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ARITHMETIC FOR CLOSED BALLS

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ABSTRACT: Inspired by circular complex interval arithmetic, an arithmetic for closed balls in $\mathbb{R}^n$ is pursued. In this sense, the properties of certain operations on closed balls in $\mathbb{R}^n$, some of which related either to the Hadamard product of vectors or to the 2-fold vector cross product when $n \in \{3, 7\}$, are studied. In particular, known results for operations on closed balls in $\mathbb{C}$, which can be identified with $\mathbb{R}^2$, are extended to closed balls in $\mathbb{R}^n$.

KEYWORDS: closed ball, operation, 2-fold vector cross product, Hadamard product of vectors, algebraic structure.

MATH. SUBJECT CLASSIFICATION (2010): 15A69, 15A72, 15A99, 08A40, 15A66, 17A75.

1. Introduction

Interval mathematics, a part of set theory, is involved in the computing paradigm of information processing known as granular computing – concerned with constructing and processing using information granules, to intelligently comprehend and interact with the world. As stated in [12], the appearance of electronic computers enhanced the practicality and the interest in calculating with sets of numbers, also in identifying associated algebraic structures, for error control purposes.

The works [7], due to Gargantini and Henrici, and [2], by Alefeld and Herzberger, are devoted to circular complex interval arithmetic which deals with the so-called circular complex intervals (closed balls in $\mathbb{C}$). More recent research related to interval mathematics, in the context of granular computing, can be found in [12] and in the journal with the same name as the context. In addition, an overview of ball arithmetic as a tool for rigorous algebraic computation with real numbers was presented in [11].

Inspired by [2] and [7], we extend some known results for certain operations on closed balls in $\mathbb{C}$, which can be identified with $\mathbb{R}^2$, to closed balls in $\mathbb{R}^n$. To begin with, in section 2, we recall definitions and state results related to closed balls, vector cross products and the Hadamard product of vectors. As

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highlighted in [3], the former products can be found in mathematical models of physical processes, namely in control theory, and in the description of spacecraft attitude control. The latter product, as mentioned in [5] and references therein, appears in applications to: lossy compression algorithms for JPEG images; machine learning, namely for describing the architecture of neural networks.

In section 3, we study the properties of certain operations on closed balls in $\mathbb{R}^n$, some of which related either to the Hadamard product of vectors or to the 2-fold vector cross product when $n \in \{3, 7\}$. As a consequence, several algebraic structures, with one or two binary operations, are presented. In particular, recalling a sufficient condition over fields of characteristic 0 given in [1], examples of power associative algebras are exhibited. Inclusion monotonicity, which according to [2] is the foundation for many applications of interval arithmetic, is satisfied by some of the mentioned operations. For other aspects connected with vector cross products and the Hadamard product, see, for instance, the references [6], [8] and [9], [13].

2. Preliminaries

Let F be a field of characteristic different from 2. Let V be a d-dimensional vector space over F, equipped with a nondegenerate symmetric bilinear form $(\cdot|\cdot)$. A bilinear map $\times : V \times V \to V$ is a 2-*fold vector cross product* if, for any $u, v \in V$,

(1) $(u \times v|u) = (u \times v|v) = 0$,

(2) $(u \times v|u \times v) = \begin{vmatrix} (u|u) & (u|v) \\ (v|u) & (v|v) \end{vmatrix}$.

It is well known that if $\times$ is a 2-fold vector cross product on V, then $d \in \{1, 3, 7\}$ and, in particular, $d = 1$ leads to the trivial case, [4]. In fact, the restriction on d is a consequence of the generalized Hurwitz Theorem which asserts that, over a field of characteristic different from 2, if $\mathcal{A}$ is a finite dimensional composition algebra with identity (also called *Hurwitz algebra*), then its dimension is equal to $1, 2, 4$ or 8. Moreover, $\mathcal{A}$ is isomorphic either to the base field, a separable quadratic extension of the base field, a generalized quaternion algebra or a generalized octonion algebra, [10].

Throughout the article, we work in the Euclidean vector space $\mathbb{R}^n$ and denote the *Euclidean norm* of a vector, the ∞-*norm* of a vector, the *Hadamard (componentwise) product of vectors* and, whenever $n \in \{3, 7\}$, the 2-fold vector cross product by, respectively, $\|\cdot\|$, $\|\cdot\|_\infty$, $\circ$ and $\times$.

Let $a \in \mathbb{R}^n$ and let $r \in \mathbb{R}_0^+$. We call $A = \{x \in \mathbb{R}^n : \|x - a\| \leq r\}$ the *closed ball* in $\mathbb{R}^n$ with *center* a and *radius* r, also written as $A = \langle a; r \rangle$. The set of closed balls in $\mathbb{R}^n$ is denoted by $\mathcal{B}$, and by $\mathcal{B}^+$ or $\mathcal{B}^0$ if, respectively, $r \in \mathbb{R}^+$ or $r = 0$.

Let $A = \langle a; r_1 \rangle, B = \langle b; r_2 \rangle \in \mathcal{B}$. The closed balls A and B are *equal* ($A = B$) if there is set-theoretic equality between them, that is, $a = b$ and $r_1 = r_2$. A is *contained* in B ($A \subseteq B$) if set-theoretic inclusion holds, which is equivalent to the condition established in the subsequent result.

Theorem 2.1. *Let $A = \langle a; r_1 \rangle, B = \langle b; r_2 \rangle \in \mathcal{B}$. Then $A \subseteq B$ if and only if $\|a - b\| \leq r_2 - r_1$.*

Proof: ($\Leftarrow$) Let $x \in A$. Then $\|x - a\| \leq r_1$. Observe that $\|x - b\| = \|x - a + a - b\| \leq \|x - a\| + \|a - b\| \leq r_1 + r_2 - r_1 = r_2$. We conclude that $x \in B$.

($\Rightarrow$) Suppose that $A \subseteq B$ and, by way of contradiction, that $\|a - b\| > r_2 - r_1$. Consider the line passing through a and b. This line intersects the border of A at a point x such that $\|x - b\| = \|a - b\| + \|x - a\| > r_2 - r_1 + r_1 = r_2$. Hence, $x \notin B$ – a contradiction. ∎

Corollary 2.2. *Let $A = \langle a; r_1 \rangle, B = \langle a; r_2 \rangle \in \mathcal{B}$. Then $A \subseteq B$ if and only if $r_1 \leq r_2$.*

Proof: A direct consequence of Theorem 2.1 since the closed balls are concentric. ∎

3. Operations

3.1. Addition and Subtraction. Consider the binary operation $+_{\mathcal{B}} : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ given by

$$A +_{\mathcal{B}} B = \langle a; r_1 \rangle +_{\mathcal{B}} \langle b; r_2 \rangle := \langle a + b; r_1 + r_2 \rangle.$$

The subsequent results establish several properties related to the addition $+_{\mathcal{B}}$.

Theorem 3.1. *$(\mathcal{B}, +_{\mathcal{B}})$ is a commutative monoid.*

Proof: Let $A = \langle a; r_1 \rangle, B = \langle b; r_2 \rangle, C = \langle c; r_3 \rangle \in \mathcal{B}$.

Owing to the associativity of the addition in $\mathbb{R}^n$ and to the associativity of the addition in $\mathbb{R}$, we obtain

$$\begin{aligned}(A +_{\mathcal{B}} B) +_{\mathcal{B}} C &= \langle (a+b)+c; (r_1+r_2)+r_3 \rangle \\ &= \langle a+(b+c); r_1+(r_2+r_3) \rangle \\ &= A +_{\mathcal{B}} (B +_{\mathcal{B}} C).\end{aligned}$$

Thus, $+_{\mathcal{B}}$ is associative.

Due to the commutativity of the addition in $\mathbb{R}^n$ and to the commutativity of the addition in $\mathbb{R}$, we get

$$A +_{\mathcal{B}} B = \langle a+b; r_1+r_2 \rangle = \langle b+a; r_2+r_1 \rangle = B +_{\mathcal{B}} A.$$

Hence, $+_{\mathcal{B}}$ is commutative.

Taking into account the commutativity of $+_{\mathcal{B}}$, and the neutral elements of $\mathbb{R}^n$ and $\mathbb{R}$ relative to the respective additions, we have

$$\langle a; r_1 \rangle +_{\mathcal{B}} \langle 0; 0 \rangle = \langle a; r_1 \rangle.$$

Thus, the neutral element of $(\mathcal{B}, +_{\mathcal{B}})$ is $E = \langle 0; 0 \rangle$. ■

Corollary 3.2. *The set of elements of $\mathcal{B}$ which possess reciprocal relative to $+_{\mathcal{B}}$ is $\mathcal{B}^0$.*

Proof: Let $A = \langle a; r_1 \rangle \in \mathcal{B}$. Suppose that $A' = \langle a'; r_1' \rangle \in \mathcal{B}$ is the reciprocal of A relative to $+_{\mathcal{B}}$. Then we have

$$A +_{\mathcal{B}} A' = E \Leftrightarrow \langle a+a'; r_1+r_1' \rangle = \langle 0; 0 \rangle \Leftrightarrow a+a' = 0 \wedge r_1 + r_1' = 0.$$

Hence, $a' = -a$ and, taking into account the definition of $\mathcal{B}$, $r_1 = 0$. Therefore, the reciprocal of $\langle a; 0 \rangle$ relative to $+_{\mathcal{B}}$ is $\langle -a; 0 \rangle$. ■

Lemma 3.3. *Let $A = \langle a; r_1 \rangle, B = \langle b; r_2 \rangle \in \mathcal{B}$. Then*

$$A +_{\mathcal{B}} B = \{x + y : x \in A \wedge y \in B\}.$$

Proof: $(\supseteq)$ Let $x \in A$ and $y \in B$. Then, $\|x-a\| \leq r_1$, $\|y-b\| \leq r_2$ and $\|x+y-(a+b)\| \leq \|x-a\| + \|y-b\| \leq r_1+r_2$. Consequently, $x+y \in A +_{\mathcal{B}} B$.

$(\subseteq)$ Let $u \in A +_{\mathcal{B}} B$. Then $\|u-(a+b)\| \leq r_1+r_2$. If $r_1+r_2 = 0$ then $u = a+b$ and the inclusion follows. If $r_1+r_2 \neq 0$, then consider u written as

$v+(u-v)$ with $v=\alpha u+(1-\alpha)(a+b)-b$, where $\alpha=\frac{r_1}{r_1+r_2}$. Then we have

$$\|v-a\|=\alpha\|u-(a+b)\|\leq\frac{r_1}{r_1+r_2}(r_1+r_2)=r_1$$
$$\|u-v-b\|=(1-\alpha)\|u-(a+b)\|\leq\frac{r_2}{r_1+r_2}(r_1+r_2)=r_2$$

Therefore, $v\in A$ and $u-v\in B$. ∎

The next result shows that inclusion monotonicity is satisfied by $+_{\mathcal{B}}$.

Theorem 3.4. *Let $A_m, B_m\in\mathcal{B}, m\in\{1,2\}$. If $A_m\subseteq B_m$, $m\in\{1,2\}$, then $A_1+_{\mathcal{B}}A_2\subseteq B_1+_{\mathcal{B}}B_2$.*

Proof: Let $A_m, B_m\in\mathcal{B}$ such that $A_m\subseteq B_m$, $m\in\{1,2\}$. From Lemma 3.3, we have

$$\begin{aligned}A_1+_{\mathcal{B}}A_2&=\{x+y: x\in A_1\wedge y\in A_2\}\\&\subseteq\{x+y: x\in B_1\wedge y\in B_2\}\\&=B_1+_{\mathcal{B}}B_2.\end{aligned}$$

∎

Taking into account Corollary 3.2, it is not possible to define the subtraction of elements in $\mathcal{B}$ as the addition with the reciprocal relative to $+_{\mathcal{B}}$. In this sense, we finish the section with an alternative definition. Consider the binary operation $-_{\mathcal{B}}:\mathcal{B}\times\mathcal{B}\to\mathcal{B}$ given by

$$A-_{\mathcal{B}}B=\langle a;r_1\rangle-_{\mathcal{B}}\langle b;r_2\rangle:=\langle a-b;r_1+r_2\rangle.$$

Theorem 3.5. $(\mathcal{B},-_{\mathcal{B}})$ *is a groupoid with right neutral element.*

Proof: Let $A=\langle a;r_1\rangle\in\mathcal{B}$. Then $A-_{\mathcal{B}}E=\langle a;r_1\rangle-_{\mathcal{B}}\langle 0;0\rangle=\langle a;r_1\rangle$. Hence, the right neutral element of $(\mathcal{B},-_{\mathcal{B}})$ is the neutral element $E=\langle 0;0\rangle$ of $(\mathcal{B},+_{\mathcal{B}})$. ∎

3.2. Multiplication $\times_{\mathcal{B},r}$. Consider the binary operation $\times_{\mathcal{B},r}:\mathcal{B}\times\mathcal{B}\to\mathcal{B}$ given by

$$A\times_{\mathcal{B},r}B=\langle a;r_1\rangle\times_{\mathcal{B},r}\langle b;r_2\rangle:=\langle a\times b+r_2a+r_1b;r_1r_2\rangle.$$

Let $\{e_i: i=1,2,3\}$ be the canonical basis of $\mathbb{R}^3$. From Theorem 2.1, $\langle e_1;1\rangle\subseteq\langle e_1;2\rangle$ and $\langle e_2;1\rangle\subseteq\langle 3e_2;3\rangle$. However, by the same result, $\langle e_1;1\rangle\times_{\mathcal{B},r}\langle e_2;1\rangle=\langle e_1+e_2+e_3;1\rangle\nsubseteq\langle e_1;2\rangle\times_{\mathcal{B},r}\langle 3e_2;3\rangle=\langle 3e_1+6e_2+3e_3;6\rangle$. Thus, inclusion monotonicity is not satisfied by $\times_{\mathcal{B},r}$.

Despite not satisfying commutativity, anti-commutativity, and associativity, there are some properties which hold for the multiplication $\times_{\mathcal{B},r}$.

Theorem 3.6. *$(\mathcal{B}, \times_{\mathcal{B},r})$ is a groupoid with neutral element.*

Proof: Let $A = \langle a; r_1 \rangle \in \mathcal{B}$. Then

$$\langle a; r_1 \rangle \times_{\mathcal{B},r} \langle 0; 1 \rangle = \langle a; r_1 \rangle = \langle 0; 1 \rangle \times_{\mathcal{B},r} \langle a; r_1 \rangle.$$

Therefore, the neutral element of $(\mathcal{B}, \times_{\mathcal{B},r})$ is $\langle 0; 1 \rangle$. ■

Corollary 3.7. *The set of elements of $\mathcal{B}$ which possess reciprocal relative to $\times_{\mathcal{B},r}$ is $\mathcal{B}^+$.*

Proof: Let $A = \langle a; r_1 \rangle \in \mathcal{B}^+$. As

$$\langle a; r_1 \rangle \times_{\mathcal{B},r} \langle -\frac{1}{r_1^2} a; \frac{1}{r_1} \rangle = \langle 0; 1 \rangle = \langle -\frac{1}{r_1^2} a; \frac{1}{r_1} \rangle \times_{\mathcal{B},r} \langle a; r_1 \rangle,$$

then the reciprocal of $A = \langle a; r_1 \rangle$ relative to $\times_{\mathcal{B},r}$ is $A' = \langle -\frac{1}{r_1^2} a; \frac{1}{r_1} \rangle$. ■

Taking into account the previous result, now consider the binary operation $:_{\mathcal{B}^+,r}\colon \mathcal{B}^+ \times \mathcal{B}^+ \to \mathcal{B}^+$ given by

$$A :_{\mathcal{B}^+,r} B := A \times_{\mathcal{B},r} B',$$

where B' denotes the reciprocal of B relative to $\times_{\mathcal{B},r}$.

Corollary 3.8. *$(\mathcal{B}^+, :_{\mathcal{B}^+,r})$ is a groupoid with right neutral element.*

Proof: As a consequence of Theorem 3.6, by the definition of $:_{\mathcal{B}^+,r}$, we conclude that the right neutral element of $(\mathcal{B}^+, :_{\mathcal{B}^+,r})$ is $\langle 0; 1 \rangle$. ■

The following results concern powers. We define the right powers of an element $A \in \mathcal{B}$ by $A^0 = \langle 0; 1 \rangle$ and $A^k = A^{k-1} \times_{\mathcal{B},r} A$ for $k \in \mathbb{N}$.

Theorem 3.9. *$(\mathcal{B}, \times_{\mathcal{B},r})$ is a power associative algebra.*

Proof: We want to prove that, for all $m, s \in \mathbb{N}$ and for all $A \in \mathcal{B}$, $A^s \times_{\mathcal{B},r} A^m = A^{s+m}$. By [1], it suffices to show that $A^2 \times_{\mathcal{B},r} A = A \times_{\mathcal{B},r} A^2$ and $(A^2 \times_{\mathcal{B},r} A) \times_{\mathcal{B},r} A = A^2 \times_{\mathcal{B},r} A^2$.

Let $A = \langle a; r_1 \rangle \in \mathcal{B}$. We obtain

$$\begin{aligned} A^2 \times_{\mathcal{B},r} A &= \langle 2r_1 a; r_1^2 \rangle \times_{\mathcal{B},r} \langle a; r_1 \rangle \\ &= \langle 3r_1^2 a; r_1^3 \rangle \\ &= \langle a; r_1 \rangle \times_{\mathcal{B},r} \langle 2r_1 a; r_1^2 \rangle \\ &= A \times_{\mathcal{B},r} A^2 \end{aligned}$$

and

$$
\begin{aligned}
(A^2 \times_{\mathcal{B},r} A) \times_{\mathcal{B},r} A &= \langle 3r_1^2 a; r_1^3 \rangle \times_{\mathcal{B},r} \langle a; r_1 \rangle \\
&= \langle 4r_1^3 a; r_1^4 \rangle \\
&= \langle 2r_1 a; r_1^2 \rangle \times_{\mathcal{B},r} \langle 2r_1 a; r_1^2 \rangle \\
&= A^2 \times_{\mathcal{B},r} A^2.
\end{aligned}
$$

■

Theorem 3.10. *Let $A = \langle a; r_1 \rangle \in \mathcal{B}$. For all $k \in \mathbb{N}$, $A^k = \langle k r_1^{k-1} a; r_1^k \rangle$.*

Proof: Let $A = \langle a; r_1 \rangle \in \mathcal{B}$. It is straightforward to see that the base case holds. As for the induction step, we have

$$A^k = A^{k-1} \times_{\mathcal{B},r} A = \langle (k-1) r_1^{k-2} a; r_1^{k-1} \rangle \times_{\mathcal{B},r} \langle a; r_1 \rangle = \langle k r_1^{k-1} a; r_1^k \rangle.$$

■

Theorem 3.11. *Let $A = \langle a; r \rangle \in \mathcal{B}^+$. The square root of A is given by $A^{1/2} = \langle \frac{1}{2r^{1/2}} a; r^{1/2} \rangle$.*

Proof: Let $A = \langle a; r \rangle \in \mathcal{B}^+$. Let $B = \langle b; s \rangle \in \mathcal{B}$ such that $B^2 = A$. Hence, $s^2 = r$ and $b \times b + 2sb = a$, from where the result follows. ■

We end the section with a result that relates $\times_{\mathcal{B},r}$ to $+_{\mathcal{B}}$.

Theorem 3.12. *$(\mathcal{B}, +_{\mathcal{B}}, \times_{\mathcal{B},r})$ is a ringoid.*

Proof: Let $A = \langle a; r_1 \rangle, B = \langle b; r_2 \rangle, C = \langle c; r_3 \rangle \in \mathcal{B}$. As $\times$ is bilinear, we have

$$
\begin{aligned}
A \times_{\mathcal{B},r} (B +_{\mathcal{B}} C) &= \langle a; r_1 \rangle \times_{\mathcal{B},r} \langle b + c; r_2 + r_3 \rangle \\
&= \langle a \times (b + c) + (r_2 + r_3) a + r_1 (b + c); r_1 (r_2 + r_3) \rangle \\
&= \langle a \times b + r_2 a + r_1 b + a \times c + r_3 a + r_1 c; r_1 r_2 + r_1 r_3 \rangle \\
&= A \times_{\mathcal{B},r} B +_{\mathcal{B}} A \times_{\mathcal{B},r} C
\end{aligned}
$$

An analogous reasoning provides the proof of the right distributivity. Thus, $\times_{\mathcal{B},r}$ is distributive relative to $+_{\mathcal{B}}$. ■

3.3. Multiplication $\times_{\mathcal{B},c}$. Consider the binary operation $\times_{\mathcal{B},c} : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ given by

$$A \times_{\mathcal{B},c} B = \langle a; r_1 \rangle \times_{\mathcal{B},c} \langle b; r_2 \rangle := \langle a \times b; r_2 \|a\| + r_1 \|b\| + r_1 r_2 \rangle.$$

Although commutativity, anti-commutativity, associativity, existence of neutral element and power associativity do not hold, other properties are satisfied by the multiplication $\times_{\mathcal{B},c}$.

Theorem 3.13. *Let $A = \langle 0; r\rangle \in \mathcal{B}$. The square roots of A are given by $A^{1/2} = \langle b; -\|b\| + \sqrt{\|b\|^2 + r}\rangle$, with $b \in \mathbb{R}^n$.*

Proof: Let $A = \langle 0; r\rangle \in \mathcal{B}$. Let $B = \langle b; s\rangle \in \mathcal{B}$ such that $B \times_{\mathcal{B},c} B = A$. We obtain $s^2 + 2\|b\|s - r = 0$ and, as $r, s \in \mathbb{R}_0^+$, $s = -\|b\| + \sqrt{\|b\|^2 + r}$. ■

The next result shows that inclusion monotonicity is satisfied by $\times_{\mathcal{B},c}$.

Theorem 3.14. *Let $A_m, B_m \in \mathcal{B}, m \in \{1,2\}$. If $A_m \subseteq B_m$, $m \in \{1,2\}$, then $A_1 \times_{\mathcal{B},c} A_2 \subseteq B_1 \times_{\mathcal{B},c} B_2$.*

Proof: Let $A_m = \langle a_m; r_m\rangle$, $B_m = \langle b_m; s_m\rangle \in \mathcal{B}$ such that $A_m \subseteq B_m$, $m \in \{1,2\}$. Then, by Theorem 2.1, $\|a_m - b_m\| \leq s_m - r_m$, $m \in \{1,2\}$. We also have

$$\begin{aligned} A_1 \times_{\mathcal{B},c} A_2 &= \langle a_1 \times a_2; r_2\|a_1\| + r_1\|a_2\| + r_1 r_2\rangle, \\ B_1 \times_{\mathcal{B},c} B_2 &= \langle b_1 \times b_2; s_2\|b_1\| + s_1\|b_2\| + s_1 s_2\rangle. \end{aligned}$$

From

$$\begin{aligned} &\|a_1 \times a_2 - b_1 \times b_2\| \\ &= \| - b_2 \times (a_1 - b_1) + b_1 \times (a_2 - b_2) + (a_1 - b_1) \times (a_2 - b_2)\| \\ &\leq \|b_2\|\|a_1 - b_1\| + \|b_1\|\|a_2 - b_2\| + \|a_1 - b_1\|\|a_2 - b_2\| \\ &\leq \|b_2\|(s_1 - r_1) + \|b_1\|(s_2 - r_2) + (s_1 - r_1)(s_2 - r_2) \end{aligned}$$

and

$$-\|b_m\| \leq -\|a_m\| + \|a_m - b_m\| \leq -\|a_m\| + s_m - r_m, m \in \{1,2\},$$

we get $\|a_1 \times a_2 - b_1 \times b_2\| \leq \beta - \alpha$, where $\beta = s_2\|b_1\| + s_1\|b_2\| + s_1 s_2$ and $\alpha = r_2\|a_1\| + r_1\|a_2\| + r_1 r_2$. Invoking once again Theorem 2.1, the result follows. ■

We finish the section with a result that relates $\times_{\mathcal{B},c}$ to $+_{\mathcal{B}}$.

Theorem 3.15. *The operation $\times_{\mathcal{B},c}$ is subdistributive with respect to $+_{\mathcal{B}}$.*

Proof: Let $A = \langle a; r_1\rangle, B = \langle b; r_2\rangle, C = \langle c; r_3\rangle \in \mathcal{B}$. Invoking Corollary 2.2, we have

$$\begin{aligned} A \times_{\mathcal{B},c} (B +_{\mathcal{B}} C) &= \langle a; r_1\rangle \times_{\mathcal{B},c} \langle b + c; r_2 + r_3\rangle \\ &= \langle a \times (b + c); (r_2 + r_3)\|a\| + r_1\|b + c\| + r_1(r_2 + r_3)\rangle \\ &\subseteq \langle a \times b + a \times c; r_2\|a\| + r_1\|b\| + r_1r_2 + r_3\|a\| + r_1\|c\| + r_1r_3\rangle \\ &= A \times_{\mathcal{B},c} B +_{\mathcal{B}} A \times_{\mathcal{B},c} C \end{aligned}$$

Hence, left subdistributivity is valid. Through a similar reasoning, the right subdistributivity also holds. ∎

3.4. Multiplication $\circ_{\mathcal{B},r}$. Consider the binary operation $\circ_{\mathcal{B},r} : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ given by

$$A \circ_{\mathcal{B},r} B = \langle a; r_1\rangle \circ_{\mathcal{B},r} \langle b; r_2\rangle := \langle a \circ b + r_2a + r_1b; r_1r_2\rangle.$$

As, by Theorem 2.1, $A_1 = \langle(0, 1); 1\rangle \subseteq B_1 = \langle(-2, 1); 3\rangle$, $A_2 = \langle(1, 0); 1\rangle \subseteq B_2 = \langle(1, -2); 3\rangle$, but $A_1 \circ_{\mathcal{B},r} A_2 = \langle(1, 1); 1\rangle \nsubseteq B_1 \circ_{\mathcal{B},r} B_2 = \langle(-5, -5); 9\rangle$, then $\circ_{\mathcal{B},r}$ does not satisfy the inclusion monotonicity. However, the following properties hold for the multiplication $\circ_{\mathcal{B},r}$.

Theorem 3.16. *$(\mathcal{B}, \circ_{\mathcal{B},r})$ is a commutative monoid.*

Proof: Since the Hadamard product $\circ$ of vectors is associative and commutative on $\mathbb{R}^n$, so is the operation $\circ_{\mathcal{B},r}$. It is straightforward to verify that $\langle a; r_1\rangle = \langle a; r_1\rangle \circ_{\mathcal{B},r} \langle 0; 1\rangle$ and, thus, $\langle 0; 1\rangle$ is the neutral element of $(\mathcal{B}, \circ_{\mathcal{B},r})$. ∎

Theorem 3.17. *The set of elements of $\mathcal{B}$ which possess reciprocal relative to $\circ_{\mathcal{B},r}$ is $\{A = \langle a; r_1\rangle \in \mathcal{B}^+ : a = (a_1, \dots, a_n) \in \mathbb{R}^n \wedge a_i \neq -r_1, i \in \{1, \dots, n\}\}$.*

Proof: Let A=$\langle a; r_1\rangle \in \mathcal{B}^+$. Let $b = (b_1, \dots, b_n) \in \mathbb{R}^n$ such that $\langle a; r_1\rangle \circ_{\mathcal{B},r} \langle b; 1/r_1\rangle = \langle 0; 1\rangle$. Then, from $a \circ b + \frac{1}{r_1}a + r_1b = 0$ it follows that

$$a_ib_i + \frac{1}{r_1}a_i + r_1b_i = 0, i \in \{1, \dots, n\}.$$

This leads to a linear system of n equations in n unknowns b_i, whose unique solution is b with $b_i = -\frac{a_i}{r_1(a_i+r_1)}$, $i \in \{1, \dots, n\}$. ∎

The subsequent results concern powers. We define the powers of an element $A = \langle a; r_1\rangle \in \mathcal{B}$ by $A^0 = \langle 0; 1\rangle$ and $A^k = A^{k-1} \circ_{\mathcal{B},r} A$ for $k \in \mathbb{N}$. Analogously, let us denote $(1, \dots, 1)$ by $a^{\circ 0}$ and $a^{\circ(k-1)} \circ a$ by $a^{\circ k}$ for $k \in \mathbb{N}$.

Theorem 3.18. *$(\mathcal{B}, \circ_{\mathcal{B},r})$ is a power associative algebra.*

Proof: Invoking Theorem 3.16, due to the commutativity and to the associativity of $\circ_{\mathcal{B},r}$, $A^2 \circ_{\mathcal{B},r} A = A \circ_{\mathcal{B},r} A^2$ and $(A^2 \circ_{\mathcal{B},r} A) \circ_{\mathcal{B},r} A = A^2 \circ_{\mathcal{B},r} A^2$ hold for all $A \in \mathcal{B}$. ∎

Theorem 3.19. *Let $A = \langle a; r_1\rangle \in \mathcal{B}$. For all $k \in \mathbb{N}$, $A^k = \langle \sum_{i=1}^{k} \binom{k}{i} r_1^{k-i} a^{\circ i}; r_1^k\rangle$.*

Proof: We proceed by induction on k. Clearly, the equality is valid for $k = 1$. Let us suppose that it is true for k. Then,

$$\begin{aligned} A^{k+1} &= A^k \circ_{\mathcal{B},r} A = \langle \textstyle\sum_{i=1}^{k} \binom{k}{i} r_1^{k-i} a^{\circ i}; r_1^k\rangle \circ_{\mathcal{B},r} \langle a; r_1\rangle \\ &= \langle \textstyle\sum_{i=1}^{k} \binom{k}{i} r_1^{k-i} a^{\circ(i+1)} + \sum_{i=1}^{k} \binom{k}{i} r_1^{k+1-i} a^{\circ i} + r_1^k a; r_1^{k+1}\rangle \\ &= \langle a^{\circ(k+1)} + \textstyle\sum_{i=2}^{k} \left[\binom{k}{i-1} + \binom{k}{i}\right] r_1^{k+1-i} a^{\circ i} + (k+1) r_1^k a; r_1^{k+1}\rangle \\ &= \langle \textstyle\sum_{i=1}^{k+1} \binom{k+1}{i} r_1^{k+1-i} a^{\circ i}; r_1^{k+1}\rangle. \end{aligned}$$

∎

Theorem 3.20. *Let $A = \langle a; r_1\rangle \in \mathcal{B}$ with $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. If $r_1 + a_i \in \mathbb{R}_0^+$ for $i \in \{1, \ldots, n\}$, then the square roots of A are given by $A^{1/2} = \langle b; r_1^{1/2}\rangle$, where $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$ with $b_i = -\sqrt{r_1} \pm \sqrt{r_1 + a_i}$ for $i \in \{1, \ldots, n\}$.*

Proof: Let $B = \langle b; s\rangle \in \mathcal{B}$ such that $A = B^2$. Then, $\langle a; r_1\rangle = \langle b^{\circ 2} + 2sb; s^2\rangle$. So, we have $s^2 = r_1$ and $b_i^2 + 2sb_i - a_i = 0$ for $i \in \{1, \ldots, n\}$. Since, for each i, the discriminant belongs to $\mathbb{R}_0^+$, then $b_i = -s \pm \sqrt{s^2 + a_i}$ and the result follows. ∎

The last result of the section shows an algebraic structure which relates $+_{\mathcal{B}}$ to $\circ_{\mathcal{B},r}$.

Theorem 3.21. *$(\mathcal{B}, +_{\mathcal{B}}, \circ_{\mathcal{B},r})$ is a commutative semiring.*

Proof: Since, from Theorem 3.1 and Theorem 3.16, $(\mathcal{B}, +_{\mathcal{B}})$ and $(\mathcal{B}, \circ_{\mathcal{B},r})$ are commutative monoids, it only remains to prove that $\circ_{\mathcal{B},r}$ is distributive with respect to $+_{\mathcal{B}}$.

Let $A = \langle a; r_1\rangle$, $B = \langle b; r_2\rangle$ and $C = \langle c; r_3\rangle$, then

$$\begin{aligned} A \circ_{\mathcal{B},r} (B +_{\mathcal{B}} C) &= \langle a; r_1\rangle \circ_{\mathcal{B},r} (\langle b; r_2\rangle +_{\mathcal{B}} \langle c; r_3\rangle) \\ &= \langle a \circ (b + c) + (r_2 + r_3)a + r_1(b + c); r_1(r_2 + r_3)\rangle \\ &= \langle a \circ b + a \circ c + r_2 a + r_3 a + r_1 b + r_1 c; r_1 r_2 + r_1 r_3\rangle \\ &= \langle a \circ b + r_2 a + r_1 b; r_1 r_2\rangle +_{\mathcal{B}} \langle a \circ c + r_3 a + r_1 c; r_1 r_3\rangle \\ &= (A \circ_{\mathcal{B},r} B) +_{\mathcal{B}} (A \circ_{\mathcal{B},r} C). \end{aligned}$$

∎

3.5. Multiplication $\circ_{\mathcal{B},c}$. Consider the binary operation $\circ_{\mathcal{B},c} : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ given by

$$A \circ_{\mathcal{B},c} B = \langle a; r_1 \rangle \circ_{\mathcal{B},c} \langle b; r_2 \rangle := \langle a \circ b; r_1 \|b\|_\infty + r_2 \|a\|_\infty + r_1 r_2 \rangle.$$

Since $\|a \circ b\|_\infty \neq \|a\|_\infty \|b\|_\infty$ for some $a, b \in \mathbb{R}^n$, the multiplication $\circ_{\mathcal{B},c}$ is not associative. Despite this, there are several properties satisfied by $\circ_{\mathcal{B},c}$.

Theorem 3.22. $(\mathcal{B}, \circ_{\mathcal{B},c})$ *is a commutative groupoid with neutral element.*

Proof: It is clear that $\circ_{\mathcal{B},c}$ is commutative. Let us denote $(1, \ldots, 1) \in \mathbb{R}^n$ by 1. If $A = \langle a; r \rangle \in \mathcal{B}$, then

$$A \circ_{\mathcal{B},c} \langle 1; 0 \rangle = \langle a \circ 1; r \rangle = A.$$

So, $\langle 1; 0 \rangle$ is the neutral element of $(\mathcal{B}, \circ_{\mathcal{B},c})$. ■

Theorem 3.23. *The set of elements of $\mathcal{B}$ which possess reciprocal relative to $\circ_{\mathcal{B},c}$ is $\{A = \langle a; 0 \rangle \in \mathcal{B}^0 : a = (a_1, \ldots, a_n) \in \mathbb{R}^n \wedge a_i \neq 0, i \in \{1, \ldots, n\}\}$.*

Proof: Let $A = \langle a; r \rangle \in \mathcal{B}$. Suppose that $B = \langle b; s \rangle$ is the reciprocal of A relative to $\circ_{\mathcal{B},c}$. Then,

$$A \circ_{\mathcal{B},c} B = \langle a; r \rangle \circ_{\mathcal{B},c} \langle b; s \rangle = \langle a \circ b; r\|b\|_\infty + s\|a\|_\infty + rs \rangle = \langle 1; 0 \rangle.$$

So, $a \circ b = 1$ and $b_i = a_i^{-1}$ for $1 \leq i \leq n$ whenever $a_i \neq 0$. On the other hand, $r\|b\|_\infty + s\|a\|_\infty + rs = 0$ with $r, s \in \mathbb{R}_0^+$, which implies $r = s = 0$. The result follows. ■

The following results concern powers. We define the powers of an element $A = \langle a; r \rangle \in \mathcal{B}$ by $A^0 = \langle 1; 0 \rangle$ and $A^k = A^{k-1} \circ_{\mathcal{B},c} A$ for $k \in \mathbb{N}$. As in the previous section, let us denote $(1, \ldots, 1)$ by $a^{\circ 0}$ and $a^{\circ(k-1)} \circ a$ by $a^{\circ k}$ for $k \in \mathbb{N}$.

Theorem 3.24. $(\mathcal{B}, \circ_{\mathcal{B},c})$ *is a power associative algebra.*

Proof: Since $\circ_{\mathcal{B},c}$ is commutative by Theorem 3.22, $A^2 \circ_{\mathcal{B},c} A = A \circ_{\mathcal{B},c} A^2$ holds for all $A \in \mathcal{B}$. Hence, it suffices to show that $(A^2 \circ_{\mathcal{B},c} A) \circ_{\mathcal{B},c} A = A^2 \circ_{\mathcal{B},c} A^2$ for all $A \in \mathcal{B}$. Let $A = \langle a; r \rangle \in \mathcal{B}$. Then

$$\begin{aligned}
A^2 \circ_{\mathcal{B},c} A &= \langle a^{\circ 2}; 2r\|a\|_\infty + r^2 \rangle \circ_{\mathcal{B},c} \langle a; r \rangle \\
&= \langle a^{\circ 3}; r\|a^{\circ 2}\|_\infty + \|a\|_\infty (2r\|a\|_\infty + r^2) + r(2r\|a\|_\infty + r^2) \rangle \\
&= \langle a^{\circ 3}; 3r\|a\|_\infty^2 + 3r^2\|a\|_\infty + r^3 \rangle \\
(A^2 \circ_{\mathcal{B},c} A) \circ_{\mathcal{B},c} A &= \langle a^{\circ 3}; 3r\|a\|_\infty^2 + 3r^2\|a\|_\infty + r^3 \rangle \circ_{\mathcal{B},c} \langle a; r \rangle \\
&= \langle a^{\circ 4}; 4r\|a\|_\infty^3 + 6r^2\|a\|_\infty^2 + 4r^3\|a\|_\infty + r^4 \rangle.
\end{aligned}$$

On the other hand, we have that

$$\begin{aligned} A^2 \circ_{\mathcal{B},c} A^2 &= \langle a^{\circ 2}; 2r\|a\|_\infty + r^2\rangle \circ_{\mathcal{B},c} \langle a^{\circ 2}; 2r\|a\|_\infty + r^2\rangle \\ &= \langle a^{\circ 4}; 4r\|a\|_\infty^3 + 6r^2\|a\|_\infty^2 + 4r^3\|a\|_\infty + r^4\rangle. \end{aligned}$$

■

Theorem 3.25. *Let $A \in \mathcal{B}$. For all $k \in \mathbb{N}$, $A^k = \langle a^{\circ k}; (\|a\|_\infty + r)^k - \|a\|_\infty^k\rangle$.*

Proof: Let us proceed by induction on k. Clearly, the equality is valid for $k = 1$. Suppose that it also holds for k. Then

$$\begin{aligned} A^{k+1} &= A^k \circ_{\mathcal{B},c} A \\ &= \langle a^{\circ k}; \textstyle\sum_{i=1}^k \binom{k}{i} r^i \|a\|_\infty^{k-i}\rangle \circ_{\mathcal{B},c} \langle a; r\rangle \\ &= \langle a^{\circ(k+1)}; r\|a^{\circ k}\|_\infty + \textstyle\sum_{i=1}^k \binom{k}{i} r^i \|a\|_\infty^{k+1-i} + \sum_{i=1}^k \binom{k}{i} r^{i+1}\|a\|_\infty^{k-i}\rangle \\ &= \langle a^{\circ(k+1)}; (k+1)r\|a\|_\infty^k + \textstyle\sum_{i=2}^k \left[\binom{k}{i} + \binom{k}{i-1}\right] r^i \|a\|_\infty^{k+1-i} + r^{k+1}\rangle \\ &= \langle a^{\circ(k+1)}; \textstyle\sum_{i=1}^{k+1} \binom{k+1}{i} r^i \|a\|_\infty^{k+1-i}\rangle \\ &= \langle a^{\circ(k+1)}; (\|a\|_\infty + r)^{k+1} - \|a\|_\infty^{k+1}\rangle. \end{aligned}$$

■

Theorem 3.26. *Let $A = \langle a; r\rangle \in (\mathcal{B}, \circ_{\mathcal{B},c})$ with $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ such that $a_i \in \mathbb{R}_0^+$ for $i \in \{1, \ldots, n\}$. The square roots of A are given by $A^{1/2} = \langle a^{\circ 1/2}; \sqrt{r + \|a\|_\infty} - \|a\|_\infty^{1/2}\rangle$, where $a^{\circ 1/2} = (\pm a_1^{1/2}, \ldots, \pm a_n^{1/2})$.*

Proof: Let us suppose that $B = \langle b; s\rangle \in \mathcal{B}$ is such that $A = B^2$. Then, we obtain $\langle a; r\rangle = \langle b; s\rangle \circ_{\mathcal{B},c} \langle b; s\rangle = \langle b^{\circ 2}; s^2 + 2s\|b\|_\infty\rangle$. So, $b = a^{\circ 1/2}$ and s is a solution of $s^2 + 2s\|a\|_\infty^{1/2} - r = 0$. Since $s \in \mathbb{R}_0^+$, $s = \sqrt{r + \|a\|_\infty} - \|a\|_\infty^{1/2}$ and the result follows. ■

The next result shows that inclusion monotonicity is satisfied by $\circ_{\mathcal{B},c}$.

Theorem 3.27. *Let $A_m, B_m \in \mathcal{B}, m \in \{1, 2\}$. If $A_m \subseteq B_m$, $m \in \{1, 2\}$, then $A_1 \circ_{\mathcal{B},c} A_2 \subseteq B_1 \circ_{\mathcal{B},c} B_2$.*

Proof: Let $A_m = \langle a_m; r_m\rangle$, $B_m = \langle b_m; s_m\rangle \in \mathcal{B}$ such that $A_m \subseteq B_m$, $m \in \{1, 2\}$. From Theorem 2.1, $\|b_m - a_m\| \le s_m - r_m$, $m \in \{1, 2\}$, and it suffices to prove that

$$\|b_1 \circ b_2 - a_1 \circ a_2\| \le s_1\|b_2\|_\infty + s_2\|b_1\|_\infty + s_1 s_2 - r_1\|a_2\|_\infty - r_2\|a_1\|_\infty - r_1 r_2.$$

On the one hand, we obtain

$$\begin{aligned}\|b_1 \circ b_2 - a_1 \circ a_2\| &= \|b_1 \circ b_2 - b_1 \circ a_2 + b_1 \circ a_2 - a_1 \circ a_2\| \\ &\leq \|b_1 \circ (b_2 - a_2)\| + \|(b_1 - a_1) \circ a_2\| \\ &\leq \|b_1\|_\infty \|b_2 - a_2\| + \|a_2\|_\infty \|b_1 - a_1\| \\ &\leq \|b_1\|_\infty (s_2 - r_2) + \|a_2\|_\infty (s_1 - r_1).\end{aligned}$$

On the other hand, we have

$$\begin{aligned} s_1\|a_2\|_\infty &\leq s_1\|b_2\|_\infty + s_1\|a_2 - b_2\|_\infty \leq s_1\|b_2\|_\infty + s_1(s_2 - r_2), \\ -r_2\|b_1\|_\infty &\leq -r_2\|a_1\|_\infty + r_2\|a_1 - b_1\|_\infty \leq -r_2\|a_1\|_\infty + r_2(s_1 - r_1).\end{aligned}$$

Taking into account the former and the latter inequalities, we arrive at the result. ∎

The final result of the work relates $+_{\mathcal{B}}$ to $\circ_{\mathcal{B},c}$.

Theorem 3.28. *The operation $\circ_{\mathcal{B},c}$ is subdistributive with respect to $+_{\mathcal{B}}$.*

Proof: Let $A = \langle a; r_1 \rangle$, $B = \langle b; r_2 \rangle$ and $C = \langle c; r_3 \rangle$. Then, by Corollary 2.2,

$$\begin{aligned} & A \circ_{\mathcal{B},c} (B +_{\mathcal{B}} C) = \\ &\quad = \langle a; r_1 \rangle \circ_{\mathcal{B},c} (\langle b; r_2 \rangle +_{\mathcal{B}} \langle c; r_3 \rangle) \\ &\quad = \langle a \circ (b + c); r_1\|b + c\|_\infty + (r_2 + r_3)\|a\|_\infty + r_1(r_2 + r_3) \rangle \\ &\quad = \langle a \circ b + a \circ c; r_2\|a\|_\infty + r_1 r_2 + r_3\|a\|_\infty + r_1 r_3 + r_1\|b + c\|_\infty \rangle \\ &\quad \subseteq \langle a \circ b + a \circ c; r_1\|b\|_\infty + r_2\|a\|_\infty + r_1 r_2 + r_1\|c\|_\infty + r_3\|a\|_\infty + r_1 r_3 \rangle \\ &\quad = (A \circ_{\mathcal{B},c} B) +_{\mathcal{B}} (A \circ_{\mathcal{B},c} C).\end{aligned}$$

∎

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