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SOME GENERAL ASPECTS OF EXACTNESS AND STRONG EXACTNESS OF MEETS

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Dedicated to the memory of Hans-Peter Künzi

Abstract: Exact meets in a distributive lattice are the meets $\bigwedge_i a_i$ such that for all b, $(\bigwedge_i a_i) \vee b = \bigwedge_i (a_i \vee b)$, strongly exact meets in a frame are preserved by all frame homomorphisms. Finite meets are, trivially, (strongly) exact; this naturally leads to the concepts of exact resp. strongly exact filters closed under all exact resp. strongly exact meets. In [2, 12] it was shown that the subsets of all exacts resp. strongly exact filters are sublocales of the frame of upsets on a frame, hence frames themselves, and, somewhat surprisingly, that they are isomorphic to the useful frame $\mathsf{S}_\mathfrak{c}(L)$ of sublocales join-generated by closed sublocales resp. the dual of the coframe meet-generated by open sublocales.

In this paper we show that these are special instances of much more general facts. The latter concerns the free extension of join-semilattices to coframes; each coframe homomorphism lifting a general join-homomorphism $\varphi \colon S \to C$ (where S is a join-semilattice and C a coframe) and the associated (adjoint) colocalic maps create a frame of generalized strongly exact filters (φ-precise filters) and a closure operator on C (and – a minor point – any closure operator on C is thus obtained). The former case is slightly more involved: here we have an extension of the concept of exactness (ψ-exactness) connected with the lifts of $\psi \colon S \to C$ with complemented values in more general distributive complete lattices C creating, again, frames of ψ-exact filters; as an application we learn that if such a C is join-generated (resp. meet-generated) by its complemented elements then it is a frame (resp. coframe) explaining, e.g., the coframe character of the lattice of sublocales, and the (seemingly paradoxical) embedding of the frame $\mathsf{S}_\mathfrak{c}(L)$ into it.

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Introduction

Exact meets in a distributive lattice L are the meets $\bigwedge_i a_i$ such that for all $b \in L$, $(\bigwedge_i a_i) \vee b = \bigwedge_i (a_i \vee b)$. This concept appeared first in Mac Neille [10] and was shown to be useful for instance in the study of injective hulls in [6], or of Cauchy lattices in [1]; see also [3]. In frames L one has a stronger concept of a *strongly exact meet* $\bigwedge_i a_i$ ([15], see also [11] or [3]) characterized by stability under all frame homomorphisms $h\colon L \to M$, that is, for every such h, $h(\bigwedge a_i) = \bigwedge h(a_i)$ (for another characterization see 1.6 below).

The obvious fact that in particular finite meets are (strongly) exact naturally leads to the concepts of an *exact filter* resp. of a *strongly exact filter*, a filter closed under all exact resp. strongly exact meets. The subsets $\mathrm{Filt}_{\mathsf{Ex}}$ resp. $\mathrm{Filt}_{\mathsf{sEx}}$ of all exact resp. strongly exact filters turn out to be sublocales (natural subobjects) of the frame $\mathfrak{U}(L)$ of all up-sets of L, in fact quite interesting ones (see [2, 12]). To explain a fairly surprising feature of thus obtained frames we have to recall a construction from another nook of frame theory.

The system $\mathsf{S}(L)$ of all sublocales of a frame L is (in the natural order of inclusion) a coframe. The subset $\mathsf{S}_{\mathfrak{c}}(L)$ of this coframe join-generated by closed sublocales constitutes a frame (which might be somehow surprising, but has a natural explanation). The frame $\mathsf{S}_{\mathfrak{c}}(L)$ is of some interest: e.g., for a large class of frames it constitutes a Boolean envelope (playing the role of the discrete cover), it is a maximal essential extension, or pick up for T_1-spaces the subspace-induced sublocales among the general ones (see [4, 14, 5]). Although its motivation and origin has not even a remote connection with the concepts of exactness, it turned out recently ([2]) that it is naturally isomorphic with $\mathrm{Filt}_{\mathsf{Ex}}$. Similarly, it was proved in [12] that $\mathrm{Filt}_{\mathsf{sEx}}$ is naturally isomorphic with the dual of $\mathsf{S}_{\mathfrak{o}}(L)$, the subcolocale of $\mathsf{S}(L)$ meet-generated by open sublocales.

In this paper we show that these two isomorphisms are part of a fairly general phenomena.

The strong exactness case: The coframes $\mathfrak{U}(S)$ together with the natural maps $\gamma_S = (a \to \uparrow a)\colon S \to \mathfrak{U}(S)^{\mathrm{op}}$ constitute a free extension of the category of join-semilattices into that of coframes; that is, for any coframe C and any sublocale homomorphism $\varphi\colon S \to C$ we have a coframe homomorphism $\vec{\varphi}\colon \mathfrak{U}(S)^{\mathrm{op}} \to C$ such that $\vec{\varphi} \cdot \gamma = \varphi$. Each coframe homomorphism is a right

Galois adjoint; the kernel of the adjunction associated with $\vec{\varphi}$ is a sublocale of $\mathfrak{U}(L)$, and there is a natural concept of strong exactness connected with φ, the φ-precision*, such that this sublocale is constituted by φ-precise filters. The fact above is a special case for the φ associating with an element a of a frame L the respective open sublocale $\mathfrak{o}(a) \in \mathsf{S}(L)$.

The exactness is treated with less general ψ's (only those with complemented $\psi(a)$ seem to make enough sense) and more general targets (distributive complete lattices). A $\psi\colon S \to C$ now naturally extends to an adjunction

$$\begin{tikzcd} \mathfrak{U}(S) \arrow[r, bend left, "\psi_l"] \arrow[r, phantom, "\perp"] & C, \arrow[l, bend left, "\psi_r"] \end{tikzcd}$$

and we have a concept of ψ-exactness identifying $\psi_r\psi_l[\mathfrak{U}(S)]$ as the frame of ψ-exact filters. The fact above is a special case for the ψ associating with an element a of a frame L the respective closed sublocale $\mathfrak{c}(a) \in \mathsf{S}(L)$.

The two cases are linked by pairing the φ from the former with $\psi = \varphi^{\mathrm{op}}$ in the latter (provided the values of φ are complemented).

Also the right hand sides of the adjunction isomorphisms may be of some interest. In the former case we get some information on closure-type operators on the frame C. In the latter case we learn, in particular, that a distributive complete lattice join-generated by complemented elements is necessarily a frame, and if it is meet-generated by complemented elements, it is a coframe (explaining for instance the seeming discrepancy of the *coframe* $\mathsf{S}(L)$ naturally containing the *frame* $\mathsf{S}_{\mathfrak{c}}(L)$).

The paper is organized as follows:

After necessary Preliminaries we discuss in Section 2 the property of φ-precision connected with the free co-frame extension of join semilattices; it turns out to be a generalization of the concept of strong exactness. Next, Section 3 is devoted to a similar notion of ψ-exactness, extending that of exactness. Here the generality range is somewhat different, but in a common context it is weaker than the precision. The facts of these two sections are associated with specific adjunctions. The right hand sides of the induced isomorphism are discussed in Section 4; in particular we mention a

*Strong exactness is indeed stronger than exactness, but it turns out that in the context of our investigation it is the simpler of the two and taking it as derived from the former can be misleading. Therefore we decided to coin a one-word term.

consequence of the previous results (which may be known, but we have not known it before), namely that a distributive complete lattice join-generated (resp. meet-generated) by complemented elements is always a frame (resp. coframe).

1. Preliminaries

1.1. We use the standard notation for meets (infima) and joins (suprema) in posets: $a \wedge b$, $\bigwedge A$ or $\bigwedge_{a\in A} a$, $a \vee b$, $\bigvee A$ or $\bigvee_{a\in A} a$.

The least resp. largest element (if it exists) will be denoted by 0 resp. 1. We write

$$\uparrow\! a \text{ for } \{x \mid x \geq a\} \quad \text{and} \quad \uparrow\! A = \{x \mid \exists\, a \in A,\ x \geq a\}.$$

The subsets $A \subseteq (X, \leq)$ such that $\uparrow\! A = A$ will be referred to as *up-sets*.

1.2. Monotone maps $f\colon X \to Y$ and $g\colon Y \to X$ between posets are (*Galois*) *adjoint, f to the left and g to the right* if

$$f(x) \leq y \iff x \leq g(y),$$

equivalently, if $fg \leq \mathrm{id}$ and $gf \geq \mathrm{id}$. It is standard that

(1) left adjoints preserve all existing suprema and right adjoints preserve all existing infima,

(2) and on the other hand, if X, Y are complete lattices then each $f\colon X \to Y$ preserving all suprema is a left adjoint, and each $g\colon Y \to X$ preserving all infima is a right adjoint.

1.3. A *frame* (*coframe*) is a complete lattice L satisfying the distributivity law

$$\big(\bigvee A\big) \wedge b = \bigvee\{a \wedge b \mid a \in A\} \tag{frm}$$

$$\big(\big(\bigwedge A\big) \vee b = \bigwedge\{a \vee b \mid a \in A\}\big) \tag{cofrm}$$

for all $A \subseteq L$ and $b \in L$. A *frame homomorphism* preserves all joins and all finite meets.

1.3.1. The rule (frm) makes by 1.2 each map $(-) \wedge b$ a left adjoint; consequently a frame has a *Heyting structure* with $\to$ satisfying

$$a \wedge b \leq c \quad \text{iff} \quad a \leq b \to c. \tag{hey}$$

Dually, a coframe has a *co-Heyting operator* $\setminus$ satisfying $c\setminus b \leq a$ iff $c \leq a \vee b$.

In particular, in a frame resp. coframe we have the *pseudocomplements* $x^* = x \to 0$ resp. *supplements* $x^\# = 1 \setminus x$ with the De Morgan laws

$$(\bigvee_i a_i)^* = \bigwedge_i a_i^* \quad \text{resp.} \quad (\bigwedge_i a_i)^\# = \bigvee_i a_i^\#.$$

Recall that a *complement* of an element x in a distributive lattice, that is, a y such that $x \vee y = 1$ and $x \wedge y = 0$, is both a pseudocomplement and a supplement. For this case we will also use the symbol x^*.

1.3.2. The following is a well-known (although seldom explicitly mentioned) fact.

Proposition. *Let b be a complemented element in a distributive lattice L. Then we have $(\bigvee A) \wedge b = \bigvee\{a \wedge b \mid a \in A\}$ and $(\bigwedge A) \vee b = \bigwedge\{a \vee b \mid a \in A\}$ for any $A \subseteq L$ for which the indicated join or meet exists.*

Proof immediately follows from the fact that $(-) \wedge b$ has a right adjoint $b^* \vee (-)$, and $(-) \vee b$ has a left adjoint $b^* \wedge (-)$. $\square$

1.3.3. A few Heyting rules. By the adjunction (hey) we have

$$a \to (\bigwedge_i b_i) = \bigwedge_i (a \to b_i) \quad \text{and} \quad (\bigvee_i a_i) = \bigwedge_i (a_i \to b).$$

Further, we ill use the following rules (we present them with proofs; in spite of their simplicity they are sometimes of surprising help):

(1) $a \to b = 1$ iff $a \leq b$ (since $a \to b = 1$ iff $1 \leq a \to b$).
(2) $b \leq a \to b$ (since $a \wedge b \leq b$).
(3) $a \wedge (a \to b) \leq b$ (since $a \to b \leq a \to b$).
(4) $a \wedge (a \to b) = a \wedge b$ ($\leq$ by (3) and $a \wedge b \leq a \wedge (a \to b)$ by(2)).
(5) $(b \vee a) \wedge (b \to a) = a$ (we have $(b \vee a) \wedge (b \to a) = (b \wedge (b \to a)) \vee (a \wedge (b \to a)) = (a \wedge b) \vee a$ by (4) and (2)).
(6) $a \to b = a \to c$ iff $a \wedge b = a \wedge c$ ($\Rightarrow$ since $a \wedge (a \to x) = a \wedge x$, $\Leftarrow$ since $a \to (a \wedge x) = (a \to a) \wedge (a \to x)$).

1.4. A typical frame is the lattice $\Omega(X)$ of open sets of a topological space X, and for continuous maps $f \colon X \to Y$ there are frame homomorphisms $\Omega(f) = (U \mapsto f^{-1}[U]) \colon \Omega(Y) \to \Omega(X)$. Thus we have a contravariant functor

$$\Omega \colon \mathbf{Top} \to \mathbf{Frm},$$

where **Top** is the category of topological spaces, and **Frm** that of frames. To make it covariant one considers the category of locales $\mathbf{Frm}^{\mathrm{op}}$, denoted **Loc**.

It is of advantage to work with **Loc** as with a concrete category representing its morphisms, the *localic maps* $f\colon M \to L$ opposite to a frame homomorphism $h\colon L \to M$ as the right Galois adjoints of h.

1.5. Sublocales. The extremal epimorphisms in **Frm** are precisely the onto frame homomorphisms, and hence, the extremal monomorphisms in **Loc** are the one-to-one localic maps. This leads to the following natural approach to subobjects of locales.

Sublocales of a frame L, that is, subsets $S \subseteq L$ such that the embedding maps $j\colon S \subseteq L$ are localic ones are characterized by the requirements that (see e.g. [13])

(S1) for every $M \subseteq S$ the meet $\bigwedge M$ lies in S, and

(S2) for every $s \in S$ and every $x \in L$, $x \to s$ lies in S.

The system $\mathsf{S}(L)$ of all sublocales of L, ordered by inclusion, is a complete lattice with a fairly transparent structure:

$$\bigwedge_i S_i = \bigcap_i S_i \quad \text{and} \quad \bigvee_i S_i = \{\bigwedge M \mid M \subseteq \bigcup_i S_i\}.$$

Note that a sublocale S of a frame L is a frame itself, with joins typically distinct from those in L. The least sublocale $\bigvee \emptyset = \{1\}$ is designated by O and referred to as the *void sublocale*. It is a fundamental fact that

the lattice $\mathsf{S}(L)$ *is a coframe.*

1.5.1. Open and closed sublocales. With each element $a \in L$ there is associated an *open sublocale*

$$\mathfrak{o}(a) = \{x \mid x = a \to x\} = \{a \to x \mid x \in L\}$$

and a *closed sublocale*

$$\mathfrak{c}(a) = {\uparrow} a.$$

In case of a space X (represented as the frame $\Omega(X)$) they precisely correspond to the open and closed subspaces (and to the *open* and *closed parts* from the pioneering article [8]).

One has the following identities:

$$\mathfrak{o}(0) = \mathsf{O},\ \mathfrak{o}(1) = L,\ \mathfrak{o}(a \wedge b) = \mathfrak{o}(a) \cap \mathfrak{o}(b) \text{ and } \mathfrak{o}(\bigvee_i a_i) = \bigvee_i \mathfrak{o}(a_i),$$

$$\mathfrak{c}(0) = L,\ \mathfrak{c}(1) = \mathsf{O},\ \mathfrak{c}(a \wedge b) = \mathfrak{c}(a) \vee \mathfrak{c}(b) \text{ and } \mathfrak{c}(\bigvee_i a_i) = \bigcap_i \mathfrak{c}(a_i).$$

Thus in particular one has a frame embedding $\mathfrak{c} = (a \mapsto \mathfrak{c}(a))\colon L \to \mathsf{S}(L)^{\mathrm{op}}$.

1.5.2. Nuclei. Sublocales $S \subseteq L$ are associated with the so called *nuclei* $\nu \colon L \to L$, monotone mappings satisfying $a \leq \nu(a)$, $\nu\nu(a) = \nu(a)$ and $\nu(a \wedge b) = \nu(a) \wedge \nu(b)$. The one-to-one correspondence is given by

$$S \mapsto \nu_S(a) = \bigwedge\{s \in S \mid a \leq s\} \quad \text{and} \quad \nu \mapsto S_\nu = \nu[L].$$

One has

Proposition. *For every $s \in S$ and every $x \in L$,*

$$x \to s = \nu_S(x) \to s.$$

Proof: $y \leq x \to s$ iff $x \leq y \to s$ iff $\nu(x) \leq y \to s$ iff $y \leq \nu(x) \to s$. ∎

For more about frames and locales see, e.g., [13] or [9].

1.6. Exact and strongly exact meets. A meet $\bigwedge A$ in a frame L is said to be *exact* (see e.g. [10, 6, 1, 3, 2]) if for every $b \in L$,

$$(\bigwedge A) \vee b = \bigwedge\{a \vee b \mid a \in A\}.$$

This is well-known to be equivalent with the assumption that

the sublocale $\bigvee_{a\in A} \mathfrak{c}(a)$ is closed

(for a short proof see 3.7.2 - 3.7.3 below).

A meet $\bigwedge A$ is said to be *strongly exact* ([15], see also [11, 3]) if

the sublocale $\bigwedge_{a\in A} \mathfrak{o}(a)$ is open.

1.6.1. Observation. *Each strongly exact meet is exact.*

(By the coframe De Morgan law in $\mathsf{S}(L)$, if $\bigcap_{a\in A} \mathfrak{o}(a) = \mathfrak{o}(b)$ then $\bigvee_{a\in A} \mathfrak{c}(a) = \bigvee_{a\in A} \mathfrak{o}(a)^{\#} = (\bigcap_{a\in A} \mathfrak{o}(a))^{\#} = \mathfrak{o}(b)^{\#} = \mathfrak{c}(b)$.)

1.6.2. Notes. 1. An interesting characterization of strong exactness (which we will not need here; for a proof see e.g. [3]) is that the strongly exact meets $\bigwedge A$ in L are precisely those for which $h(\bigwedge A) = \bigwedge_{a\in A} h(a)$ for every frame homomorphism $h : L \to M$.

2. In the context of this paper it turned out that viewing strong exactness as a stronger modification of exactness is misleading: it is a property in its own right, in fact somewhat simpler than the weaker one. Therefore we will use in the sequel for the generalized concept the one-word term "precise".

2. φ-precision and coframe extension of semilattices

2.1. For a join-semilattice S consider the up-set frame

$$\mathfrak{U}(S) = \{A \mid A \subseteq S, A = \uparrow A\}.^{\dagger}$$

2.1.1. Proposition. *The maps*

$$\gamma_S = (a \mapsto \uparrow a) \colon S \to \mathfrak{U}(S)^{op}$$

constitute a free extension of $\vee$**SLat** *in* **Cofrm**. *The unique coframe homomorphism* $\overrightarrow{\varphi}$ *lifting a* $\vee$**SLat**-*morphism* $\varphi \colon S \to C$ *in*

$$\begin{tikzcd} \mathfrak{U}(S)^{op} \arrow[drr, "\overrightarrow{\varphi}"] & & \\ S \arrow[u, "\gamma_S"] \arrow[rr, "\varphi"'] & & C \end{tikzcd}$$

is defined by $\overrightarrow{\varphi}(A) = \bigwedge\{\varphi(a) \mid a \in A\}$.

Proof: $A = \bigcup\{\uparrow a \mid a \in A\} = \bigcup\{\gamma(a) \mid a \in A\}$ which is the meet in $\mathfrak{U}(S)^{\mathrm{op}}$ and hence the $\overrightarrow{\varphi}$ is uniquely determined by the formula mentioned. Obviously, $\overrightarrow{\varphi}$ preserves all meets. For joins we have

$$\begin{aligned} \overrightarrow{\varphi}(A) \vee \overrightarrow{\varphi}(B) &= \bigwedge\{\varphi(a) \mid a \in A\} \vee \bigwedge\{\varphi(b) \mid b \in B\} = \\ &= \bigwedge\{\varphi(a) \vee \varphi(b) \mid a \in A, b \in B\} = \bigwedge\{\varphi(a \vee b) \mid a \in A, b \in B\} \geq \\ &\geq \bigwedge\{\varphi(c) \mid c \in A \cap B\} = \overrightarrow{\varphi}(A \vee B) \geq \overrightarrow{\varphi}(A) \vee \overrightarrow{\varphi}(B) \end{aligned}$$

(since $A \vee B = A \cap B$ in $\mathfrak{U}(S)^{\mathrm{op}}$). ■

2.1.2. Observation. *The colocalic map* $\overleftarrow{\varphi}$ *associated with the frame homomorphism* $\overrightarrow{\varphi}$ *is defined by*

$$\overleftarrow{\varphi}(c) = \{a \mid c \leq \varphi(a)\}.$$

(We have $A \subseteq \{a \mid c \leq \varphi(a)\}$ iff $c \leq \bigwedge\{\varphi(a) \mid a \in A\}$, and the adjoint is uniquely determined.)

[†]For join-semilattices with top it is of a technical advantage to consider $\mathfrak{U}(A)$ as the set of all *non-empty* upsets of S. Here the distinction is of no importance.

2.2. φ-precise sets and φ-precise filters. Let S be a join-semilattice and let $\varphi\colon S \to C$ be a join-homomorphism with C a coframe.

A subset $M \subseteq S$ is said to be *φ-precise* if $\bigwedge_{m\in M}\varphi(m) = \varphi(a)$ for some a.

An up-set $A \in \mathfrak{U}(S)$ is said to be a *φ-precise filter* if for every φ-precise $M \subseteq A$ every a such that $\bigwedge_{m\in M}\varphi(m) = \varphi(a)$ is in A.

2.2.1. Lemma. *Each $\overleftarrow{\varphi}\overrightarrow{\varphi}(A)$ is a φ-precise filter.*

Proof: Let $M \subseteq \overleftarrow{\varphi}\overrightarrow{\varphi}(A)$. Then for each $m \in M$, $\varphi(m) \geq \bigwedge_{a\in A}\varphi(a)$; hence $\bigwedge_{m\in M}\varphi(m) \geq \bigwedge_{a\in A}\varphi(a)$. Now if M is φ-precise and $\bigwedge_{m\in M}\varphi(m) = \varphi(b)$ we obtain $\varphi(b) \geq \bigwedge_{a\in A}\varphi(a)$, that is, $b \in \overleftarrow{\varphi}\overrightarrow{\varphi}(A)$. ■

2.2.2. Theorem. $A = \overleftarrow{\varphi}\overrightarrow{\varphi}(A)$ *iff A is a φ-precise filter.*

Proof: In view of 2.1 and the adjunction inclusion it suffices to prove that for each φ-precise filter A, $\overleftarrow{\varphi}\overrightarrow{\varphi}(A) \subseteq A$. Thus let $b \in \overleftarrow{\varphi}\overrightarrow{\varphi}(A)$. Then $\varphi(b) \geq \bigwedge_{a\in A}\varphi(a)$ and hence

$$\varphi(b) = \Big(\bigwedge_{a\in A}\varphi(a)\Big) \vee \varphi(b) = \bigwedge_{a\in A}(\varphi(a)\vee\varphi(b)) = \bigwedge_{a\in A}\varphi(a\vee b).$$

Thus, $M = \{a \vee b \mid a \in A\}$ is φ-precise. We have $M \subseteq A$ and $\bigwedge_{m\in M}\varphi(m) = \varphi(b)$ and since A is a φ-precise filter, $b \in A$. ■

2.3. The structure of $\overleftarrow{\varphi}\overrightarrow{\varphi}[\mathfrak{U}(A)]$. The system

$$\mathrm{pFilt}_\varphi$$

of all φ-precise filters will be now considered as a subset of the frame $\mathfrak{U}(S)$. Thus we will have a (frame homomorphism-localic map) pair

$$\begin{tikzcd} \mathfrak{U}(S) \arrow[r, bend left, "\overrightarrow{\varphi}"] & C^{\mathrm{op}} \arrow[l, bend left, "\overleftarrow{\varphi}"] \end{tikzcd}$$

2.3.1. Observations. *Let $h\colon L \to M$ be a frame homomorphism and let $f\colon M \to L$ be the associated localic map. Then*

1. $f[M] = fh[L]$ *and*
2. fh *is a nucleus.*

(1: $\supseteq$ is trivial and $\subseteq$ follows from $f[M] = fhf[M]$. For 2, $a \subseteq fh(a)$ and $(fh)(fh) = fh$ follows from the adjunction, and $fh(a\wedge b) = fh(a)\wedge fh(b)$ since h is a homomorphism and f a right adjoint.)

From any of the two observations one immediately obtains

2.3.2. Theorem. *The set* pFilt_φ *of all the* φ*-precise filters is a sublocale of* $\mathfrak{U}(S)$.

2.4. The case of an embedding φ, in particular $\varphi = \mathfrak{o}\colon L \to \mathsf{S}(L)$. The special case of (order) embeddings φ, that is, φ such that

$$\varphi(a) \le \varphi(b) \quad \text{iff} \quad a \le b$$

are of special interest.

2.4.1. Proposition. *Let* φ *be an embedding. Then a subset* $M \subseteq S$ *is* φ*-precise iff* $\bigwedge M$ *exists and* $\varphi(\bigwedge M) = \bigwedge_{m\in M} \varphi(m)$.

Proof: $\Leftarrow$ is obvious. Now let M be φ-precise, and let $\bigwedge_{m\in M} \varphi(m) = \varphi(a)$. Then for all $m \in M$, $\varphi(a) \le \varphi(m)$ and hence $a \le m$, and if $x \le m$ for all $m \in M$ then $\varphi(x) \le \varphi(m)$ for all $m \in M$ and hence $\varphi(x) \le \varphi(a)$, and finally $x \le a$. ■

2.4.2. Proposition. *Let* φ *be an embedding and let a subset* $M \subseteq S$ *be* φ*-precise. Then for every* b,

$$\bigwedge M \vee b = \bigwedge_{m\in M} (m \vee b).$$

Proof: Using the coframe distributivity we obtain

$$\begin{aligned}\bigwedge_{m\in M} \varphi(m \vee b) &= \bigwedge_{m\in M} (\varphi(m) \vee \varphi(b)) = \\ &= (\bigwedge_{m\in M} \varphi(m)) \vee \varphi(b) = \varphi(\bigwedge M) \vee \varphi(b) = \varphi(\bigwedge M \vee b)\end{aligned} \qquad (*)$$

and hence, first $\{m \vee b \mid m \in M\}$ is φ-precise. Next, by 2.4.1,

$$\bigwedge_{m\in M} \varphi(m \vee b) = \varphi(\bigwedge_{m\in M} (m \vee b)),$$

hence $\varphi(\bigwedge M \vee b) = \varphi(\bigwedge_{m\in M}(m \vee b))$ and finally $\bigwedge M \vee b = \bigwedge_{m\in M}(m \vee b)$. ■

2.4.3. Remark. Note that the reasoning in $(*)$ did not need the embedding property. Thus we have that

> *for any join-semilattice morphism* φ, *if* M *is* φ*-precise then so is every* $\{m \vee b \mid m \in M\}$.

2.4.4. Open sublocales and strong exactness. Consider the embedding

$$\varphi = \mathfrak{o} = (a \mapsto \mathfrak{o}(a)) \colon L \to \mathsf{S}(L).$$

Then a set $M \subseteq L$ is φ-precise iff the meet $\bigwedge M$ is strongly exact, and strongly exact filters are the upsets closed under strongly exact meets and, since $\overrightarrow{\varphi}(A) = \bigcap\{\mathfrak{o}(a) \mid a \in A\}$ and $\overleftarrow{\varphi}(S) = \{a \mid S \subseteq \mathfrak{o}(a)\}$, we have learned that (in accordance to the result of [12]) the strongly exact filters are those up-sets A for which $\overleftarrow{\varphi}\overrightarrow{\varphi}(A) = \{b \mid \bigcap\{\mathfrak{o}(a) \mid a \in A\} \subseteq \mathfrak{o}(b)\}$ and that the strongly exact filters constitute a sublocale of $\mathfrak{U}(L)$.

3. ψ-exactness in particular φ^{op}-exactness and exactness

3.1. For technical reasons, but also for easier comparison with previous facts, we will change the nature and generality of the determining morphism. We had join-homomorphisms $\varphi \colon S \to C$ with S a join-semilattice and C a coframe. Now we will consider meet-homomorphisms

$$\psi \colon S^{\mathrm{op}} \to C$$

where S is a join-semilattice again, C (more generally) a distributive complete lattice, such that

each $\psi(a)$ is complemented.

3.1.1. Notes. 1. Hence (recall 1.3.2)

each $\psi(a)$ distributes over meets and joins.

2. If a φ from the previous section happens to be such that every $\varphi(a)$ is complemented then in particular the

$$\varphi^{\mathrm{op}} = (a \mapsto \varphi(a)^*) \colon S^{\mathrm{op}} \to C$$

qualifies as a ψ above.

3.2. ψ-exact sets and ψ-exact filters. A subset $M \subseteq S$ is said to be *ψ-exact* if $\bigvee_{m\in M} \psi(m) = \psi(a)$ for some a.

An up-set $A \in \mathfrak{U}(S)$ is said to be a *ψ-exact filter* if for every ψ-exact $M \subseteq A$ every a such that $\bigvee_{m\in M} \psi(m) = \psi(a)$ is in A.

3.3. An adjunction. Define $\psi_l\colon \mathfrak{U}(S) \to C$ and $\psi_r\colon C \to \mathfrak{U}(S)$ by setting

$$\psi_l(A) = \bigvee\{\psi(a) \mid a \in A\},$$
$$\psi_r(c) = \{b \mid \psi(b) \le c\}.$$

Obviously they are adjoint maps, ψ_l to the left, ψ_r to the right.

3.3.1. Note. Again we have here a free lifting, this time to the category of (distributive) $\bigvee$-lattices,

$$\begin{tikzcd} \mathfrak{U}(S) \arrow[dr, "\psi_l"] & \\ S^{\mathrm{op}} \arrow[u, "\gamma_S"] \arrow[r, "\psi"'] & C \end{tikzcd}$$

with γ_S now understood as $S^{\mathrm{op}} \to \mathfrak{U}(S)$, and again the ψ_r is the morphism associated with ψ_l analogously as the co-localic map with the coframe homomorphisms (now in representing the dual of the category of $\bigvee$-lattices). But we do not see so far any use for this, the ψ being rather special in the context.

3.4. We have a characteristic of ψ-exact filters quite analogous to that of φ-precise filters in 2.2. But keep in mind that while the Lemma is just the 2.2.1 repeated, the Theorem is based on 1.3.2, not on coframe distributivity.

3.4.1. Lemma. *Each $\psi_r\psi_l(A)$ is a ψ-exact filter.*

Proof: Let $M \subseteq \psi_r\psi_l(A) = \{b \mid \psi(b) \le \bigvee\{\psi(a) \mid a \in A\}\}$. Then for each $m \in M$, $\psi(m) \le \bigvee_{a\in A}\psi(a)$; hence $\bigvee_{m\in M}\psi(m) \le \bigvee_{a\in A}\psi(a)$. Now if M is φ-exact and $\bigvee_{m\in M}\psi(m) = \psi(b)$ we obtain $\psi(b) \le \bigvee_{a\in A}\psi(a)$, that is, $b \in \psi_r\psi_l(A)$. ■

3.4.2. Theorem. *$A = \psi_r\psi_l(A)$ iff A is a ψ-exact filter.*

Proof: In view of 3.4.1 and the adjunction inclusion it suffices to prove that for each ψ-exact filter A, $\psi_r\psi_l(A) \subseteq A$. Thus let $b \in \psi_r\psi_l(A)$. Then $\psi(b) \le \bigvee_{a\in A}\psi(a)$ and hence

$$\psi(b) = (\bigvee_{a\in A}\psi(a)) \wedge \psi(b) = \bigvee_{a\in A}(\psi(a) \wedge \psi(b) = \bigvee_{a\in A}\psi(a \vee b).$$

Thus, $M = \{a \vee b \mid a \in A\}$ is ψ-exact. We have $M \subseteq A$ and $\bigvee_{m\in M}\psi(m) = \psi(b)$ and since A is a ψ-exact filter, $b \in A$. ■

3.5. Analogously with 2.3, the system

$$\mathrm{eFilt}_\psi$$

of all ψ-exact filters constitutes a sublocale of $\mathfrak{U}(L)$. Now, however, we do not have a (homomorphism, localic map) pair between $\mathfrak{U}(L)$ and a frame (C is only distributive) and hence we have not the immediate consequence of trivial observations like in 2.3.1.

3.5.1. It is easy to check that the Heyting operation in $\mathfrak{U}(S)$ is given by

$$B \to C = \{x \mid \forall b \in B,\ b \vee x \in C\}.$$

3.5.2. Lemma. *Let M be ψ-exact. Then for any b,*

$$M \vee b = \{m \vee b \mid m \in M\}$$

is ψ-exact with $\bigvee_{m\in M} \psi(m\vee b) = \psi(a\vee b)$ for each a such that $\bigvee_{m\in M} \psi(m) = \psi(a)$.

Proof: Let $\bigvee_{m\in M} \psi(m) = \psi(a)$. Then

$$\bigvee\{\psi(m \vee b) \mid m \in M\} = \bigvee\{\psi(m) \wedge \psi(b) \mid m \in M\} =$$
$$= (\bigvee\{\psi(m) \mid m \in M\}) \wedge \psi(b) = \psi(a) \wedge \psi(b) = \psi(a \vee b)$$

(note that unlike in 2.1 the distributivity was not that of a coframe: we had to use the complementarity of $\psi(b)$). ■

Note. Recall 2.4.3. Unlike there, we have (again) used 1.3.2, not coframe distributivity.

3.5.3. Theorem. *The set* eFilt_ψ *of all the ψ-exact filters is a sublocale of $\mathfrak{U}(S)$.*

Proof: Obviously it is closed under meets. If $M \subseteq B \to C$ then for each $m \in M$ and $b \in B$, $m \vee b \in C$, hence for each $b \in B$, $M \vee b \subseteq C$. If M is a ψ-exact set with $\bigvee_{m\in M} \psi(m) = \psi(a)$ then, by 3.5.2, $M \vee b$ is ψ-exact with $\bigvee_{m\in M} \psi(m\vee b) = \psi(a\vee b)$. Thus, if C is a ψ-exact filter then for each $b \in B$, $a \vee b \in C$ and hence $a \in B \to C$. ■

3.6. Observation. *Recall 3.1.1. Let $\varphi\colon S \to C$ be a join-homomorphism into a coframe such that each $\varphi(a)$ is complemented. Then*

- *each φ-precise set $M \subseteq S$ is φ^{op}-exact, and consequently*
- *each φ^{op}-exact filter is a φ-precise filter, so that*
- *the frame* $\mathrm{eFilt}_{\varphi^{op}}$ *is a sublocale of the frame* pFilt_φ.

(By the coframe De Morgan rule we have $\bigvee_{m\in M}\varphi(m)^{\#}=(\bigwedge_{m\in M}\varphi(m))^{\#}$ and hence if $\bigwedge_{m\in M}\varphi(m)=\varphi(a)$, $\bigvee_{m\in M}\varphi(m)^{\#}=\varphi(a)^{\#}=\varphi(a)^{*}=\varphi^{\mathrm{op}}(a)$.)

3.7. The case of an embedding ψ, in particular $\psi=\mathfrak{c}$.

3.7.1. Proposition. *Let ψ be an embedding. Then a subset $M\subseteq S$ is ψ-exact iff $\bigwedge M$ exists and $\psi(\bigwedge M)=\bigvee_{m\in M}\psi(m)$.*

Proof: $\Leftarrow$ is obvious. Now let M be ψ-precise, and let $\bigvee_{m\in M}\psi(m)=\psi(a)$. Then for all $m\in M$, $\psi(a)\geq\psi(m)$ and hence $a\leq m$. If $x\leq m$ for all $m\in M$ then $\psi(x)\geq\psi(m)$ for all $m\in M$ and hence $\psi(x)\geq\psi(a)$, and finally $x\leq a$. ■

3.7.2. Proposition. *Let ψ be an embedding and let a subset $M\subseteq S$ be ψ-exact. Then for every b,*

$$\bigwedge M\vee b=\bigwedge_{m\in M}(m\vee b).$$

Proof: If M is exact then by 3.5.2 also $M\vee b=\{m\vee b\mid m\in M\}$ is exact and we have

$$\psi(\bigwedge M)=\bigvee_{m\in M}\psi(m)\quad\text{and}\quad\psi(\bigwedge_{m\in M}(m\vee b))=\bigvee_{m\in M}\psi(m\vee b).$$

Thus, by 1.3.2,

$$\begin{aligned}\psi(\bigwedge M\vee b)&=\psi(\bigwedge M)\wedge\psi(b)=(\bigvee_{m\in M}\psi(m))\wedge\psi(b)=\\&=\bigvee_{m\in M}(\psi(m)\wedge\psi(b))=\bigvee_{m\in M}\psi(m\vee b)=\psi(\bigwedge_{m\in M}(m\vee b))\end{aligned}$$

and since ψ is one-to-one, $\bigwedge M\vee b=\bigwedge_{m\in M}(m\vee b)$. ■

3.7.3. Theorem. *Consider the mapping $\mathfrak{c}=(a\mapsto\mathfrak{c}(a))\colon L\to\mathsf{S}(L)$. Then a subset $M\subseteq L$ is $\mathfrak{c}$-exact iff $\bigwedge M$ is an exact meet.*

Proof: $\Rightarrow$ follows from 3.7.2.
$\Leftarrow$: Let $\bigwedge M$ be exact. We want to prove that $S=\bigvee_{m\in M}\mathfrak{c}(m)=\bigvee_{m\in M}{\uparrow}m$ is closed. Obviously, $\bigwedge M$ is the smallest element of S and $S\subseteq{\uparrow}\bigwedge M$. Let $b\geq\bigwedge M$. Then by exactness, $b=\bigwedge M\vee b=\bigwedge_{m\in M}(m\vee b)$, and since $m\vee b\in{\uparrow}m$, $b\in S$. ■

4. The right hand sides

4.1. A closure operator. Consider a homomorphism φ and the adjunction from 2.1.2. We have an isomorphism

$$\mathrm{pFilt}_\varphi \cong \overrightarrow{\varphi}\overleftarrow{\varphi}[C].$$

For $c \in C$ set $\widetilde{c} = \overrightarrow{\varphi}\overleftarrow{\varphi}(c)$.

4.1.1. Proposition. 1. *$c \mapsto \widetilde{c}$ is a closure-type operator (that is, $a \leq \widetilde{a}$, $\widetilde{\widetilde{a}} = \widetilde{a}$ and $\widetilde{a \vee b} = \widetilde{a} \vee \widetilde{b}$).*

2. *Every closure operator u on a coframe C is induced by a homomorphism as in 2.1.*

Proof: 1. $a \leq \widetilde{a}$, $\widetilde{\widetilde{a}} = \widetilde{a}$ follow immediately from the adjunction, and $\widetilde{a \vee b} = \widetilde{a} \vee \widetilde{b}$ since $\overrightarrow{\varphi}$ is a homomorphism and $\overleftarrow{\varphi}$ is a left adjoint.

2. Consider $S = \{a \mid a = u(a)\}$ and $\varphi = a \mapsto u(a)\colon S \to C$. Then $\overrightarrow{\varphi}\overleftarrow{\varphi}(a) = \bigwedge\{x = u(x) \mid a \leq x\} = u(a)$. ■

4.1.2. From 2.3.2 we immediately obtain

Corollary. *$\overrightarrow{\varphi}\overleftarrow{\varphi}[C] = \{c \mid c = \widetilde{c}\}$ is a coframe.*

4.1.3. Note. For $\varphi = \mathfrak{o}$ (recall 2.4), $\widetilde{S}$ is the fitting (the "other closure" from [7]).

4.2. Consider the adjunction from 3.3. Here we have

4.2.1. Proposition. *The $\psi_l\psi_r[S] \cong \mathrm{eFilt}_\psi$ is the system of all $\bigvee_{m\in M}\psi(m)$ with $M \subseteq S$. Consequently, by 3.5.3,*

$$\{\bigvee_{m\in M}\psi(m) \mid M \subseteq S\}$$

is a frame.

Proof: First, since $a \leq b$ implies $\psi(a) \geq \psi(b)$ we have

$$\bigvee_{m\in M}\psi(m) = \bigvee_{m\in\uparrow M}\psi(m),$$

hence it suffices to consider the joins indexed by up-sets $A \in \mathfrak{U}(S)$.

Next, for every $A \in \mathfrak{U}(A)$, $\bigvee_{a\in A}\psi(a) = \psi_l(A) = \psi_l\psi_r\psi_l(A) \in \psi_l\psi_r[C]$ and on the other hand each $c \in \psi_l\psi_r[C]$ is a join of $\psi(m)$'s. ■

4.3. From 4.2.1 we see that

if a distributive complete C is generated by the elements of the form $\psi(m)$ then it is a frame.

In particular, taking into account the map

$$(c \mapsto c)\colon \{c \mid c \text{ complemented}\} \to C$$

we obtain

4.3.1. Corollary. *If a distributive complete lattice C is join-generated by its complemented elements then it is a frame.*

Taking the opposite order we obtain

4.3.2. Corollary. *If a distributive complete lattice C is meet-generated by its complemented elements then it is a coframe.*

4.4. Corollary 4.3.2 suggests a new proof that $\mathsf{S}(L)$ is a coframe. It is a well known fact that it is meet generated by complemented elements (we will present a proof – and the same about the distributivity below – to show that the new observation is based on simple facts only).

4.4.1. Theorem. *In $\mathsf{S}(L)$, $S = \bigcap\{\mathfrak{c}(\nu_S(a)) \vee \mathfrak{o}(a) \mid a \in L\}$ for every sublocale $S \subseteq L$.*

Proof: $\subseteq$: If $s \in S$ then for each a,

$$s = (\nu_S(a) \vee s) \wedge (\nu_s(a) \to s) = (\nu_S(a) \vee s) \wedge (a \to s) \in \mathfrak{c}(\nu_S(a)) \vee \mathfrak{o}(a)$$

by 1.3.3(5) and 1.6.2.
$\supseteq$: If $x \in \bigcap\{\mathfrak{c}(\nu_S(a)) \vee \mathfrak{o}(a) \mid a \in L\}$ then in particular $x \in \mathfrak{c}(\nu_S(x)) \vee \mathfrak{o}(x)$. Hence $x = u \wedge v$ with $u \geq \nu_S(x)$ and $x \to v = v$. Then $x \leq v$ and $v = x \to v = 1$ so that $x = u \geq \nu_S(x)$, and $x \in S$. ∎

4.4.2. Proposition. *$\mathsf{S}(L)$ is distributive.*

Proof: $(A \vee B) \cap C \supseteq (A \cap C) \vee (B \cap C)$ is trivial.
$\subseteq$: Let $a \in A$, $b \in B$ and $a \wedge b \in C$. We have by 1.3.3(4)

$$(b \to a) \wedge ((b \to a) \to b) = b \wedge (b \to a) = b \wedge a$$

where

$$b \to a = (b \to a) \wedge (b \to b) = b \to (a \wedge b) \in A \cap B$$

and $(b \to a) \to b \in B \cap C$. For the last:

$$(b \to a) \wedge b = a \wedge b = a \wedge b \wedge a = (b \to a) \wedge (b \wedge a),$$

hence $(b \to a) \to b = a \wedge b = (b \to a) \to (b \wedge a)$ by 1.3.3(6). ■

4.4.3. Remarks. 1. Note that 4.3.1 and 4.3.2 elucidate the seeming discrepancy of the *coframe* $\mathsf{S}(L)$ naturally containing the *frame* $\mathsf{S}_{\mathfrak{c}}(L)$.

2. Finitizing the proof of the coframe distributivity from (e.g.) [13] is also very short:
$(A_1 \cap A_2) \vee B \subseteq (A_1 \vee B) \cap (A_2 \vee B)$ is trivial.
$\supseteq$: Let $x = a_1 \wedge b_1 = a_2 \wedge b_2 \in (A_1 \vee B) \cap (A_2 \vee B)$. Set $b = b_1 \wedge b_2$. Then

$$a_i \wedge b \leq a_i \wedge b_i = x = x \wedge x = a_1 \wedge b_1 \wedge a_2 \wedge b_2 \leq a_j \wedge b,$$

hence $a_1 \wedge b = a_2 \wedge b$ and hence, by 1.3.3(6), $b \to a_1 = b \to a_2 = a \in A_1 \cap A_2$, and $b \wedge a = b \wedge (b \to a_1) = b \wedge a_1 = x$. □

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