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DIAMETER OF THE COMMUTATION CLASSES GRAPH OF A PERMUTATION

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Abstract: We define a statistic on the graph of commutation classes of a permutation of the symmetric group which is used to show that these graphs are equipped with a ranked poset structure, with a minimum and maximum. This characterization also allows us to compute the diameter of the commutation graph for any permutation, from which the results for the longest permutation and for fully commutative permutations are recovered.

Keywords: Reduced words, balanced tableaux, commutation graph, diameter.

1. Introduction

Given an integer $n \geq 2$, we let $\mathcal{S}_n$ denote the symmetric group on the alphabet $[n] = \{1, \ldots, n\}$, with composition of permutations performed from right to left. We usually write permutations in one-line notation $w = w_1 w_2 \cdots w_n$, where $w_i = w(i)$.

The symmetric group $\mathcal{S}_n$ is an example of the more general concept of a Coxeter group [2, 4], which are groups G that can be generated by a set $S = \{s_1, \ldots, s_m\} \subset G$ satisfying relations $(s_i s_j)^{m_{ij}} = 1$, where $m_{ii} = 1$ and $m_{ij} \geq 2$ for $i \neq j$. Any element $w \in G$ can be written as a finite product of elements of S. If $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ with ℓ minimal, the word $i_1 i_2 \cdots i_\ell$ is called a *reduced word* (or *reduced decomposition*) of g. In this case, we define the *length* of w by $\ell(w) = \ell$. The set of all reduced words of w is denoted by $\mathrm{R}(w)$.

The symmetric group has a Coxeter representation with generators s_i, the adjacent transposition interchanging the elements i and $i+1$, for $1 \leq i \leq n-1$, which satisfy the Coxeter relations

$$s_i s_j = s_j s_i \text{ for } |i - j| \geq 2, \tag{1}$$

$$s_i s_{i+1} s_1 = s_{i+1} s_i s_{i+1} \text{ for } 1 \leq i \leq n-2, \tag{2}$$

and $s_i^2 = 1$, the identity element. The relations (1) are known as *commutations* or *short braid relations*, and the relations (2) are called *long braid relations*.

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The graph $G(w)$, having vertex set $\mathrm{R}(w)$ and a edge connecting two reduced words if they differ by a single Coxeter relation has been considered by several authors. Tits [13] showed that $G(w)$ is connected, and Reiner and Roichman [10] computed the diameter for the particular case of the longest permutation $w_{\mathrm{o}} = n \cdots 21$, using hyperplane arrangements. This result was later obtained by Assaf [11] using balanced tableaux. The number of reduced words in $\mathrm{R}(w_{\mathrm{o}})$ was first computed algebraically by Stanley [12] using generating functions, and latter proved bijectively by Edelman and Greene [5], by establishing a bijection between reduced words for w_{o} and balanced labelings of the Rothe diagram of w_{o}. Fomin, Greene, Reiner and Shimozono [7] generalized this result, proving a one to one correspondence between reduced words for $w \in \mathcal{S}_n$ and standard balanced labellings of the Rothe diagram of w.

Contracting the commutation edges of $G(w)$ leads to the associated graph $C(w)$, known as the commutation graph of w, which has also received some attention. Elnitsky [6] established a bijection between the vertices of $C(w)$ and rhombic tilings of certain polygons, and proved that $C(w)$ is bipartite. The diameter of $C(w_{\mathrm{o}})$ has been computed in [9], and a connection to geometric representation theory was explored in [3].

In this paper, we establish a statistic on the classes of $C(w)$, inducing a rank poset structure on $C(w)$ with a unique minimal and a unique maximal element. This allows us to give a precise formula for the diameter of the graph $C(w)$. We recover, as special cases, the diameter of the commutation graph for the longest element w_{o} and the characterization of fully commutative permutations obtained by Billey, Jockusch and Stanley [1].

2. Reduced Words and Balanced Tableaux

The length of a permutation $w \in \mathcal{S}_n$ can also be given be the number of *inversions* of w [2], that is the number of pairs (w_j, w_i) such that $i < j$ and $w_j > w_i$:

$$\ell(w) = \big|\{(w_j, w_i) : i < j \text{ and } w_j > w_i\}\big|.$$

Thus, if $w = s_{i_1} s_{i_2} \cdots s_{i_{\ell(w)}}$, the word $i_1 i_2 \cdots i_{\ell(w)}$ is a reduced word for w. We consider adjacent transpositions s_{i_j} acting on positions i_j and $i_j + 1$, and perform their composition left to right to mirror the composition of permutations.

When $n \geq 4$ the largest possible diameters for a graph $C(w)$, with $w \in \mathcal{S}_n$, are, by decreasing order, $\delta, \delta - n + 2, \delta - 2n + 5, \delta - 2n + 4$, corresponding to permutations with length ℓ, $\ell - 1$ and $\ell - 2$. The longest permutation w_{o} is the only permutation whose graph has length δ. The unimodal permutations $(n-1)n(n-2)\cdots 21$ and $(n-2)(n-1)n(n-3)\cdots 21$, of lengths $\ell - 1$ and $\ell - 2$, are examples of permutations whose graphs have diameters $\delta - n + 2$ and $\delta - 2n + 5$, respectively. Finally, the permutation $w = (n-1)n(n-2)(n-3)\cdots 4312$ has length $\ell - 2$ and the graph $C(w)$ has diameter $\delta - 2n + 4$.

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Theorem 5.3. *The diameter of the commutation graph of an unimodal permutation $w_1 w_2 \cdots w_n$ of $\mathcal{S}_n$ with peak i is* $\binom{n-i+1}{3} + \sum_{k=1}^{i-1} \binom{w_k - k}{2}$.

Proof: If w is unimodal with peak i, then the set T_w is the union of sets $A \cup B$, where A is formed by all triples $(w_{j_3}, w_{j_2}, w_{j_1})$ with $i \leq j_1 < j_2 < j_3 \leq n$, and B is formed by all triples (w_k, w_{j_2}, w_{j_3}) with $k < i < j_2 < j_3 \leq n$ and $w_k > w_{j_2}$. The set A has precisely $\binom{n-i+1}{3}$ elements, while in B for each fixed k there are $\binom{w_k - k}{2}$ triples, since there are precisely $w_k - k$ letters less than w_k after the peak. The result now follows from Theorem 5.1. ∎

Since the longest permutation $w_\circ$ is the only unimodal permutation with peak 1, we recover the following result from [9] for the diameter of $C(w_\circ)$. This is the largest diameter for the commutation graph of a permutation in $\mathcal{S}_n$.

Corollary 5.4. *The diameter of the commutation graph for the longest permutation w_o of $\mathcal{S}_n$ is* $\binom{n}{3}$.

The next proposition will help us to establish the maximal cardinalities that a graph $C(w)$ can have.

Proposition 5.5. *Let $\ell := \ell(w_o)$ be the length of the longest permutation of $\mathcal{S}_n$, $\delta := \binom{n}{3}$ the diameter of $C(w_o)$, and $w \in \mathcal{S}_n$. If $\ell(w) = \ell - k$, then the diameter of $C(w)$ belongs to the interval* $\left[\delta - k(n-2), \delta - k(n-2) + \binom{k}{2}\right]$.

Proof: If $\ell(w) = \ell - k$, then every pair (a, b), with $1 \leq a < b \leq n$ is transposed by w except for k pairs, (a_i, b_i), with $1 \leq i \leq k$. Thus, the set T_w contains every triple (a, b, c), with $1 \leq a < b < c \leq n$, except for the ones for which two of the elements are a_i and b_i. For each i there are $n - 2$ triples of this form, and then, by Theorem 5.1, the diameter of $C(w)$ is at least $\delta - k(n-2)$.

Each two pairs (a_i, b_i) may have a common element, and if this is the case for every two pairs then the number of triples in T_w is $\delta - k(n-2) + \binom{k}{2}$, which imply that the diameter of $C(w)$ is at most $\delta - k(n-2) + \binom{k}{2}$. ∎

For example, the permutation $s_1 s_2 = 231$ acts on 123 as

$$\begin{aligned} s_1 s_2 &= 123 \cdot s_1 s_2 \\ &= 213 \cdot s_2 \\ &= 231. \end{aligned}$$

Another useful notion is the *descent set* of a permutation, which is defined as the set

$$\mathrm{Des}(w) = \{i : w_i > w_{i+1}\}.$$

The elements of the descent set of w are called *descents* and can be used to obtain a reduced word for w. Start with $w^0 = w$ and construct a sequence of permutations

$$w^0, w^1, \ldots, w^{\ell(w)}, \tag{3}$$

with $w^{j+1} = s_{i_j} \cdot w^j$ where i_j is an element of $\mathrm{Des}(w^j)$, for $j \in [n]$. Since $i_j \in \mathrm{Des}(w^j)$, we have $\ell(w^{j+1}) = \ell(w^j) - 1$, and thus the last permutation $w^{\ell(w)}$ in (3) is the identity. Therefore, the sequence $s_{i_{\ell(w)-1}} \cdots s_{i_1} s_{i_0}$ is a reduced word for w.

There are in general, several possibilities for the index $i_j \in \mathrm{Des}(w^j)$ in step j of the procedure above. We denote by $a_{\min}$ (resp. $a_{\max}$) the reduced word for w obtained by choosing in each step j the smallest (resp. greatest) index in $\mathrm{Des}(w^j)$.

Example 2.1. The permutation $w = 25431$ has length $\ell(w) = 7$, corresponding to the inversions $(2,1), (3,1), (4,1), (5,1), (4,3), (5,3)$ and $(5,4)$. We construct the reduced word $a_{\max} = 4321434$ for w using the procedure above, displaying the sequence of permutations in Table 1, where the labels on the left are descents (in this case the largest one) of the permutation in the line above. The action of each adjacent transposition s_i, corresponding to descent i, is illustrated by the numbers in bold.

The *Rothe diagram* of a permutation $w \in \mathcal{S}_n$, denoted by $\mathbb{D}(w)$ is the subset of cells in the first quadrant of the plane defined by

$$\mathbb{D}(w) = \{(i, w_j) : i < j \text{ and } w_i > w_j\} \subseteq [n] \times [n].$$

The cells of $\mathbb{D}(w)$ correspond to inversions in w, namely $(p, q) \in \mathbb{D}(w)$ if and only if (w_p, q) is an inversion of w. Therefore, the Rothe diagram of w gives a graphical representation of the inversion pairs of w. In particular, the number of cells in $\mathbb{D}(w)$ is the length $\ell(w)$ of w.

4	2	5	4	**3**	**1**	w^0
3	2	5	**4**	**1**	3	w^1
2	2	**5**	**1**	4	3	w^2
1	**2**	**1**	5	4	3	w^3
4	1	2	5	**4**	**3**	w^4
3	1	2	**5**	**3**	4	w^5
4	1	2	3	**5**	**4**	w^6
	1	2	3	4	5	w^7

TABLE 1. A reduced word for 25431.

The Rothe diagram for $w = w_1 w_2 ... w_n$ can be obtained by writing w vertically along the y-axis, with w_i at height i, and writing the positive numbers along the x-axis. Then, with this numerical arrangement of rows and columns, place a cell in position (w_j, i) whenever this is an inversion pair for w, for each $i, j \in n$. For instance, the Rothe diagrams for the permutations 25431 and 54321 are given in Figure 1.

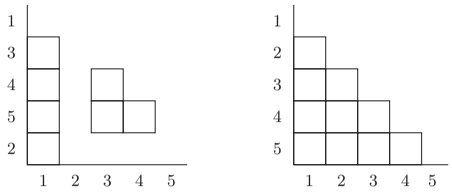

FIGURE 1. Rothe diagrams for 25431 and 54321.

Note that the Rothe diagram of w have the *southwest* property: if (z, y) and (y, x) are cells in $\mathbb{D}(w)$, then (z, x) is also in $\mathbb{D}(w)$. These three cells form a hook with end cells (z, y) and (y, x).

A labelling of the cells of the Rothe diagram for a permutation $w \in \mathcal{S}_n$ with the positive integers in $[\ell(w)]$ is called a *standard balanced tableaux* if for any entry of the diagram, the number of entries to its right that are greater is equal to the number of entries above it that are smaller. Figure 2 shows two standard balanced tableaux on $\mathbb{D}(25431)$ and $\mathbb{D}(54321)$.

Denote the set of all standard balanced tableaux on $\mathbb{D}(w)$ by $\mathrm{SBT}(w)$.

5. Diameter of commutation graphs

We can now give a formula for the diameter of $C(w)$, for any permutation $w \in \mathcal{S}_n$.

Theorem 5.1. *The diameter of $C(w)$ is equal to the cardinality of T_w.*

Proof: If T_w is nonempty, then by Lemma 4.3 we have

$$d([a_{\min}], [a_{\max}]) = |\mathrm{T}_w|.$$

Let $a, b \in \mathrm{R}(w)$ such that $a \notin [b]$. By equations (4) and (5), we have

$$d([a_{\min}], [a]) + d([a], [a_{\max}]) + d([a_{\min}], [b]) + d([b], [a_{\max}]) = 2|\mathrm{T}_w|.$$

Using the triangle inequality, we conclude that

$$\begin{aligned} d([a], [b]) &\leq \min\{d([a_{\min}], [a]) + d([a_{\min}], [b]),\ d([a], [a_{\max}]) + d([b], [a_{\max}])\} \\ &\leq |\mathrm{T}_w|, \end{aligned}$$

proving that the distance between any two commutative classes $[a]$ and $[b]$ is at most $|\mathrm{T}_w|$. Since this number is the largest possible distance between any two classes in $C(w)$, it is the diameter of the graph. ∎

Let $w = w_1 \cdots w_n \in \mathcal{S}_n$ and let $p \in \mathcal{S}_r$, for $r \leq n$. We say that w *contains the pattern* p if there exists a subsequence $w_{i_1} \cdots w_{i_r}$ whose elements are in the same relative order as the elements in p. If w does not contain p, then we say that w *avoids* p, or that w is p-avoiding.

A permutation having only one commutative class is said to be a fully commutative permutation. If $w \in \mathcal{S}_n$ is fully commutative, the distance between any two reduced words for w must be zero, which implies that T_w is the empty set. That is, there is no triple $i < j < k$ with $w_i > w_j > w_k$, *i.e.* w is 321-avoiding. Thus we have recover a result of Billey, Jockusch, and Stanley [1].

Theorem 5.2. *A permutation $w \in \mathcal{S}_n$ is fully commutative if and only if it is* 321*-avoiding.*

A permutation $w = w_1 \cdots w_n$ is *unimodal* if there exists an index i, called the peak, such that $w_1 < w_2 < \cdots < w_i > w_{i+1} > \cdots > w_n$. Unimodal permutations are characterized as avoiding the patterns 312 and 213, and are enumerated by 2^{n-1} [8]. Using Theorem 5.1, we can derive an explicit formula for the diameter of the commutation classes of unimodal permutations.

descent corresponding to the pair (z, y) will appear first, followed by the descent corresponding to (z, x), and finally (y, x). This means that the labels i, j, k of cells (y, x), (z, x) and (z, y) satisfy $i < j < k$, proving that $\Gamma(w, (x, y, z)) = 0$. ∎

Definition 4.4. Let $w \in \mathcal{S}_n$ and $a, b \in \mathrm{R}(w)$. Denote by $\sup(a)$ the set of all triples $t \in \mathrm{T}_w$ for which $\Gamma(a, t) = 1$.

By Lemma 4.3, we have $\sup(a_{\min}) = \emptyset$ and $\sup(a_{\max}) = \mathrm{T}_w$. Moreover, we have $|\sup(a)| = t(a_{\min}, a)$.

Lemma 4.4. *Let $a \in R(w)$ such that $a \notin [a_{\min}]$. Then, there is $b \in R(w)$ such that $[b] \underset{\llcorner}{\sim} [a]$ and $sup(b) \subset sup(a)$.*

Proof: Assume there is no reduced word $b \in \mathrm{R}(w)$ such that $[b] \underset{\llcorner}{\sim} [a]$ and $\sup(b) \subset \sup(a)$. This means that a cannot have a factor $(i+1)i(i+1)$. Therefore, each factor $(i+1)i$ in any word in the class $[a]$ can only be followed by a letter $i+1$ if there is a letter $i+2$ between them. But this implies $T(a, t) = 0$ for all triple $t \in \mathrm{T}_w$. That is, $a = a_{\min}$, contradicting our assumption. ∎

Note that in the condition of the lemma above, we have $\sup(b) = \sup(a) \setminus \{t\}$, for some $t \in \sup(a)$, and thus $d([a_{\min}], [b]) = d([a_{\min}], [a]) - 1$. Therefore, by successive applications of Lemma 4.4 and Proposition 4.2 it follows that

$$d([a_{\min}], [a]) = t(a_{\min}, a) = |\sup(a)|. \tag{4}$$

An analogous result to Lemma 4.4 can be stated for the word $a_{\max}$, that is, if $a \notin [a_{\max}]$, then there is $b \in \mathrm{R}(w)$ such that $[b] \underset{\llcorner}{\sim} [a]$ and $\sup(a) \subset \sup(b)$. It follows that

$$d([a], [a_{\max}]) = t(a, a_{\max}) = |\mathrm{T}_w| - |\sup(a)|. \tag{5}$$

This result shows that the map $t : \mathrm{R}(w) \to [|\mathrm{T}_w|]$ defined by $t(a) = t(a_{\min}, a)$ is a rank function for the graph $C(w)$, making it into a ranked partially ordered set with maximum and minimum. This partial order induces an orientation on $C(w)$.

Proposition 4.5. *Let $w \in \mathcal{S}_n$. The partial order defined on the commutation classes of $C(w)$ given by the transitive closure of covering relations*

$$[a] < [b] \text{ if } a \underset{\llcorner}{\sim} b \text{ and } t(b) = t(a) + 1,$$

makes $C(w)$ into a ranked partially ordered set with a unique minimal element $[a_{\min}]$ and a unique maximal element $[a_{\max}]$.

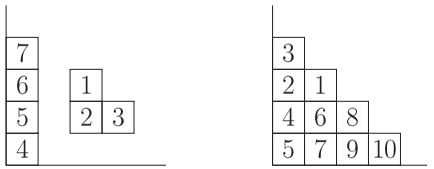

FIGURE 2. Two standard balanced tableaux.

Let $w \in \mathcal{S}_n$ be a permutation of length ℓ and $a = a_1 a_2 \cdots a_\ell$ a reduced decomposition of w. Note that each a_i corresponds to a unique inversion in w, namely the pair of numbers transposed by a_i in the product

$$w = s_{a_1} s_{a_2} \cdots s_{a_\ell},$$

and a is determined uniquely by the order in which these inversions are carried out. Since $(p, q) \in \mathbb{D}(w)$ if and only if (p, q) is an inversion of w, we define the labelling P_a of $\mathbb{D}(w)$ by setting $P_a(p, q) = i$ if a_i transposes p and q, where $p > q$. The tableau P_a is a standard balanced tableau called the *canonical labelling* of $\mathbb{D}(w)$ induced by a.

S. Fomin *et al.* [7] proved that the map $a \mapsto P_a$ defines a bijection between $\mathrm{R}(w)$ and $\mathrm{SBT}(w)$.

Example 2.2. Let $w = 4321 \in S_4$ and let $a = 213213 \in \mathrm{R}(w)$ be a reduced decomposition of w. The action of each a_i is illustrated in following table, from which we get the standard balanced tableau D_a.

	1	2	3	4
$a_1 = 2$	1	**3**	**2**	4
$a_2 = 1$	**3**	**1**	2	4
$a_3 = 3$	3	1	**4**	**2**
$a_4 = 2$	3	**4**	**1**	2
$a_5 = 1$	**4**	**3**	1	2
$a_6 = 3$	4	3	**2**	**1**

$P_a =$

1				
2	6			
3	2	1		
4	4	3	5	
	1	2	3	4

3. The commutation graph of a permutation

We define a relation $\sim$ on the set $\mathrm{R}(w)$ by setting $a \sim b$ if and only if a and b differ by a sequence of commutations. This is an equivalence relation and the classes it defines are the commutation classes of w, denoted by $[a]$. We write

$a \underset{\mathsf{S}}{\sim} b$ (resp. $a \underset{\mathsf{L}}{\sim} b$) when a and b differ by a single commutation (resp. long braid relation), and $[a] \underset{\mathsf{L}}{\sim} [b]$ when those classes differ by a single long braid relation, *i.e.* when exist $a' \in [a]$ and $b' \in [b]$ such that $a' \underset{\mathsf{L}}{\sim} b'$.

Definition 3.1. Let $w \in \mathcal{S}_n$. The *commutation graph* $C(w)$ has vertex set the commutation classes of $\mathrm{R}(w)$, and an edge connecting two classes when they differ by a long braid relation.

Note that since $C(w)$ can be obtained from $G(w)$ by contracting commutation edges, the connectivity of $G(w)$ implies the connectivity of $C(w)$. The *distance* $d([a],[b])$ between two commutation classes $[a]$ and $[b]$ on $C(w)$ is the length of a shortest path joining $[a]$ and $[b]$, that is the number of edges in such path. The *eccentricity* of a class $[a]$ is the distance to a farthest commutative class from $[a]$. The *radius* and *diameter* of $C(w)$ are the minimum and maximum eccentricities, respectively. Figure 3 shows the commutation graph $C(456312)$, which has diameter 6 and radius 3.

Next, following [11], we define analogs of commutation and braid relations for balanced tableaux.

Definition 3.2. Given a permutation w and an integer $1 \leq i < \ell(w)$, the map $\mathfrak{c}_i$ acts on the tableaux in $\mathrm{SBT}(w)$ for which the labels i and $i+1$ are not in the same row nor the same column, by interchanging the labels i and $i+1$.

It easy to check that the map $\mathfrak{c}_i$ is a well-defined, since for $R \in \mathrm{SBT}(w)$, if i and $i+1$ are not in the same row nor the same column, then interchanging them keep the balanced condition true since all other entries compare the same with i and with $i+1$. Thus, $\mathfrak{c}_i(R) \in \mathrm{SBT}(w)$. The map $\mathfrak{c}_i$ is clearly an involution.

Let $a = pa_ia_{i+1}q$ be a reduced decompositions of $w \in \mathcal{S}_n$ with $|a_i - a_{i+1}| > 1$. Then, a_i and a_{i+1} correspond to inversions on disjoint set of integers, say (y,x) and (k,z), with $x < y$ and $z < k$. This means that the cells (y,x) and (k,w) of tableau P_a are in distinct rows and columns. Swapping the labels i and $i+1$ of these cells gives the tableau $P_{a'}$, where $a' = pa'_ia'_{i+1}q \underset{\mathsf{S}}{\sim} a$, with $a'_i = a_{i+1}$ and $a'_{i+1} = a_i$. That is, we have $a \underset{\mathsf{S}}{\sim} a'$ if and only if $\mathfrak{c}_i(P_a) = P_{a'}$.

Definition 3.3. Given a permutation w and an integer $1 < i < \ell(w)$, the map $\mathfrak{b}_i$ acts on the tableaux in $\mathrm{SBT}(w)$ having one of the labels $i-1$ or $i+1$ in the same column and above i and the other in the same row and right of i, by interchanging the labels $i-1$ and $i+1$.

Assume now that $t([a],[b]) = 1$, and let $(x,y,z) \in \mathrm{T}_w$ be the only triple having distinct Γ-values for a and b. Using the same argument of the proof of Proposition 4.1, we may assume that all cells of P_a and P_b have the same value, with the exception of cells in positions (z,y), (z,x) and (y,x). Suppose $T(a,(x,y,z)) = 0$ and $T(b,(x,y,z)) = 1$. Then, we must have $P_a(y,x) = i$, $P_a(z,x) = j$ and $P_a(z,y) = k$, with $i < j < k$, while $P_b(y,x) = k$, $P_b(z,x) = j$ and $P_b(z,y) = i$. We will show that if $j > i+1$ we can use commutative relations to swap the integer j with the integer $i+1$ in P_a.

When $j > i+1$ the permutation $a_1 \cdots a_{i-1}$ has the factor xy since $P_a(y,x) = i$, and $b_1 \cdots b_{i-1}$ has the factor yz since $P_b(z,y) = i$. Since these two permutations coincide, we conclude that xyz is a factor in this permutation, with $x < y < z$. If $j-1$ is in row z of P_a, and also P_b, say in position (z,u), then the permutation $a_1 \cdots a_i \cdots a_{j-1}a_j$ implies the inversion of (u,x) and (u,y) by the action of $a_{i+1} \cdots a_{j-2}$. But then, we cannot have $P_b(z,u) = j-1$, contradicting our assumption. Thus, $j-1$ cannot be in row z of P_a. The same argument shows that $j-1$ cannot be in column x of P_a. Therefore, we can swap the integers j and $j-1$ using the map $\mathfrak{b}_j$, corresponding to a commutation relation in a. Repeating the argument, we may assume that $j = i+1$, and by Lemma 3.1 we also have $k = i+2$.

Therefore, we may use commutation relations to find reduced words $a', b' \in \mathrm{R}(w)$ such that $a \underset{\mathsf{S}}{\sim} a'$ and $b \underset{\mathsf{S}}{\sim} b'$, where all cells of $P_{a'}$ and $P_{b'}$ have the same value, with the exception of cells in positions (z,y), (z,x) and (y,x), where we have $P_{a'}(y,x) = i$, $P_{a'}(z,x) = i+1$, $P_{a'}(z,y) = i+2$, and $P_{b'}(y,x) = i+2$, $P_{a'}(z,x) = i+1$, $P_{a'}(z,y) = i$. It follows that $\mathfrak{b}_{i+1}(P_{a'}) = P_{b'}$, that is $d([a],[b]) = 1$. ∎

Proposition 4.2 shows that each triple in T_w where two reduced decompositions $a, b \in \mathrm{R}(w)$ have distinct Γ-values only for this triple, corresponds to a long braid relation between a and b, or equivalently, to a mapping $\mathfrak{b}_i$ of P_a into P_b. Also, it follows that $t([a],[b]) \leq d([a],[b])$ for any $a, b \in \mathrm{R}(w)$.

Lemma 4.3. *The reduced word $a_{\min}$ (resp. $a_{\max}$) for $w \in \mathcal{S}_n$ have Γ-value equal to 0 (resp. 1), for all triples in T_w.*

Proof: We prove the result for $a_{\min}$ only. The other case is analogous. Let (x,y,z), $x < y < z$, be a triple in T_w and consider cells (y,x), (z,x) and (z,y) in the tableau $P_{a_{\min}}$. Since these cells correspond to inversions in w, the word zyx is a subword of w. Thus, in the process of construction $a_{\min}$, the

same row nor the same column. If one of these corresponds to a cell of a triple $(x, y, z) \in \mathrm{T}_w$, then the change of the label does not modify the relative values of the end points of the cells (z, y), (z, x) and (y, x). It follows that $\Gamma(a, (x, y, z)) = \Gamma(b, (x, y, z))$ for all $(x, y, z) \in \mathrm{T}_w$.

Reciprocally, suppose the Γ-value of a and b is the same for all triples in T_w. Assume all labels $1, \ldots, i-1$ are in the same cells in both tableaux P_a and P_b, and $P_a(z, y) = i$ and $P_b(z, y) = i + k$ for some $k \geq 1$. Then, the permutation associated with $a_1 \cdots a_{i-1} = b_1 \cdots b_{i-1}$ has the factor yz.

If the label $i+k-1$ in P_b is in row z, say in cell (z, x), then the permutation associated with $b_1 \cdots b_{i+k-1}$ has the factor yxz. This means that (y, x) is also an inversion for w, and therefore (x, y, z) is a inversion triple with $\Gamma(a, (x, y, z)) = 1$ and $\Gamma(b, (x, y, z)) = 0$, contradicting our assumption. The same reasoning shows that the label $i + k - 1$ cannot be in column y of P_b, and therefore it must be in a cell which is not in row z nor in column x. It follows that we can exchange the labels $i + k$ and $i + k - 1$ using a commutation. That is $\mathfrak{c}_{i+k}(P_b) = P_{b'}$, such that all cells $1, \ldots, i$ are in same cells in both P_a and $P_{b'}$. Repeating the argument, there is a sequence of integers $i_1, \ldots, i_j$ such that $\mathfrak{c}_{i_1} \cdots \mathfrak{c}_{i_j}(P_b) = P_a$, showing that a and b are in the same commutation class. ∎

Definition 4.3. Given $a, b \in R(w)$, let

$$t(a, b) = \sum_{(x,y,z) \in \mathrm{T}_w} \Gamma(a, (x, y, z)) \oplus_2 \Gamma(b, (x, y, z)),$$

where $\oplus_2$ represents the sum modulo 2.

The number $t(a, b)$ gives the number of triples in T_w for which the Γ-value of a and b are distinct. Note that by Proposition 4.1, we have $t(a, b) = t(a', b')$ for any $a' \in [a]$ and $b' \in [b]$. In particular, if a and b are in the same commutation class we get $t(a, b) = 0$.

Proposition 4.2. *Let $a, b \in R(w)$. Then, $d([a], [b]) = 1$ if and only if $t([a], [b]) = 1$.*

Proof: If $d([a], [b]) = 1$, then we may assume without loss of generality that $a \underset{\mathsf{L}}{\sim} b$, which means that $\mathfrak{b}_i(P_a) = P_b$ for some integer i. It follows from Lemma 3.1 that the only triple having distinct Γ-values for a and b is (x, y, z), $x < y < z$, corresponding to the cells (z, y), (z, x) and (y, x) having labels $i - 1$, i, and $i + 1$, and therefore $t([a], [b]) = 1$.

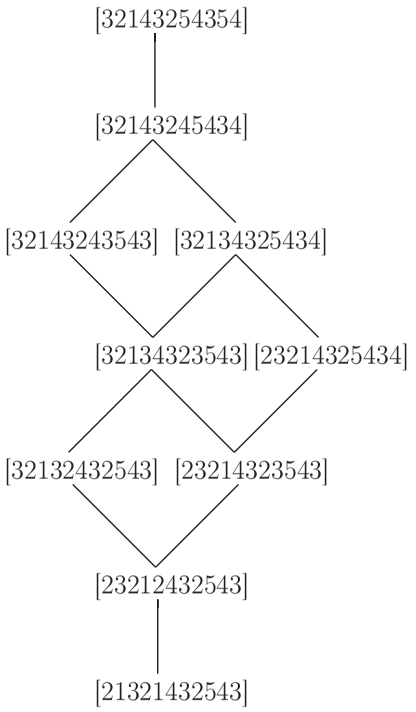

FIGURE 3. The graph $C(456312)$.

For $R \in \mathrm{SBT}(w)$, if $i \pm 1$ is in the same row as i, and $i \mp 1$ is in the same column, then swapping them maintains the tableaux balanced since all entries $j \notin \{i-1, i, i+i\}$ compares the same with both integers $i \pm 1$. Thus, the map $\mathfrak{b}_i$ is an involution.

Let $a = pa_{i-1}a_ia_{i+1}q$ be a reduced decompositions of $w \in \mathcal{S}_n$, with $a_{i-1} = a_{i+1} = a_i \pm 1$. Then, a_{i-1}, a_i and a_{i+1} correspond to inversions (y, x), (z, x) and (z, y), with $x < y < z$. Interchanging the labels $i+1$ and $i-1$ in P_a gives the tableau $P_{a'}$, where $a = pa'_{i-1}a'_ia'_{i+1}q$ with $a'_{i-1} = a'_{i+1} = a_i$ and $a'_i = a_{i+1}$, showing that $a \underset{\mathsf{L}}{\sim} a'$ if and only if $\mathfrak{b}_i(P_a) = P_{a'}$.

Figure 4 shows the action of the maps $\mathfrak{c}_i$ and $\mathfrak{b}_j$ on some tableaux in SBT(456312) for the classes of the bottom three levels of graph $C(456312)$ depicted in Figure

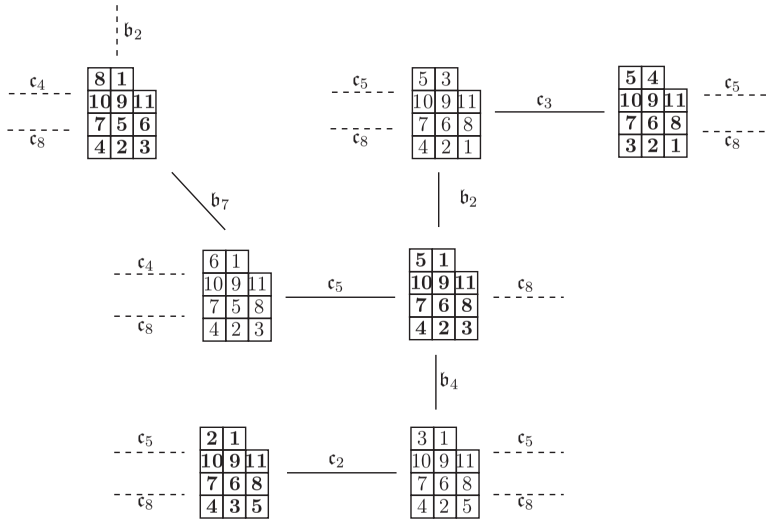

FIGURE 4. Examples of the action of maps $\mathfrak{c}_i$ and $\mathfrak{b}_j$.

3. The tableaux in bold font correspond to the words represented in the graph, and the dashed lines correspond to maps that can also act in the tableaux.

We end this section with a result that will be useful in the sequel.

Lemma 3.1. *Suppose $R \in SBT(w)$ has labels $i-1$ and i in positions (z,y) and (z,x), respectively, with $x < y < z$. Then, R cannot have a label $i+1$ in position (y',x) with $y \neq y'$.*

Proof: Let $R = P_a$, with $a = a_1 \cdots a_\ell$ a reduced decomposition of w. Then, the factor $a_{i-1}a_i$ of a acts on the factor xyz of $a_1 \cdots a_{i-2}$ transforming it into zxy. It follows that there cannot be a label $i+1$ in position $(y'x)$ of P_a with $y \neq y'$, since it does not correspond to an inversion of integers in consecutive positions of the word $a_1 \cdots a_i$. ∎

4. A statistic on $C(w)$

In this section we define a statistic on $C(w)$ which allows the computation of the diameter of the commutation graph of any permutation of the symmetric group.

Definition 4.1. Given a permutation $w \in \mathcal{S}_n$, define the set of *inversion triples*

$$\mathrm{T}_w = \{(w_k, w_j, w_i) : \ w_i > w_j > w_k \text{ and } i < j < k\},$$

form by all triples (x,y,z) in $[n]$ such that (z,y), (z,x) and (y,x) are inversions for w.

Note that each triple (x,y,z) in T_w corresponds to the endpoints cells (z,y), (z,x) and (y,x) of a hook in the Rothe diagram of w. For instance, for the permutation $456312 \in S_6$ we have

$$\mathrm{T}_{456312} = \{(1,3,4),(1,3,5),(1,3,6),(2,3,4),(2,3,5),(2,3,6)\},$$

while the set T_{w_o}, for the longest permutation $w_o = n \cdots 21$ of $\mathcal{S}_n$ is formed by all $\binom{n}{3}$ triples (x,y,z).

Definition 4.2. Given a permutation $w \in \mathcal{S}_n$, define the map Γ on the cartesian product $\mathrm{R}(w) \times T_w$ by setting

$$\Gamma(a,(x,y,z)) = \begin{cases} 1, & \text{if } P_a(y,x) > P_a(z,y) \\ 0, & \text{if } P_a(y,x) < P_a(z,y) \end{cases}.$$

In other words, $\Gamma(a,(x,y,z))$ is 0 whenever the inversion of the pair (y,x) occur before the inversion of the pair (z,y) in the process of transforming the identity into the permutation w by the action of each adjacent transposition corresponding to the letters of $a = a_1 a_2 \cdots a_{\ell(w)}$. Note also that since the subword xyz of the identity is transformed into the subword zyx of w by the action of the letters of a, we must have $P_a(y,x) > P_a(z,x) > P_a(z,y)$ when $\Gamma(a,(x,y,z)) = 1$, and $P_a(z,y) > P_a(z,x) > P_a(y,x)$ when $\Gamma(a,(x,y,z)) = 0$. For instance, the analysis of the diagram in Example 2.2 for the reduced word $a = 213213$ of the longest permutation of $\mathcal{S}_4$ shows that $\Gamma(a,(123)) = 1$, $\Gamma(a,(124)) = 1$, $\Gamma(a,(134)) = 0$ and $\Gamma(a,(234)) = 0$.

In [9] it is shown that the map Γ is invariant for the commutation classes of $C(w_o)$. We generalize this result for any permutation w.

Proposition 4.1. *Two reduced words $a, b \in R(w)$ are in the same commutation class if and only if $\Gamma(a,(x,y,z)) = \Gamma(b,(x,y,z))$, for all triple $(x,y,z) \in T_w$.*

Proof: If a and b are in the same commutation class, we may assume without loss of generality that there is an integer $i \in [\ell(w)-1]$ such that $\mathfrak{c}_i(P_a) = P_b$. The map $\mathfrak{c}_i$ changes the labels i and $i+1$ of two cells that are not in the