Recall from [8] that Schreier split epimorphisms are also called **right homogenous** split epimorphisms. A split epimorphism as in (15) is called **left homogenous** if, for every $x \in X$, there exists a unique element $a \in K$ such that $x = sf(x) \cdot k(a)$.

Proposition 5.3. [8, Proposition 2.1.5] *Given a Schreier split epimorphism as in* (15)*, the following hold:*

(S3) $qk = 1_K$*;*
(S4) $qs = 0$*;*
(S5) $q(1) = 1$*;*
(S6) $kq(s(y) \cdot k(a)) \cdot s(y) = s(y) \cdot k(a)$*, for all* $a \in K, y \in Y$*;*
(S7) $q(x \cdot x') = q(x) \cdot q(sf(x) \cdot kq(x'))$*, for all* $x, x' \in X$.

A split epimorphism as in (15) is said to be **strong** when (k, s) is a jointly extremal-epimorphic pair. It is **stably strong** if every pullback of it along any

split monomorphism is isomorphic to the binary coproduct

$$\begin{array}{ccccc}
(X\backslash Y \cup \{*\}, *) & \xleftarrow[k]{} & (X, *) & \underset{f}{\overset{s}{\leftrightarrows}} & (Y, *) \\
\| & & \downarrow\wr & & \| \\
(X\backslash Y \cup \{*\}, *) & \xleftarrow[\langle 1_K\ 0\rangle]{} & ((X\backslash Y \cup \{*\}) + Y, *) & \underset{\iota_2}{\overset{\langle 0\ 1_Y\rangle}{\leftrightarrows}} & (Y, *)
\end{array}$$

It is easy to see that the opposite of equality **(iS2)** always holds.

We have just proved that in $\mathsf{Set}_*^{\mathrm{op}}$, with respect to the natural imaginary splitting (24), the only intrinsic Schreier split epimorphisms correspond to binary product projections. Moreover, a pointed set $(Y, *)$ is an intrinsic Schreier special object if and only if (23) is a product projection, i.e., if and only if it is the zero object.

Note that we could also apply the same approach to the finite powerset monad in Set_*.

10. Intrinsic Schreier special objects vs. protomodular objects

Recall from [19] that an object Y in a finitely complete category is called a **protomodular object** when all points over it $(f\colon X \to Y, s\colon Y \to X)$ are stably strong. More precisely, for any pullback

$$\begin{array}{ccc}
Z \times_Y X & \longrightarrow & X \\
{\scriptstyle \pi_Z}\downarrow\uparrow{\scriptstyle \langle 1_Z, sg\rangle} & & {\scriptstyle f}\downarrow\uparrow{\scriptstyle s} \\
Z & \underset{g}{\longrightarrow} & Y,
\end{array}$$

the pair $(\langle 0, k\rangle, \langle 1_Z, sg\rangle)$, where k is the kernel of f, is a jointly extremal-epimorphic pair. If the point (f, s) is stably strong, then it is strong, i.e., (k, s) is a jointly extremal-epimorphic pair.

In the category Mon of monoids the notion of a Schreier special object and the notion of a protomodular object both coincide with that of a group: a monoid is a Schreier special object if and only if it is a group [10] if and only if it is a protomodular object [19].

The question of understanding under which conditions these two notions coincide arises naturally. In general, neither of these notions implies the other, as we showed in [20]. Indeed, the variety HSLat of Heyting semilattices provides

It is easy to check that t is a natural transformation and that it satisfies the opposite of equality (5), for all pointed sets $(A,*)$ and $(B,*)$

$$\begin{array}{ccc} & & (P(A+B),\{*\}) \\ & \overset{t_{A,B}}{\nearrow} & \uparrow{\scriptstyle \varepsilon_{A+B}} \\ (A\times B,(*,*)) & \xleftarrow[\begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix}]{} & (A+B,*). \end{array}$$

An intrinsic Schreier split epimorphism in $\mathsf{Set}_*^{\mathrm{op}}$ corresponds to a split monomorphism in Set_*. The following diagram represents a split monomorphism, given by an injection f, and its cokernel in Set_*

$$(K,*) \xleftarrow[k]{} (X,*) \underset{f}{\overset{s}{\rightleftarrows}} (Y,*),$$

where $K = X\backslash Y \cup \{*\}$, $k(y) = *$, for all $y \in Y$ and $k(x) = x$, for all $x \in X\backslash Y$. It is an intrinsic Schreier split monomorphism if there exists a morphism of pointed sets $q\colon (K,*) \to (P(X),\{*\})$ such that the opposite of equalities **(iS1)** and **(iS2)** hold. Note that $q(x) \in P(X)$, i.e., $* \in q(x) \subseteq X$, for all $x \in X\backslash Y$.

The opposite of **(iS1)** is given by the commutativity of the following diagram (we omit the fixed points to make it easier to read)

$$\begin{array}{ccccc} P^2(X) & \xleftarrow{P([1\ 1])} & P(P(X)+P(X)) & \xleftarrow{t_{P(X),P(X)}} & P(X)\times P(X) \\ {\scriptstyle \delta_X}\downarrow & & & & \uparrow{\scriptstyle \langle qk,\varepsilon_X fs\rangle} \\ P(X) & & \xleftarrow[\varepsilon_X]{} & & X. \end{array}$$

The commutativity of the diagram above always holds for any element $y \in Y$. For any element $x \in X\backslash Y$, we get

$$\begin{array}{ccccc} \{q(x),\{fs(x),*\},\{*\}\} & \xleftarrow{P([1\ 1])} & \{\underline{q(x)},\overline{\{fs(x),*\}},\{*\}\} & \xleftarrow{t_{P(X),P(X)}} & (q(x),\{fs(x),*\}) \\ {\scriptstyle \delta_X}\downarrow & & & & \uparrow{\scriptstyle \langle qk,\varepsilon_X fs\rangle} \\ q(x)\cup\{fs(x),*\} = \{x,*\} & & \xleftarrow[\varepsilon_X]{} & & x. \end{array}$$

From the equality $q(x) \cup \{fs(x),*\} = \{x,*\}$, and the fact that $s(x) \in Y$ and $x \in X\backslash Y$, we deduce that $s(x) = *$ and $q(x) = \{x,*\}$. As a consequence the

morphism is strong. Any Schreier split epimorphism is (stably) strong (see [8], Lemma 2.1.6 and Proposition 2.3.4), thus f is the cokernel of its kernel k. So, such a split epimorphism is in fact a *Schreier split extension*.

As shown in [17], the definition of a Schreier split epimorphism makes sense also in the wider context of Jónsson-Tarski varieties.

5.4. Intrinsic Schreier split extensions [20]**.** We recall our approach towards Schreier extensions. Here $\mathbb{C}$ will denote a regular unital category with binary coproducts, comonadic covers and a natural imaginary splitting t.

Definition 5.5. A split epimorphism f with chosen section s and kernel k

$$K \underset{k}{\rightarrowtail} X \underset{f}{\overset{s}{\rightleftarrows}} Y, \tag{16}$$

is called an **intrinsic Schreier split epimorphism** (with respect to t) if there exists an imaginary morphism $q\colon X \dashrightarrow K$ (i.e., a morphism $q\colon P(X) \to K$), called the **imaginary (Schreier) retraction**, such that

(iS1) $\mu^X \circ \langle k\circ q, \overline{sf}\rangle = \overline{1_X}$, i.e., the diagram

$$\begin{array}{ccc} X & \overset{\langle k\circ q,\overline{sf}\rangle}{\dashrightarrow} & X\times X \\ & {\scriptstyle \overline{1_X}}\searrow & \downarrow{\scriptstyle \mu^X} \\ & & X \end{array}$$

commutes;

(iS2) $q \circ \mu^X \circ (k \times s) = \overline{\pi_K}$, i.e., the diagram

$$\begin{array}{ccccc} K\times Y & \xrightarrow{k\times s} & X\times X & \overset{\mu^X}{\dashrightarrow} & X \\ & {\scriptstyle \overline{\pi_K}}\searrow & & & \downarrow{\scriptstyle q} \\ & & & & K \end{array}$$

commutes.

The original definition in [20] expressed the above axioms through their corresponding morphisms and equalities in $\mathbb{C}$. However, using the composition in the Kleisli category $\mathbb{K}$, as above, gives a better understanding of the link with **(S1)** and **(S2)**.

The imaginary retraction of an intrinsic Schreier split epimorphism is necessarily unique (see [20, Proposition 5.3]) and we also have (by [20, Proposition 5.4]):

(iS3) $K \xrightarrow{k} X \underset{q\circ k=\overline{1_K}}{\overset{q}{\rightleftarrows}} K$, i.e., $qP(k) = \varepsilon_K$;

(iS4) $Y \xrightarrow{s} X \underset{q\circ s=\overline{0}}{\overset{q}{\rightleftarrows}} K$, i.e., $qP(s) = 0$;

(iS5) $0 \xrightarrow{0_X} X \underset{q\circ 0_X=\overline{0_K}}{\overset{q}{\rightleftarrows}} K$, i.e., $qP(0_X)(= q0_{P(X)}) = 0_K$;

(iS6)
$$\begin{array}{ccc} Y \times K & \xrightarrow{\langle k\circ q\circ \mu^X\circ(s\times k),\overline{s\pi_Y}\rangle} & X \times X \\ {\scriptstyle s\times k}\downarrow & & \downarrow{\scriptstyle \mu^X} \\ X \times X & \xrightarrow[\mu^X]{} & X, \end{array}$$
i.e., $\mu^X \circ \langle k \circ q \circ \mu^X \circ (s \times k), \overline{s\pi_Y}\rangle = \mu^X \circ (s \times k)$.

In order to get the intrinsic version of **(S7)**, we will need a further assumption, that will be discussed in the next section.

If we apply this intrinsic definition to the category Mon of monoids, we regain the original definition of a Schreier split epimorphism (= right homogeneous split epimorphism). Also, left homogeneous split epimorphisms (see Remark 5.2) fit the picture. Indeed:

Theorem 5.6. [20, Theorem 5.10] *In the case of monoids, the intrinsic Schreier split epimorphisms with respect to the direct imaginary splitting t^d are precisely the Schreier split epimorphisms. Similarly, the intrinsic Schreier split epimorphisms with respect to the twisted imaginary splitting t^w are the left homogeneous split epimorphisms.*

This result extends to Jónsson–Tarski varieties [20].

5.7. $\mathscr{S}$-protomodular categories [8]**.** We recall now the definition of an $\mathscr{S}$-protomodular category, with respect to a class $\mathscr{S}$ of points (i.e., of split epimorphisms with a fixed section) in a pointed category $\mathbb{C}$ with finite limits.

We denote by $\mathsf{Pt}(\mathbb{C})$ the category of points in $\mathbb{C}$, whose morphisms are pairs of morphisms which form commutative squares with both the split epimorphisms

also

$$\begin{array}{rl} & \pi_2 \circ \mu^{Y\times Y} \circ \langle\langle 1_Y, 0\rangle \circ q, \overline{\langle 1_Y, 1_Y\rangle\pi_2}\rangle \\ \overset{(14)}{=} & \mu^Y \circ (\pi_2 \times \pi_2) \circ \langle\langle 1_Y, 0\rangle \circ q, \overline{\langle 1_Y, 1_Y\rangle\pi_2}\rangle \\ = & \mu^Y \circ \langle \pi_2\langle 1_Y, 0\rangle \circ q, \overline{\pi_2\langle 1_Y, 1_Y\rangle\pi_2}\rangle \\ = & \mu^Y \circ \langle 0, \overline{\pi_2}\rangle \\ = & \mu^Y \circ \langle 0, 1_Y\rangle \circ \overline{\pi_2} \\ \overset{(13)}{=} & \overline{\pi_2} \\ = & \pi_2 \circ \overline{1_{Y\times Y}}. \end{array}$$

Combining these two equalities we get **(iSs1)**. Hence

Theorem 8.2. *In a regular unital category with binary coproducts, comonadic covers and a natural imaginary splitting, an object is an intrinsic Schreier special object if and only if its canonical imaginary magma structure is a one-sided loop structure.*

9. A non-varietal example

In this section we give an example of a non-varietal category for which there exists a natural imaginary splitting, and we analyse what are the intrinsic Schreier split epimorphisms and the intrinsic Schreier special objects in that context.

Take $\mathbb{C} = \mathsf{Set}_*^{\mathrm{op}}$, which is a semi-abelian category [14, 4, 1], so it is a regular unital category with binary coproducts. We consider the powerset monad (P, δ, ε) in Set_*, where:

$$\begin{aligned} P(X, x_0) &= (P(X) = \{A \subseteq X \mid x_0 \in A\}, \{x_0\}), \\ \varepsilon_{(X,x_0)}(x) &= \{x, x_0\}, \\ \delta_{(X,x_0)}(\{A_i\}_{i\in I}) &= \bigcup_{i\in I} A_i, \text{where each } A_i \in P(X). \end{aligned}$$

The monad (P, δ, ε) may be seen as a comonad in $\mathsf{Set}_*^{\mathrm{op}}$. Moreover, it is easy to check that each $P(X, x_0)$ is projective in $\mathsf{Set}_*^{\mathrm{op}}$, so that $\mathsf{Set}_*^{\mathrm{op}}$ is equipped with comonadic projective covers; we are in the conditions of Subsection 4.3. A natural imaginary splitting in $\mathsf{Set}_*^{\mathrm{op}}$ corresponds to a natural transformation $t\colon (\cdot)\times(\cdot) \Rightarrow P((\cdot)+(\cdot))$ in Set_*. We define, for any pair of pointed sets $(A, *)$ and $(B, *)$,

$$t_{A,B}\colon (A \times B, (*,*)) \to (P(A+B), \{*\})\colon (a, b) \mapsto \{\underline{a}, \overline{b}, *\} \qquad (24)$$

right loop axiom	"imaginary" commutative diagram
$(x-y)+y=x$	**(iL1)** $Y\times Y \overset{\langle q,\overline{\pi_2}\rangle}{\dashrightarrow} Y\times Y$, μ^Y, $\overline{\pi_1}$, Y
$(x+y)-y=x$	**(iL2)** $Y\times Y \overset{\langle \mu^Y,\overline{\pi_2}\rangle}{\dashrightarrow} Y\times Y$, q, $\overline{\pi_1}$, Y

TABLE 2. The right loop axioms and their corresponding diagrams

$(\langle 1_Y, 0\rangle \times \langle 1_Y, 1_Y\rangle)$ corresponds to the real morphism

$$\begin{array}{rcl} (1_{Y\times Y}\ \ 1_{Y\times Y})t_{Y\times Y,Y\times Y}P(\langle 1_Y,0\rangle\times\langle 1_Y,1_Y\rangle) & \overset{(6)}{=} & (\langle 1_Y,0\rangle\ \langle 1_Y,1_Y\rangle)t_{Y,Y} \\ & = & \langle (1_Y\ \ 1_Y), (0\ \ 1_Y)\rangle t_{Y,Y} \\ & \overset{(11),(8)}{=} & \langle \mu^Y, \pi_2\varepsilon_{Y\times Y}\rangle \\ & = & \langle \mu^Y, \overline{\pi_2}\rangle. \end{array}$$

The converse is also true. Indeed, suppose the object Y has the structure of an imaginary one-sided loop, in the sense that it is equipped with an imaginary morphism $q\colon Y\times Y \dashrightarrow Y$ which satisfies, together with the imaginary addition μ^Y, **(iL1)** and **(iL2)**. Then q is the imaginary Schreier retraction for the split epimorphism (23). To show this, we need to show that **(iSs1)** and **(iSs2)** hold. **(iSs2)** follows immediately from **(iL2)**, because, as we already observed, $\mu^{Y\times Y}\circ(\langle 1_Y,0\rangle\times\langle 1_Y,1_Y\rangle) = \langle \mu^Y, \overline{\pi_2}\rangle$. In order to prove **(iSs1)**, we use the previous equality $(\pi_1\times\pi_1)\circ\langle\langle 1_Y,0\rangle\circ q, \overline{\langle 1_Y,1_Y\rangle\pi_2}\rangle = \langle q, \overline{\pi_2}\rangle$ to get

$$\begin{array}{rcl} \pi_1\circ\mu^{Y\times Y}\circ\langle\langle 1_Y,0\rangle\circ q, \overline{\langle 1_Y,1_Y\rangle\pi_2}\rangle & = & \mu^Y\circ\langle q,\overline{\pi_2}\rangle \\ & \overset{\textbf{(iL1)}}{=} & \overline{\pi_1} \\ & = & \pi_1\circ\overline{1_{Y\times Y}}; \end{array}$$

and their sections. The functor $\mathrm{cod}\colon \mathsf{Pt}(\mathbb{C})\to\mathbb{C}$ associates with every split epimorphism its codomain. It is a fibration, usually called the **fibration of points**. For each object Y of $\mathbb{C}$, we denote by $\mathsf{Pt}_Y(\mathbb{C})$ the fibre of this fibration, whose objects are the points with codomain Y.

Let $\mathscr{S}$ be a class of points in $\mathbb{C}$ which is stable under pullbacks along any morphism. If we look at it as a full subcategory $\mathscr{S}\text{-}\mathsf{Pt}(\mathbb{C})$ of $\mathsf{Pt}(\mathbb{C})$, then it gives rise to a subfibration $\mathscr{S}\text{-}\mathrm{cod}$ of the fibration of points.

Definition 5.8. [8, Definition 8.1.1] Let $\mathbb{C}$ be a pointed finitely complete category, and $\mathscr{S}$ a pullback-stable class of points. We say that $\mathbb{C}$ is **$\mathscr{S}$-protomodular** when:

1. every point in $\mathscr{S}\text{-}\mathsf{Pt}(\mathbb{C})$ is a strong point;
2. $\mathscr{S}\text{-}\mathsf{Pt}(\mathbb{C})$ is closed under finite limits in $\mathsf{Pt}(\mathbb{C})$.

As shown in [10], $\mathscr{S}$-protomodular categories satisfy, relatively to the class $\mathscr{S}$, many of the properties of protomodular categories [2]. In particular, a relative version of the Split Short Five Lemma holds: given a morphism of $\mathscr{S}$-split extensions, i.e., a diagram

$$\begin{array}{ccccc} K' & \overset{k'}{\rightarrowtail} & X' & \underset{f'}{\overset{s'}{\leftrightarrows}} & Y' \\ {\scriptstyle\gamma}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ K & \overset{k}{\rightarrowtail} & X & \underset{f}{\overset{s}{\leftrightarrows}} & Y, \end{array}$$

such that the two rows are $\mathscr{S}$-split extensions (points in $\mathscr{S}$ with their kernel) and the three squares involving, respectively, the split epimorphisms, the kernels, and the sections commute, g is an isomorphism if and only if both γ and h are isomorphisms. In Section 7 we will show that, when $\mathscr{S}$ is the class of intrinsic Schreier split extensions, a stronger version of this lemma holds. Moreover, we will discuss the validity of other homological lemmas.

Example 5.9. [8] The category Mon of monoids is $\mathscr{S}$-protomodular with respect to the class $\mathscr{S}$ of Schreier split epimorphisms.

Example 5.10. [17] Every Jónsson–Tarski variety is an $\mathscr{S}$-protomodular category with respect to the class of Schreier split epimorphisms.

Example 5.11. [20] Every regular unital category with binary coproducts, equipped with comonadic covers and a natural imaginary splitting, is $\mathscr{S}$-protomodular with respect to the class $\mathscr{S}$ of intrinsic Schreier split epimorphisms.

Consequently, any such split epimorphism is an *intrinsic Schreier split extension*.

The reader may find several other examples in [11].

6. The associativity axiom

In order to improve the behaviour of the intrinsic Schreier split extensions, it is useful to consider an additional assumption, concerning the associativity of the imaginary addition μ^X.

Let $\mathbb{C}$ be a regular unital category with binary coproducts, comonadic covers and a natural imaginary splitting t. Suppose that, for every object X, the imaginary addition μ^X satisfies the associativity axiom $\mu^X \circ (\mu^X \times \overline{1_X}) = \mu^X \circ (\overline{1_X} \times \mu^X)$, i.e., the diagram

$$\begin{array}{ccc} X \times X \times X & \xrightarrow{\mu^X \times \overline{1_X}} & X \times X \\ {\scriptstyle \overline{1_X} \times \mu^X} \downarrow & & \downarrow {\scriptstyle \mu^X} \\ X \times X & \xrightarrow[\mu^X]{} & X \end{array}$$

commutes. In particular, for arbitrary imaginary morphisms $a, b, c \colon A \dashrightarrow X$, we get

$$\mu^X \circ \langle \mu^X \circ \langle a, b\rangle, c\rangle = \mu^X \circ \langle a, \mu^X \circ \langle b, c\rangle\rangle. \tag{17}$$

Example 6.1. In Gp and in Mon, the direct and twisted imaginary splittings induce associative imaginary additions.

Among the properties, listed in the previous section, of Schreier split epimorphisms of monoids, there is one, namely the property given by **(S7)**, which uses the associativity of the monoid operation (for X). Hence it is not so surprising that we may prove its intrinsic version when we assume the associativity axiom.

Proposition 6.2. *Suppose that the natural addition $(\mu^X \colon X \times X \dashrightarrow X)_{X \in \mathbb{C}}$ is associative. Given an intrinsic Schreier split extension* (16) *with imaginary retraction q, the following diagram commutes*

In this context, we also have:

(iSs3) $Y \xrightarrow[q \circ \langle 1_Y, 0\rangle = \overline{1_Y}]{\langle 1_Y, 0\rangle} Y \times Y \overset{q}{\dashrightarrow} Y$, i.e., $qP(\langle 1_Y, 0\rangle) = \varepsilon_Y$;

(iSs4) $Y \xrightarrow[q \circ \langle 1_Y, 1_Y\rangle = \overline{0}]{\langle 1_Y, 1_Y\rangle} Y \times Y \overset{q}{\dashrightarrow} Y$, i.e., $qP(\langle 1_Y, 1_Y\rangle) = 0$.

So, if Y is an intrinsic Schreier special object, then the identities **(iSs3)** and **(iSs4)** make $q \colon Y \times Y \dashrightarrow Y$ an **imaginary subtraction**. Indeed, the identities **(iSs3)** and **(iSs4)** correspond to the varietal axioms for a subtraction, i.e.,

$$q(x, 0) = x, \qquad q(x, x) = 0.$$

8.1. Imaginary (one-sided) loops. Consider an intrinsic Schreier special object Y. We now show that the imaginary addition given in (11) and the imaginary subtraction $q \colon Y \times Y \dashrightarrow Y$ satisfy the axioms of a (one-sided) loop (like those of a right loop or a left loop). We say then that Y has the structure of an **imaginary one-sided loop**.

We must prove the following right loop or left loop axioms

$$\begin{cases} (x - y) + y = x \\ (x + y) - y = x \end{cases} \quad \text{or} \quad \begin{cases} x + (-x + y) = y \\ -x + (x + y) = y \end{cases}$$

in the imaginary context; we consider the left-hand side axioms. Table 2 gives the right loop axioms and their corresponding "imaginary" commutative diagrams. The only difference in the diagrams is that μ^Y and q are swapped, just as "+" and "−" are swapped in the right loop axioms.

The commutativity of **(iL1)** follows from composing **(iSs1)** with π_1. From (14) we know that $\pi_1 \circ \mu^{Y \times Y} = \mu^Y \circ (\pi_1 \times \pi_1)$. Then, we just have to prove that $(\pi_1 \times \pi_1) \circ \langle\langle 1_Y, 0\rangle \circ q, \overline{\langle 1_Y, 1_Y\rangle \pi_2}\rangle = \langle q, \overline{\pi_2}\rangle$. In fact, $(\pi_1 \times \pi_1) \circ \langle\langle 1_Y, 0\rangle \circ q, \overline{\langle 1_Y, 1_Y\rangle \pi_2}\rangle$ corresponds to the real morphism

$$(\pi_1 \times \pi_1)\langle\langle 1_Y, 0\rangle q, \langle 1_Y, 1_Y\rangle \pi_2 \varepsilon_{Y \times Y}\rangle = \langle q, \pi_2 \varepsilon_{Y \times Y}\rangle = \langle q, \overline{\pi_2}\rangle.$$

The commutativity of **(iL2)** follows from **(iSs2)**. In this case we must show that $\mu^{Y \times Y} \circ (\langle 1_Y, 0\rangle \times \langle 1_Y, 1_Y\rangle) = \langle \mu^Y, \overline{\pi_2}\rangle$. The imaginary morphism $\mu^{Y \times Y} \circ$

that an object Y is called an **$\mathscr{S}$-special object** when the split epimorphism

$$Y \rightarrowtriangle_{\langle 1_Y, 0\rangle} Y \times Y \underset{\pi_2}{\overset{\langle 1_Y, 1_Y\rangle}{\twoheadleftarrow}} Y \tag{23}$$

(or, equivalently, the split epimorphism $(\pi_1, \langle 1_Y, 1_Y\rangle)$) belongs to the class $\mathscr{S}$. If $\mathbb{C}$ is an $\mathscr{S}$-protomodular category, then the full subcategory formed by the $\mathscr{S}$-special objects is protomodular ([10], Proposition 6.2), and it is called the **protomodular core** of $\mathbb{C}$ with respect to the class $\mathscr{S}$. When $\mathbb{C}$ is the category of monoids, and $\mathscr{S}$ is either the class of Schreier split epimorphisms or the one of left homogeneous split epimorphisms, the protomodular core is the category of groups. More generally, when $\mathbb{V}$ is a Jónsson–Tarski variety, an algebra in $\mathbb{V}$ is a Schreier special object if and only if it has a right loop structure [20, Proposition 7.5] (see Subsection 8.1 for the right loop axioms). Similarly, an algebra in $\mathbb{V}$ is special with respect to the class of left homogeneous split epimorphisms (see Remark 5.2) if and only if it has a left loop structure (see Subsection 8.1 for the left loop axioms).

Now we want to study what happens in the intrinsic Schreier setting. So, let $\mathbb{C}$ be a regular unital category with binary coproducts, comonadic covers and a natural imaginary splitting t. An object is an **intrinsic Schreier special object** when the split epimorphism (23) is an intrinsic Schreier split epimorphism. This means that there exists an imaginary morphism $q\colon Y \times Y \dashrightarrow Y$ such that:

(iSs1) the diagram

$$\begin{array}{ccc} Y \times Y & \xrightarrow{\langle\langle 1_Y,0\rangle\circ q, \overline{\langle 1_Y, 1_Y\rangle\pi_2}\rangle} & Y \times Y \times Y \times Y \\ & \searrow^{\overline{1_{Y\times Y}}} & \downarrow{\scriptstyle \mu^{Y\times Y}} \\ & & Y \times Y \end{array}$$

commutes;

(iSs2) the diagram

$$\begin{array}{ccccc} Y \times Y & \xrightarrow{\langle 1_Y,0\rangle\times\langle 1_Y,1_Y\rangle} & Y \times Y \times Y \times Y & \xrightarrow{\mu^{Y\times Y}} & Y \times Y \\ & \searrow^{\overline{\pi_1}} & & & \downarrow{\scriptstyle q} \\ & & & & Y \end{array}$$

commutes.

(iS7)

$$\begin{array}{ccccc} X \times X & \xrightarrow{\langle q\circ\pi_1, \overline{sf\pi_1}, k\circ q\circ\pi_2\rangle} & K \times X \times X & \xrightarrow{\overline{1_K}\times(q\circ\mu^X)} & K \times K \\ \downarrow{\scriptstyle \mu^X} & & & & \downarrow{\scriptstyle \mu^K} \\ X & & \xrightarrow{\quad q \quad} & & K \end{array}$$

i.e., $\mu^K \circ (\overline{1_K} \times (q \circ \mu^X)) \circ \langle q \circ \pi_1, \overline{sf\pi_1}, k \circ q \circ \pi_2\rangle = q \circ \mu^X$.

Proof: Using Lemma 6.3 below, it suffices to prove that

$$\mu^X \circ \langle k \circ \mu^K \circ (\overline{1_K} \times (q \circ \mu^X)) \circ \langle q \circ \pi_1, \overline{sf\pi_1}, k \circ q \circ \pi_2\rangle, \overline{sf} \circ \mu^X\rangle = \\ = \mu^X \circ \langle k \circ q \circ \mu^X, \overline{sf} \circ \mu^X\rangle$$

which, by (**iS1**), is the same as

$$\mu^X \circ \langle k \circ \mu^K \circ (\overline{1_K} \times (q \circ \mu^X)) \circ \langle q \circ \pi_1, \overline{sf\pi_1}, k \circ q \circ \pi_2\rangle, \overline{sf} \circ \mu^X\rangle = \mu^X.$$

We have

$$\begin{aligned} & \mu^X \circ \langle k \circ \mu^K \circ (\overline{1_K} \times (q \circ \mu^X)) \circ \langle q \circ \pi_1, \overline{sf\pi_1}, k \circ q \circ \pi_2\rangle, \overline{sf} \circ \mu^X\rangle \\ \overset{(14)}{=}\ & \mu^X \circ \langle \mu^X \circ (k \times k) \circ \langle q \circ \pi_1, q \circ \mu^X \circ \langle \overline{sf\pi_1}, k \circ q \circ \pi_2\rangle\rangle, \mu^X \circ (\overline{sf} \times \overline{sf})\rangle \\ =\ & \mu^X \circ \langle \mu^X \circ \langle k \circ q \circ \pi_1, k \circ q \circ \mu^X \circ \langle \overline{sf\pi_1}, k \circ q \circ \pi_2\rangle\rangle, \mu^X \circ (\overline{sf} \times \overline{sf})\rangle \\ \overset{(17)}{=}\ & \mu^X \circ \langle k \circ q \circ \pi_1, \mu^X \circ \langle k \circ q \circ \mu^X \circ \langle \overline{sf\pi_1}, k \circ q \circ \pi_2\rangle, \mu^X \circ (\overline{sf} \times \overline{sf})\rangle\rangle \\ =\ & \mu^X \circ \langle k \circ q \circ \pi_1, \mu^X \circ \langle k \circ q \circ \mu^X \circ \langle \overline{sf\pi_1}, k \circ q \circ \pi_2\rangle, \mu^X \circ \langle \overline{sf\pi_1}, \overline{sf\pi_2}\rangle\rangle\rangle \\ \overset{(17)}{=}\ & \mu^X \circ \langle k \circ q \circ \pi_1, \mu^X \circ \langle \mu^X \circ \langle k \circ q \circ \mu^X \circ \langle \overline{sf\pi_1}, k \circ q \circ \pi_2\rangle, \overline{sf\pi_1}\rangle, \overline{sf\pi_2}\rangle\rangle \\ =\ & \mu^X \circ \langle k \circ q \circ \pi_1, \mu^X \circ \langle \mu^X \circ \langle k \circ q \circ \mu^X \circ (s \times k) \circ \langle \overline{f\pi_1}, q \circ \pi_2\rangle, \\ & \overline{s\pi_Y} \circ \langle \overline{f\pi_1}, q \circ \pi_2\rangle, \overline{sf\pi_2}\rangle\rangle \\ =\ & \mu^X \circ \langle k \circ q \circ \pi_1, \mu^X \circ \langle \mu^X \circ \langle k \circ q \circ \mu^X \circ (s \times k), \overline{s\pi_Y}\rangle \circ \\ & \langle \overline{f\pi_1}, q \circ \pi_2\rangle, \overline{sf\pi_2}\rangle\rangle \\ \overset{(\mathbf{iS6})}{=}\ & \mu^X \circ \langle k \circ q \circ \pi_1, \mu^X \circ \langle \mu^X \circ (s \times k) \circ \langle \overline{f\pi_1}, q \circ \pi_2\rangle, \overline{sf\pi_2}\rangle\rangle \\ =\ & \mu^X \circ \langle k \circ q \circ \pi_1, \mu^X \circ \langle \mu^X \circ \langle \overline{sf\pi_1}, k \circ q \circ \pi_2\rangle, \overline{sf\pi_2}\rangle\rangle \\ \overset{(17)}{=}\ & \mu^X \circ \langle k \circ q \circ \pi_1, \mu^X \circ \langle \overline{sf\pi_1}, \mu^X \circ \langle k \circ q \circ \pi_2, \overline{sf\pi_2}\rangle\rangle\rangle \end{aligned}$$

$$\begin{aligned}
&\overset{\textbf{(iS1)}}{=} \mu^X \circ \langle k \circ q \circ \pi_1, \mu^X \circ \langle \overline{sf\pi_1}, \overline{\pi_2} \rangle \rangle \\
&\overset{(17)}{=} \mu^X \circ \langle \mu^X \circ \langle k \circ q \circ \pi_1, \overline{sf\pi_1} \rangle, \overline{\pi_2} \rangle \\
&\overset{\textbf{(iS1)}}{=} \mu^X \circ \langle \overline{\pi_1}, \overline{\pi_2} \rangle \\
&= \mu^X. \qquad \blacksquare
\end{aligned}$$

Lemma 6.3. *Let* (16) *be a split epimorphism with an imaginary morphism q such that* **(iS2)** *holds. If*

$$\mu^X \circ \langle k \circ a, s \circ b \rangle = \mu^X \circ \langle k \circ c, s \circ d \rangle,$$

where $a, c\colon A \dashrightarrow K$ and $b, d\colon A \dashrightarrow Y$ are imaginary morphisms, then $a = c$.

Proof:

$$\begin{aligned}
& \mu^X \circ \langle k \circ a, s \circ b \rangle = \mu^X \circ \langle k \circ c, s \circ d \rangle \\
\Rightarrow\ & q \circ \mu^X \circ \langle k \circ a, s \circ b \rangle = q \circ \mu^X \circ \langle k \circ c, s \circ d \rangle \\
\Rightarrow\ & q \circ \mu^X \circ (k \times s) \circ \langle a, b \rangle = q \circ \mu^X \circ (k \times s) \circ \langle c, d \rangle \\
\overset{\textbf{(iS2)}}{\Rightarrow}\ & \overline{\pi_K} \circ \langle a, b \rangle = \overline{\pi_K} \circ \langle c, d \rangle \\
\Rightarrow\ & a = c. \qquad \blacksquare
\end{aligned}$$

Remark 6.4. As an immediate consequence of Lemma 6.3, we obtain the uniqueness of the imaginary retraction for any intrinsic Schreier split extension (16) (which was already known from [20, Proposition 5.3]). Given two possible imaginary retractions $q, q'\colon X \dashrightarrow K$, **(iS1)** gives

$$\mu^X \circ \langle k \circ q, \overline{sf} \rangle = \overline{1_X} = \mu^X \circ \langle k \circ q', \overline{sf} \rangle;$$

consequently, $q = q'$.

7. Stability properties and homological lemmas

In this section we prove that certain stability properties for Schreier extensions of monoids shown in [8] still hold for intrinsic Schreier extensions in our context: $\mathbb{C}$ will denote a regular unital category with binary coproducts, comonadic covers and a natural imaginary splitting t. Moreover, we will observe that some of these stability properties allow to extend the validity of some classical homological lemmas to our intrinsic context.

$\gamma q' = qP(g)$. Then γ is a regular epimorphism since so are q and $P(g)$ (by **(iS2)** and Remark 2.7).

Conversely, suppose that γ and h are regular epimorphisms. We take the (regular epimorphisms, monomorphism) factorisation $g = me$, and prove that m is an isomorphism. Since the bottom row is an intrinsic Schreier split extension, we know that (k, s) is an jointly extremal-epimorphic pair (see Subsection 5.4). Since q and $h\varepsilon_{Y'}$ are regular epimorphisms, then $(kq, sh\varepsilon_{Y'})$ is also a jointly extremal-epimorphic pair. In $\mathbb{C}$, it is easy to check the commutativity of

$$\begin{array}{ccccc}
 & & M & & \\
 & {\scriptstyle \varepsilon_M P(ek')\sigma P(q)\delta_X}\nearrow & \downarrow{\scriptstyle m} & \nwarrow{\scriptstyle \varepsilon_M P(es')} & \\
P(X) & \xrightarrow[kq]{} & X & \xleftarrow[sh\varepsilon_{Y'}]{} & P(Y'),
\end{array}$$

where σ is a splitting of the split epimorphism $P(\gamma)$ (Remark 2.7). Thus, m is an isomorphism and g is a regular epimorphism.

(b) If g is a monomorphism, then so are γ and h, from the commutativity of the diagram. For the converse, suppose that $a, b\colon U \to X'$ are morphisms such that $ga = gb$. Then, $fga = fgb$, from which we get $f'a = f'b$ (since $fg = hf'$ and h is a monomorphism). On the other hand, we deduce $kqP(g)P(a) = kqP(g)P(b)$ and $k\gamma q'P(a) = k\gamma q'P(b)$, from the compatibility for imaginary retractions ([20, Proposition 5.7]). This gives $q'P(a) = q'P(b)$, since $k\gamma$ is a monomorphism. Thus $q' \circ a = q' \circ b$, as imaginary morphisms. Then

$$\begin{aligned}
a &= \overline{1_{X'}} \circ a \\
&\overset{\textbf{(iS1)}}{=} \mu^{X'} \circ \langle k' \circ q', \overline{s'f'} \rangle \circ a \\
&= \mu^{X'} \circ \langle k' \circ q' \circ a, \overline{s'f'a} \rangle \\
&= \mu^{X'} \circ \langle k' \circ q' \circ b, \overline{s'f'b} \rangle \\
&= \mu^{X'} \circ \langle k' \circ q', \overline{s'f'} \rangle \circ b \\
&\overset{\textbf{(iS1)}}{=} \overline{1_{X'}} \circ b \\
&= b. \qquad \blacksquare
\end{aligned}$$

8. Intrinsic Schreier special objects

Let $\mathbb{C}$ be a pointed and finitely complete category and $\mathscr{S}$ a class of points in $\mathbb{C}$ which is stable under pullbacks along arbitrary morphisms. Recall from [10]

section, we can apply Proposition 6.2 and Theorem 6.7 in [11] and get the following version of the 3×3-Lemma:

Theorem 7.8. *Consider the commutative diagram*

$$\begin{array}{ccccc}
K'' & \xrightarrow{k''} & X'' & \xrightarrow{f''} & Y'' \\
{\scriptstyle w'}\downarrow & & {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle v'} \\
K' & \xrightarrow[k']{} & X' & \xrightarrow[f']{} & Y' \\
{\scriptstyle w}\downarrow & & {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\
K & \xrightarrow[k]{} & X & \xrightarrow[f]{} & Y,
\end{array}$$

where the three columns and the middle row are intrinsic Schreier special extensions. The upper row is an intrinsic Schreier special extension if and only if the lower one is.

We conclude this section by proving the stronger version of the Split Short Five Lemma we mentioned in Section 5.

Proposition 7.9. (See [8, Proposition 2.3.10]) *Suppose that the values of P are projective objects in $\mathbb{C}$. Consider the diagram*

$$\begin{array}{ccccc}
K' & \underset{k'}{\overset{q'}{\leftleftarrows}} & X' & \underset{f'}{\overset{s'}{\leftleftarrows}} & Y' \\
{\scriptstyle \gamma}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\
K & \underset{k}{\overset{q}{\leftleftarrows}} & X & \underset{f}{\overset{s}{\leftleftarrows}} & Y,
\end{array}$$

where both rows are intrinsic Schreier split extensions and the three squares involving, respectively, the split epimorphism, the kernels, and the sections commute. Then

(a) *g is a regular epimorphism if and only if γ and h are regular epimorphisms;*
(b) *g is a monomorphism if and only if γ and h are monomorphisms.*

Proof: (a) If g is a regular epimorphism, then so is h, from the commutativity of the diagram. Moreover, the compatibility for the imaginary retractions gives

In order to extend those proofs in [8] which use the associativity of the monoid operations, we assume the associativity axiom holds. This is the case, in particular, of the first stability property we consider:

Proposition 7.1. (See [8, Proposition 2.3.2]) *Suppose that the natural addition $(\mu^X \colon X \times X \dashrightarrow X)_{X \in \mathbb{C}}$ is associative. Then the intrinsic Schreier split extensions are stable under composition.*

Proof: Suppose that

$$K \underset{k}{\overset{q_f}{\leftleftarrows}} X \underset{f}{\overset{s}{\leftleftarrows}} Y$$

and

$$L \underset{l}{\overset{q_g}{\leftleftarrows}} Y \underset{g}{\overset{t}{\leftleftarrows}} Z$$

are intrinsic Schreier extensions. We want to prove that

$$M \underset{m}{\rightarrowtail} X \underset{gf}{\overset{st}{\leftleftarrows}} Z$$

is an intrinsic Schreier extension, where m is the kernel of gf

$$\begin{array}{ccc}
M & \xrightarrow{m} & X \\
{\scriptstyle f'}\downarrow\uparrow{\scriptstyle s'} & & {\scriptstyle f}\downarrow\uparrow{\scriptstyle s} \\
L & \xrightarrow{l} & Y \\
 & & {\scriptstyle g}\downarrow\uparrow{\scriptstyle t} \\
0 & \longrightarrow & Z.
\end{array}$$

We must define the imaginary retraction $q \colon X \dashrightarrow M$. The imaginary morphism $\mu^X \circ \langle k \circ q_f, sl \circ q_g \circ f \rangle \colon X \dashrightarrow X$ is such that the composition with gf gives the following equalities in $\mathbb{C}$:

$$\begin{aligned}
& gf[1_X \;\; 1_X] t_{X,X} P(\langle kq_f, slq_g P(f) \rangle) \delta_X \\
= \;& gf[1_X \;\; 1_X] t_{X,X} P((kq_f) \times (slq_g P(f))) P(\langle 1_{P(X)}, 1_{P(X)} \rangle) \delta_X \\
\overset{(6)}{=} \;& gf[1_X \;\; 1_X]((kq_f) + (slq_g P(f)) t_{P(X),P(X)} P(\langle 1_{P(X)}, 1_{P(X)} \rangle) \delta_X \\
= \;& [gfkq_f \;\; gfslq_g P(f)) t_{P(X),P(X)} P(\langle 1_{P(X)}, 1_{P(X)} \rangle) \delta_X \\
= \;& 0.
\end{aligned}$$

This gives a unique morphism $q\colon P(X) \to M$ in $\mathbb{C}$, i.e., an imaginary morphism $q\colon X \dashrightarrow M$, such that $m \circ q = \mu^X \circ \langle k \circ q_f, sl \circ q_g \circ f\rangle$. Next we prove that q is the imaginary retraction for the split epimorphism gf:
(iS1)

$$\begin{array}{rl} & \mu^X \circ \langle m \circ q, \overline{stgf}\rangle \\ = & \mu^X \circ \langle \mu^X \circ \langle k \circ q_f, sl \circ q_g \circ f\rangle, \overline{stgf}\rangle \\ \overset{(17)}{=} & \mu^X \circ \langle k \circ q_f, \mu^X \circ \langle sl \circ q_g \circ f, \overline{stgf}\rangle\rangle \\ = & \mu^X \circ \langle k \circ q_f, \mu^X \circ (s \times s) \circ \langle l \circ q_g, \overline{tg}\rangle \circ f\rangle \\ \overset{(14)}{=} & \mu^X \circ \langle k \circ q_f, s \circ \mu^Y \circ \langle l \circ q_g, \overline{tg}\rangle \circ f\rangle \\ = & \overline{1_X}, \end{array}$$

where in the last step we use **(iS1)** for g and then **(iS1)** for f.

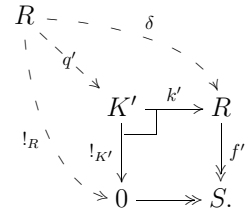
Next we prove that $m \circ q \circ \mu^X \circ (m \times (st)) = m \circ \overline{\pi_M}$ and use the fact that m is a monomorphism, to conclude **(iS2)**:

$$\begin{array}{rl} & m \circ q \circ \mu^X \circ (m \times (st)) \\ = & \mu^X \circ \langle k \circ q_f, sl \circ q_g \circ f\rangle \circ \mu^X \circ (m \times (st)) \\ = & \mu^X \circ \langle k \circ q_f \circ \mu^X \circ (m \times (st)), sl \circ q_g \circ f \circ \mu^X \circ (m \times (st))\rangle \\ \overset{(14)}{=} & \mu^X \circ \langle k \circ q_f \circ \mu^X \circ (m \times (st)), sl \circ q_g \circ \mu^Y \circ (f \times f) \circ (m \times (st))\rangle \\ = & \mu^X \circ \langle k \circ q_f \circ \mu^X \circ (m \times (st)), sl \circ q_g \circ \mu^Y \circ ((fm) \times (fst))\rangle \\ = & \mu^X \circ \langle k \circ q_f \circ \mu^X \circ (m \times (st)), sl \circ q_g \circ \mu^Y \circ ((lf') \times t)\rangle \\ = & \mu^X \circ \langle k \circ q_f \circ \mu^X \circ (m \times (st)), sl \circ q_g \circ \mu^Y \circ (l \times t) \circ (f' \times 1_Z)\rangle \\ = & \mu^X \circ \langle k \circ q_f \circ \mu^X \circ (m \times (st)), sl \circ \overline{\pi_L} \circ (f' \times 1_Z)\rangle, \end{array}$$

where in the last equality we use **(iS2)** for g. Now we use $sl \circ \overline{\pi_L} \circ (f' \times 1_Z) = \overline{sfm\pi_M}$ and **(iS7)** applied to f. This gives

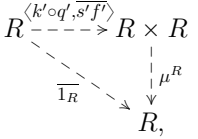
$$\begin{array}{rl} & m \circ q \circ \mu^X \circ (m \times (st)) = \cdots \\ = & \mu^X \circ \langle k \circ \mu^K \circ (\overline{1_K} \times (q_f \circ \mu^X)) \circ \langle q_f \circ \pi_1, \overline{sf\pi_1}, k \circ q_f \circ \pi_2\rangle \\ & \circ (m \times (st)), \overline{sfm\pi_M}\rangle \\ = & \mu^X \circ \langle k \circ \mu^K \circ \langle q_f \circ \pi_1 \circ (m \times (st)), q_f \circ \mu^X \circ \langle \overline{sf\pi_1}, k \circ q_f \circ \pi_2\rangle \\ & \circ (m \times (st))\rangle, \overline{sfm\pi_M}\rangle. \end{array}$$

We are now able to define the imaginary retraction q' for (f', s'):



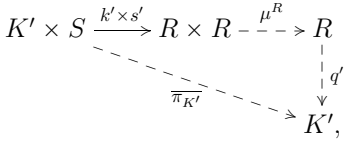
Note that $s_i f' \circ \delta = f r_i \circ \delta = fk \circ q \circ r_i = 0$, $i \in \{1, 2\}$, from which we get $f' \circ \delta = 0$.

To finish we must prove **(iS1)** and **(iS2)** for (f', s'). To obtain the equality for **(iS1)**



we prove that $r_i \circ \mu^R \circ \langle k' \circ q', \overline{s'f'}\rangle = r_i \circ \overline{1_R} = \overline{r_i}$, $i \in \{1, 2\}$, by using **(iS1)** for (f, s).

To obtain the equality for **(iS2)**



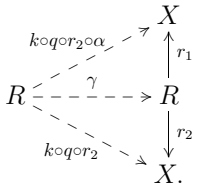
we prove that $r_i \circ k' \circ q' \circ \mu^R \circ (k' \times s') = r_i \circ k' \circ \overline{\pi_{K'}}$, $i \in \{1, 2\}$, which uses **(iS2)** for (f, s). ∎

We say that a morphism $f\colon X \to Y$ is an **intrinsic Schreier special morphism** if the split epimorphism (f_1, e_f) (or, equivalently, the split epimorphism (f_2, e_f)) is intrinsic Schreier, where $\mathrm{Eq}(f) \overset{f_2}{\underset{f_1}{\overset{\longleftarrow}{\underset{\longrightarrow}{\longleftarrow e_f \longrightarrow}}}} X$ is the kernel pair of f. If this intrinsic Schreier special morphism is a regular epimorphism, then it is automatically the cokernel of its kernel, thus it gives rise to an extension (Proposition 5.6 in [11]). Thanks to the stability properties we proved in this

and

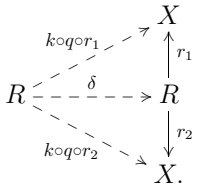
$$\begin{aligned}
& r_2 \circ \mu^R \circ \langle uk'' \circ q'' \circ \chi, e_R k \circ q \circ r_2 \circ \alpha \rangle \\
\overset{(14)}{=}\ & \mu^X \circ (r_2 \times r_2) \circ \langle uk'' \circ q'' \circ \chi, e_R k \circ q \circ r_2 \circ \alpha \rangle \\
\overset{(18),(20)}{=}\ & \mu^X \circ \langle kt_2 w \circ q'' \circ \chi, k \circ q \circ \mu^X \circ \langle sfr_2 u \circ \chi, k \circ q \circ r_1 \rangle \rangle \\
=\ & \mu^X \circ (k \times k) \circ \langle t_2 w \circ q'' \circ \chi, q \circ \mu^X \circ \langle sfr_2 u \circ \chi, k \circ q \circ r_1 \rangle \rangle \\
\overset{(19)}{=}\ & \mu^X \circ (k \times k) \circ \langle q \circ r_2 u \circ \chi, q \circ \mu^X \circ \langle sfr_2 u \circ \chi, k \circ q \circ r_1 \rangle \rangle \\
\overset{(14)}{=}\ & k \circ \mu^K \circ (\overline{1_K} \times (q \circ \mu^X)) \circ \langle q \circ r_2 u \circ \chi, \langle sfr_2 u \circ \chi, k \circ q \circ r_1 \rangle \rangle \\
=\ & k \circ \mu^K \circ (\overline{1_K} \times (q \circ \mu^X)) \circ \\
& \langle q \circ \pi_1 \circ \langle r_2 u \circ \chi, \overline{r_1} \rangle, \langle \overline{sf\pi_1} \circ \langle r_2 u \circ \chi, \overline{r_1} \rangle, k \circ q \circ \pi_2 \circ \langle r_2 u \circ \chi, \overline{r_1} \rangle \rangle \rangle \\
=\ & k \circ \mu^K \circ (\overline{1_K} \times (q \circ \mu^X)) \circ \langle q \circ \pi_1, \langle \overline{sf\pi_1}, k \circ q \circ \pi_2 \rangle \rangle \circ \langle r_2 u \circ \chi, \overline{r_1} \rangle \\
\overset{\textbf{(iS7)}}{=}\ & k \circ q \circ \mu^X \circ \langle r_2 u \circ \chi, \overline{r_1} \rangle \\
=\ & k \circ q \circ \mu^X \circ \langle r_2 u \circ \chi, r_2 \circ \overline{e_R r_1} \rangle \\
=\ & k \circ q \circ \mu^X \circ (r_2 \times r_2) \circ \langle u \circ \chi, \overline{e_R r_1} \rangle \\
\overset{(14)}{=}\ & k \circ q \circ r_2 \circ \mu^R \circ \langle u \circ \chi, \overline{e_R r_1} \rangle \\
\overset{\textbf{(iS1)}}{=}\ & k \circ q \circ r_2,
\end{aligned}$$

where the last **(iS1)** is with respect to the intrinsic Schreier extension (r_1, e_R); thus



(22)

Next we use the fact that R is transitive together with (20), (21) and (22) to deduce the existence of an imaginary morphism $\delta\colon R \dashrightarrow R$ such that the following diagram commutes



Note that, part of the composite above is

$$\begin{aligned}
& q_f \circ \mu^X \circ \langle \overline{sf\pi_1}, k \circ q_f \circ \pi_2 \rangle \circ (m \times (st)) \\
\overset{(14)}{=}\ & \mu^K \circ (q_f \times q_f) \circ \langle \overline{sf\pi_1}, k \circ q_f \circ \pi_2 \rangle \circ (m \times (st)) \\
=\ & \mu^K \circ \langle q_f \circ \overline{sf\pi_1} \circ (m \times (st)), q_f \circ k \circ q_f \circ \pi_2 (m \times (st)) \rangle \\
\overset{\textbf{(iS3)}}{=}\ & \mu^K \circ \langle q_f \circ \overline{sf\pi_1} \circ (m \times (st)), q_f \circ st\pi_2 \rangle \\
\overset{\textbf{(iS4)}}{=}\ & 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
& m \circ q \circ \mu^X \circ (m \times (st)) = \cdots \\
=\ & \mu^X \circ \langle k \circ \mu^K \circ \langle 1_K, 0 \rangle \circ q_f \circ \overline{m\pi_M}, \overline{sfm\pi_M} \rangle \\
\overset{(12)}{=}\ & \mu^X \circ \langle k \circ q_f, \overline{sf} \rangle \circ \overline{m\pi_M} \\
\overset{\textbf{(iS1)}}{=}\ & \overline{1_X} \circ \overline{m\pi_M} = m \circ \overline{\pi_M}.
\end{aligned}$$

■

Proposition 7.2. (See [8, Proposition 2.3.2]) *Consider split epimorphisms*

$$X \underset{f}{\overset{s}{\leftrightarrows}} Y \underset{g}{\overset{t}{\leftrightarrows}} Z$$

in $\mathbb{C}$. *If* (gf, st) *is an intrinsic Schreier split extension, then so is* (g, t).

Proof: We use the same notation as in Proposition 7.1. We define the imaginary retraction for g as $q_g = f' \circ q \circ s\colon Y \dashrightarrow L$. From **(iS1)** for gf, we have

$$\begin{aligned}
\mu^X \circ \langle m \circ q, \overline{stgf} \rangle = \overline{1_X} \ &\Rightarrow\ f \circ \mu^X \circ \langle m \circ q, \overline{stgf} \rangle = \overline{f} \\
&\overset{(14)}{\Rightarrow}\ \mu^Y \circ \langle fm \circ q, \overline{tgf} \rangle = \overline{f}.
\end{aligned}$$

Then

$$\begin{aligned}
\mu^Y \circ \langle l \circ q_g, \overline{tg} \rangle &= \mu^Y \circ \langle lf' \circ q \circ s, \overline{tg} \rangle \\
&= \mu^Y \circ \langle fm \circ q \circ s, \overline{tgfs} \rangle \\
&= \mu^Y \circ \langle fm \circ q, \overline{tgf} \rangle \circ s \\
&= \overline{f} \circ s \\
&= \overline{1_Y},
\end{aligned}$$

which proves **(iS1)** for g. As for **(iS2)** for g, we have

$$\begin{array}{rcl} q_g \circ \mu^Y \circ (l \times t) & = & f' \circ q \circ s \circ \mu^Y \circ (l \times t) \\ & \overset{(14)}{=} & f' \circ q \circ \mu^X \circ ((sl) \times (st)) \\ & = & f' \circ q \circ \mu^X \circ ((ms') \times (st)) \\ & = & f' \circ q \circ \mu^X \circ (m \times (st)) \circ (s' \times 1_Z) \\ & = & f' \circ \overline{\pi_M} \circ (s' \times 1_Z) \\ & = & f's' \circ \overline{\pi_L} \\ & = & \overline{\pi_L}, \end{array}$$

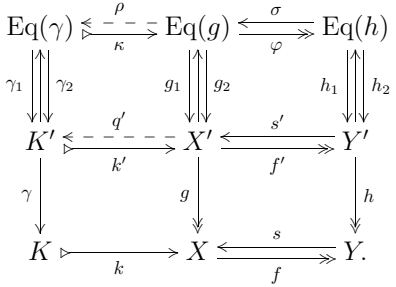
where we use **(iS2)** for gf in the fifth equality. ∎

Lemma 7.3. *Suppose that the values of P are projective objects in $\mathbb{C}$. Let $a, b\colon A \dashrightarrow X$ be imaginary morphisms and $z\colon Z \twoheadrightarrow A$ be a regular epimorphism. If $a \circ z = b \circ z$, then $a = b$.*

Proof: $a \circ z = b \circ z$ corresponds to the equality $aP(z) = bP(z)$ in $\mathbb{C}$. Then $a = b$, since $P(z)$ is a split epimorphism (see Remark 2.7). ∎

In the following $\mathrm{Eq}(f)$ denotes the kernel pair of a morphism f.

Proposition 7.4. (See [8, Proposition 2.3.5] and [11, Proposition 4.8]) *Suppose that the values of P are projective objects in $\mathbb{C}$. Consider the following commutative diagram*



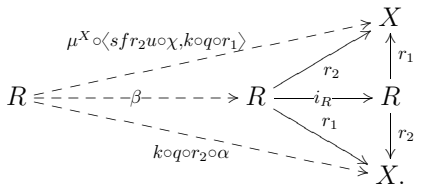
Note that, by the commutativity of limits, κ is the kernel of φ. If the top two rows are intrinsic Schreier split extensions and g and h are regular epimorphisms, then the bottom row is also an intrinsic Schreier split extension.

Proof: $\mathbb{C}$ is an $\mathscr{S}$-protomodular category (Example 5.11), thus it is an $\mathscr{S}$-Mal'tsev category [11, Theorem 5.4]. By Proposition 3.2 in [11], γ is a regular epimorphism.

and

$$\begin{array}{rcl} & & r_2 \circ \mu^R \circ \langle e_R k \circ q \circ r_2 \circ \alpha, s'f'u \circ \chi \rangle \\ & \overset{(14)}{=} & \mu^X \circ (r_2 \times r_2) \circ \langle e_R k \circ q \circ r_2 \circ \alpha, s'f'u \circ \chi \rangle \\ & \overset{(18),(20)}{=} & \mu^X \circ \langle k \circ q \circ \mu^X \circ \langle sfr_2u \circ \chi, k \circ q \circ r_1 \rangle, sfr_2u \circ \chi \rangle \\ & = & \mu^X \circ \langle k \circ q \circ \mu^X \circ (s \times k) \circ \langle fr_2u \circ \chi, q \circ r_1 \rangle, sfr_2u \circ \chi \rangle \\ & = & \mu^X \circ \langle k \circ q \circ \mu^X \circ (s \times k) \circ \langle fr_2u \circ \chi, q \circ r_1 \rangle, s\pi_Y \circ \langle fr_2u \circ \chi, q \circ r_1 \rangle \rangle \\ & = & \mu^X \circ \langle k \circ q \circ \mu^X \circ (s \times k), \overline{s\pi_Y} \rangle \circ \langle fr_2u \circ \chi, q \circ r_1 \rangle \\ & \overset{\textbf{(iS6)}}{=} & \mu^X \circ (s \times k) \circ \langle fr_2u \circ \chi, q \circ r_1 \rangle \\ & = & \mu^X \circ \langle sfr_2u \circ \chi, k \circ q \circ r_1 \rangle; \end{array}$$

thus



(21)

Third, we consider the imaginary morphism $\gamma = \mu^R \circ \langle uk'' \circ q'' \circ \chi, e_R k \circ q \circ r_2 \circ \alpha \rangle\colon R \dashrightarrow R$. We have

$$\begin{array}{rcl} & & r_1 \circ \mu^R \circ \langle uk'' \circ q'' \circ \chi, e_R k \circ q \circ r_2 \circ \alpha \rangle \\ & \overset{(14)}{=} & \mu^X \circ (r_1 \times r_1) \circ \langle uk'' \circ q'' \circ \chi, e_R k \circ q \circ r_2 \circ \alpha \rangle \\ & \overset{(18)}{=} & \mu^X \circ \langle 0, k \circ q \circ r_2 \circ \alpha \rangle \\ & = & \mu^X \circ \langle 0, 1_X \rangle \circ k \circ q \circ r_2 \circ \alpha \\ & \overset{(13)}{=} & k \circ q \circ r_2 \circ \alpha \end{array}$$

First, we consider the imaginary morphism $\alpha = \mu^R \circ \langle s'f'u \circ \chi, e_R k \circ q \circ r_1 \rangle \colon R \dashrightarrow R$. We have

$$\begin{array}{rl} & r_1 \circ \mu^R \circ \langle s'f'u \circ \chi, e_R k \circ q \circ r_1 \rangle \\ \overset{(14)}{=} & \mu^X \circ (r_1 \times r_1) \circ \langle s'f'u \circ \chi, e_R k \circ q \circ r_1 \rangle \\ \overset{(18)}{=} & \mu^X \circ \langle 0, k \circ q \circ r_1 \rangle \\ = & \mu^X \circ \langle 0, 1_X \rangle \circ k \circ q \circ r_1 \\ \overset{(13)}{=} & k \circ q \circ r_1 \end{array}$$

and

$$\begin{array}{rl} & r_2 \circ \mu^R \circ \langle s'f'u \circ \chi, e_R k \circ q \circ r_1 \rangle \\ \overset{(14)}{=} & \mu^X \circ (r_2 \times r_2) \circ \langle s'f'u \circ \chi, e_R k \circ q \circ r_1 \rangle \\ \overset{(18)}{=} & \mu^X \circ \langle sfr_2u \circ \chi, k \circ q \circ r_1 \rangle; \end{array}$$

thus

$$\begin{array}{ccc} & & X \\ & {\scriptstyle k\circ q\circ r_1} \nearrow & \uparrow {\scriptstyle r_1} \\ R & \overset{\alpha}{\dashrightarrow} & R \\ & {\scriptstyle \mu^X\circ\langle sfr_2u\circ\chi, k\circ q\circ r_1\rangle} \searrow & \downarrow {\scriptstyle r_2} \\ & & X. \end{array} \tag{20}$$

Second, we consider the imaginary morphism $\beta = \mu^R \circ \langle e_R k \circ q \circ r_2 \circ \alpha, s'f'u \circ \chi \rangle \colon R \dashrightarrow R$. We have

$$\begin{array}{rl} & r_1 \circ \mu^R \circ \langle e_R k \circ q \circ r_2 \circ \alpha, s'f'u \circ \chi \rangle \\ \overset{(14)}{=} & \mu^X \circ (r_1 \times r_1) \circ \langle e_R k \circ q \circ r_2 \circ \alpha, s'f'u \circ \chi \rangle \\ \overset{(18)}{=} & \mu^X \circ \langle k \circ q \circ r_2 \circ \alpha, 0 \rangle \\ = & \mu^X \circ \langle 1_X, 0 \rangle \circ k \circ q \circ r_2 \circ \alpha \\ \overset{(12)}{=} & k \circ q \circ r_2 \circ \alpha \end{array}$$

Since $P(X)$ is a projective object and g is a regular epimorphism, then it admits an imaginary splitting $t \colon X \dashrightarrow X'$, with $g \circ t = \overline{1_X}$ (see Remark 2.7). We define the imaginary retraction for the bottom row as $q = \gamma \circ q' \circ t \colon X \dashrightarrow K$. We must prove **(iS1)**

$$\begin{array}{rcl} \mu^X \circ \langle k \circ q, \overline{sf} \rangle & = & \mu^X \circ \langle k\gamma \circ q' \circ t, \overline{sf} \rangle \\ & = & \mu^X \circ \langle gk' \circ q' \circ t, \overline{sf} \circ g \circ t \rangle \\ & = & \mu^X \circ \langle gk' \circ q' \circ t, g \circ \overline{s'f'} \circ t \rangle \\ & = & \mu^X \circ (g \times g) \circ \langle k' \circ q', \overline{s'f'} \rangle \circ t \\ & \overset{(14)}{=} & g \circ \mu^{X'} \circ \langle k' \circ q', \overline{s'f'} \rangle \circ t \\ & = & g \circ \overline{1_{X'}} \circ t \\ & = & \overline{1_X}, \end{array}$$

where we use **(iS1)** applied to the second row in the next to last equality.

For **(iS2)**, we precompose the equality we wish to prove with the regular epimorphism $\gamma \times h$

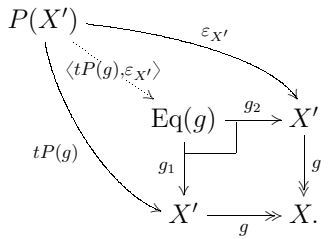
$$\begin{array}{rcl} q \circ \mu^X \circ (k \times s)(\gamma \times h) & = & \gamma \circ q' \circ t \circ \mu^X \circ ((k\gamma) \times (sh)) \\ & = & \gamma \circ q' \circ t \circ \mu^X \circ ((gk') \times (gs')) \\ & = & \gamma \circ q' \circ t \circ \mu^X \circ (g \times g)(k' \times s') \\ & \overset{(14)}{=} & \gamma \circ q' \circ t \circ g \circ \mu^{X'} \circ (k' \times s') \\ & \overset{(*)}{=} & \gamma \circ q' \circ \mu^{X'} \circ (k' \times s') \\ & = & \gamma \circ \overline{\pi_{K'}} \\ & = & \overline{\pi_K} \circ (\gamma \times h), \end{array}$$

where we use **(iS2)** for the second row in the next to last equality. Then **(iS2)** follows from Lemma 7.3.

To finish, we just need to prove the equality $\gamma \circ q' \circ t \circ g \overset{(*)}{=} \gamma \circ q'$. Actually, we prove this equality in $\mathbb{C}$ (not in $\mathbb{K}$) and to do so, we use the compatibility of the first two rows with respect to the imaginary retractions [20, Proposition 5.7]: $\gamma_i \rho = q' P(g_i)$, for $i \in \{1, 2\}$. We have

$$\begin{array}{rcl} \gamma q' P(t) \delta_X P(g) & \overset{(2)}{=} & \gamma q' P(t) P^2(g) \delta_{X'} \\ & = & \gamma q' P(tP(g)) \delta_{X'} \\ & = & \gamma q' P(g_1 \langle tP(g), \varepsilon_{X'} \rangle) \delta_{X'}, \end{array}$$

where $\langle tP(g), \varepsilon_{X'}\rangle$ is the unique morphism making the following diagram commutative

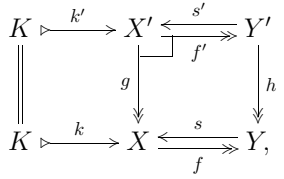


Using the compatibility mentioned earlier,

$$\begin{aligned}
\gamma q' P(t)\delta_X P(g) &= \gamma q' P(g_1) P(\langle tP(g), \varepsilon_{X'}\rangle)\delta_{X'} \\
&= \gamma\gamma_1\rho P(\langle tP(g), \varepsilon_{X'}\rangle)\delta_{X'} \\
&= \gamma\gamma_2\rho P(\langle tP(g), \varepsilon_{X'}\rangle)\delta_{X'} \\
&= \gamma q' P(g_2) P(\langle tP(g), \varepsilon_{X'}\rangle)\delta_{X'} \\
&= \gamma q' P(\varepsilon_{X'})\delta_{X'} \\
&\overset{(3)}{=} \gamma q'.
\end{aligned}$$

■

Corollary 7.5. (See [8, Corollary 2.3.6]) *Suppose that the values of P are projective objects in $\mathbb{C}$. Consider the diagram*

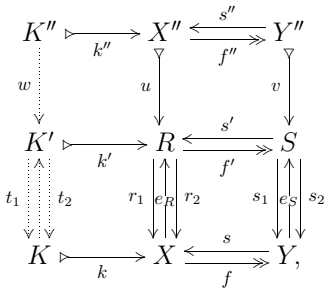


where the three squares involving, respectively, the split epimorphism, the kernels, and the sections commute. If the top row is an intrinsic Schreier split extension, then so is the bottom row.

Proof: Take the kernel pairs of the regular epimorphisms 1_K, g and h. This gives a 3×3 diagram whose top row is an intrinsic Schreier split extension, since these extensions are closed under arbitrary pullbacks (see [20, Proposition 6.1]). Applying Proposition 7.4 to this 3×3 diagram, we conclude that (f, s) is an intrinsic Schreier split extension. ■

In order to get the validity, in our context, of one of the classical homological lemmas, namely the 3×3-Lemma, we need another stability property of the class of intrinsic Schreier split epimorphisms. This property was called **equi-consistency** in [11]:

Definition 7.6. [11, Definition 6.3] Let $\mathscr{S}$ be a pullback-stable class of points. Consider any commutative diagram



(18)

where $\langle r_1, r_2\rangle\colon R \rightarrowtail X \times X$ and $\langle s_1, s_2\rangle\colon S \rightarrowtail Y \times Y$ are equivalence relations, (f, s) and (f', s') are split epimorphisms, (f'', s'') is the induced split epimorphism between the kernels of r_1 and s_1, and the diagram is completed by taking kernels and the induced dotted morphisms. $\mathscr{S}$ is **equi-consistent** if, whenever the points (f, s), (r_1, e_R) and (f'', s'') belong to $\mathscr{S}$, (f', s') is in $\mathscr{S}$, too.

Proposition 7.7. (See [11, Proposition 6.4]) *Suppose that the natural addition $(\mu^X\colon X \times X \dashrightarrow X)_{X\in\mathbb{C}}$ is associative. Then the class of intrinsic Schreier split extensions is equi-consistent.*

Proof: Consider the commutative diagram (18). Suppose that (f, s), (r_1, e_R) and (f'', s'') are intrinsic Schreier split epimorphisms, and denote by q, q'' and χ the imaginary retractions for (f, s), (f'', s'') and (r_1, e_R), respectively. In particular, we have

$$t_i w \circ q'' = q \circ r_i u,\ i \in \{1, 2\}. \tag{19}$$