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THE INVERSE OF A TRIDIAGONAL k-TOEPLITZ MATRIX

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ABSTRACT: A square matrix is k-Toeplitz if its diagonals are periodic sequences of period k. We find rational formulas for the determinant, the characteristic polynomial, and the elements of the inverse of a tridiagonal k-Toeplitz matrix (in particular, of any tridiagonal matrix) over any commutative unital ring, using only elementary linear algebra.

KEYWORDS: Tridiagonal k-Toeplitz matrix, determinant, inverse, characteristic polynomial, closed-form formulas.
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1. Introduction

Given a commutative unital ring K, a matrix $A \in \mathrm{M}_n(K)$ is tridiagonal k-Toeplitz if it is tridiagonal and the entries along the main diagonal and its adjacent diagonals are periodic sequences of period k, so that it has the form

$$A = \begin{pmatrix} a_1 & b_1 & & & & & & & & \\ c_1 & \ddots & \ddots & & & & & & & \\ & \ddots & a_k & b_k & & & & & & \\ & & c_k & a_1 & b_1 & & & & & \\ & & & c_1 & \ddots & \ddots & & & & \\ & & & & \ddots & a_k & b_k & & & \\ & & & & & c_k & a_1 & b_1 & & \\ & & & & & & c_1 & \ddots & \ddots & \\ & & & & & & & \ddots & & \end{pmatrix}.$$

Tridiagonal matrices appear frequently in many areas of pure and applied mathematics (see [21]); consequently, tridiagonal k-Toeplitz matrices appear in those contexts when some periodicity of the studied problem or physical system

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is assumed. For example, they appear in the discretization of elliptic or parabolic partial differential equations by finite difference methods ([11, 20, 23]), in classical mechanics ([22]), in chain models of quantum physics ([4]), in sound propagation theory ([6, 7]), in telecommunication system analysis ([16]), and in circuit models of wireless power transfer arrays ([1–3]). Thus the eigenvalues or the inverse of the associated matrix can be invoked to solve these problems. The case of a general complex tridiagonal matrix has been independently studied and solved many times since Egerváry and Szász's paper [9] of 1928, through different methods yielding more or less explicit formulas (see [12]). In particular, Mallik in [17] gave rational formulas for the elements of the inverse of a (nonsingular) complex tridiagonal matrix through the solution of a second-order linear inhomogeneous difference equation with variable coefficients.

The eigenproblem of a complex tridiagonal k-Toeplitz matrix was studied in small cases by Gover, and Marcellán and Petronilho ([15, 18, 19]), and a solution for the general case was given by da Fonseca and Petronilho ([14]) using tools from the theory of orthogonal polynomials. In the same paper [14] from 2005, formulas were given for the elements of the inverse of a complex tridiagonal k-Toeplitz matrix (small cases having been considered previously in [13]). These formulas are not completely in closed form, in the sense that they depend on polynomial maps of determinants that must be computed in each case. Encinas and Jiménez in 2018 ([10]), in the slightly more general context of tridiagonal (k, r)-Toeplitz matrices (see [5]), and through an elaborate use of the discrete Schrödinger operator and the Chebyshev functions, provided closed-form formulas for the elements of the inverse of a real tridiagonal k-Toeplitz matrix in the special case in which the period k divides the order n of the matrix. Actually, Wittenburg in [22, Section 4] already in 1998 produced rational formulas for the elements of the inverse of a complex tridiagonal k-Toeplitz matrix, although relating them to determinants of tridiagonal matrices of order less than k, which he did not compute explicitly. Wittenburg computes the inverse from the adjugate matrix, relates the cofactors to some determinants, and then solves, with some effort, an associated linear difference equation through the roots of an associated polynomial. He needs to study two cases separately (the defective and nondefective cases, say), depending on the multiplicity of the roots.

The second author, in [1], devised an elementary linear algebra algorithm to compute the elements of the inverse of a complex k-Toeplitz matrix with equal and constant upper and lower diagonals in the nondefective case, yielding rational formulas; with this algorithm the elements of the inverse are written as a product of determinants of tridiagonal k-Toeplitz matrices, but those determinants need to be computed a posteriori for each k, which the algorithm does through the diagonalization of an associated 2×2 matrix. In this paper we generalize said method to give actual closed-form, rational formulas for the determinant, the characteristic polynomial and the entries of the inverse of any tridiagonal k-Toeplitz matrix over any commutative unital ring, using only elementary linear algebra arguments, which in our opinion are the simplest to date.

In the first place we give an explicit polynomial formula for the determinant by generalizing the formulas from Mallik ([17]). This we achieve by solving a second-order linear difference equation with variable periodic coefficients (which appears from applying Laplace's expansion twice) by writing it as a 2×2 matrix difference equation and applying recursion, and then applying induction to prove the correctness of the associated polynomial formulas. Implicitly, we are actually computing the determinant of a generic tridiagonal k-Toeplitz matrix and then evaluating it into the commutative unital ring. The characteristic polynomial arises as a special case of the determinant of a tridiagonal k-Toeplitz matrix. Finally, the elements of the inverse are computed from the adjugate matrix: the submatrix associated to a cofactor is block-triangular with three diagonal blocks, the middle one triangular, the two in the extremes again tridiagonal k-Toeplitz matrices, allowing to write the cofactor from the product of their determinants, which we already know how to compute. In this last part our ideas are close to that of Wittenburg ([22]). We finish the paper with an example implementing the results.

2. Determinant, characteristic polynomial, and inverse

Let us quickly establish some conventions. Throughout this paper let K be any commutative unital ring. For us $0 \in \mathbb{N}$, $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$, an empty summation yields 0 and an empty product yields 1. The notation $\lfloor \cdot \rfloor$ stands for the floor function from $\mathbb{Q}$ to $\mathbb{Z}$. The notation $\binom{i}{j}$ with $i, j \in \mathbb{N}$ stands for the image of the corresponding binomial coefficient under the canonical homomorphism from $\mathbb{Z}$ to K, understanding $\binom{i}{j} = 0$ when $i < j$. For $k \in \mathbb{N}^*$, $\overline{a} := (a_1, \ldots, a_k)$ describes a vector of K^k. For $n \in \mathbb{N}^*$, $\mathrm{M}_n(K)$ denotes the ring of square matrices of order n over K, $I \in \mathrm{M}_n(K)$ denotes the identity matrix, and $\mathrm{tr}(A), \det(A)$ respectively denote the trace and the determinant of $A \in \mathrm{M}_n(K)$.

Definition 2.1 (Tridiagonal k-Toeplitz matrix).
Given $k, n \in \mathbb{N}^*$ and $\overline{a} := (a_1, \ldots, a_k), \overline{b} := (b_1, \ldots, b_k), \overline{c} := (c_1, \ldots, c_k) \in K^k$, let

$$T_n := T_n(\overline{a}; \overline{b}; \overline{c}) \in \mathrm{M}_n(K), \quad T_n = (t_{ij})_{i,j=1}^n,$$

be the tridiagonal matrix ($t_{ij} := 0$ for all i, j such that $|i - j| \geq 2$) such that $t_{ii} := a_i$, $t_{i,i+1} := b_i$, $t_{i+1,i} := c_i$ for $1 \leq i \leq k$, and $t_{i+k,j+k} := t_{ij}$ for $1 \leq i, j \leq n - k$.

In Theorem 2.11 we are going to give a polynomial formula for the determinant of a tridiagonal k-Toeplitz matrix, which will be found by solving a 2×2 matrix difference equation. To do so we need a formula for the powers of a 2×2 matrix in terms of the original matrix:

Definition 2.2. Given $t, d \in K$ and $m \in \mathbb{N}$ we define

$$s_m(t, d) := \sum_{i=0}^{\lfloor (m-1)/2 \rfloor} (-1)^i \binom{m-1-i}{i} t^{m-1-2i} d^i.$$

Lemma 2.3. *If $A \in M_2(K)$ then, for all $m \in \mathbb{N}^*$,*

$$A^m = s_m(\mathrm{tr}(A), \det(A))A - \det(A) s_{m-1}(\mathrm{tr}(A), \det(A))I.$$

Proof: We proceed by induction. Denote

$$t := \mathrm{tr}(A), \quad d := \det(A) \quad \text{and} \quad s_m := s_m(t, d).$$

The base case $A = A$ is true since $s_1 = 1$, $s_0 = 0$. By the Cayley-Hamilton theorem $A^2 = tA - dI$, so if $A^m = s_m A - d s_{m-1} I$ then

$$A^{m+1} = AA^m = s_m A^2 - d s_{m-1} A = (t s_m - d s_{m-1})A - d s_m I.$$

Now

$$\begin{aligned}
&ts_m - ds_{m-1} = \\
&= \sum_{i=0}^{\lfloor (m-1)/2 \rfloor} (-1)^i \binom{m-1-i}{i} t^{m-2i} d^i + \sum_{i=0}^{\lfloor (m-2)/2 \rfloor} (-1)^{i+1} \binom{m-2-i}{i} t^{m-2-2i} d^{i+1} = \\
&= t^m + \sum_{i=1}^{\lfloor (m-1)/2 \rfloor} (-1)^i \binom{m-1-i}{i} t^{m-2i} d^i + \sum_{i=1}^{\lfloor m/2 \rfloor} (-1)^i \binom{m-1-i}{i-1} t^{m-2i} d^i = \\
&= t^m + \sum_{i=1}^{\lfloor m/2 \rfloor} (-1)^i \left(\binom{m-1-i}{i} + \binom{m-1-i}{i-1} \right) t^{m-2i} d^i = \\
&= \sum_{i=0}^{\lfloor m/2 \rfloor} (-1)^i \binom{m-i}{i} t^{m-2i} d^i = s_{m+1},
\end{aligned}$$

since $\lfloor (m-1)/2 \rfloor = \lfloor m/2 \rfloor$ when m is odd and $\binom{m-1-\lfloor m/2 \rfloor}{\lfloor m/2 \rfloor} = 0$ when m is even. Therefore $A^{m+1} = s_{m+1}A - ds_m I$, as we needed to show. ■

Remarks 2.4 (**Powers through the eigenvalues**).
In this remark let K be a field.

(1) Given $A \in \mathrm{M}_2(K)$ we can also express $s_m(\mathrm{tr}(A), \det(A))$ in terms of its eigenvalues λ_1, λ_2 (may be equal) in an algebraic closure $\overline{K}$ of K. It is easy to show by induction that, for $m \in \mathbb{N}^*$,

$$s_m(\mathrm{tr}(A), \det(A)) = \sum_{i=0}^{m-1} \lambda_1^i \lambda_2^{m-i-1}. \tag{A}$$

Thus by Lemma 2.3 (taking into account that $\det(A) = \lambda_1\lambda_2$) we can express A^m in terms of the eigenvalues. This choice makes the formula dependent on the characteristic $\mathrm{char}(K)$ of the field: if $\mathrm{char}(K) \neq 2$, the eigenvalues of $A \in \mathrm{M}_2(K)$ can be found from the characteristic polynomial by the quadratic formula, but when $\mathrm{char}(K) = 2$ the roots of $X^2 + aX + b \in K[X]$ cannot be expressed by radicals when the polynomial is irreducible over K and $a \neq 0$, and a different approach is taken (see e.g. [8, Exercise 2.4.6]).

(2) Formula (A) can be simplified as follows: if A is nondefective ($\lambda_1 \neq \lambda_2$) then
$$s_m(\mathrm{tr}(A), \det(A)) = \frac{\lambda_2^m - \lambda_1^m}{\lambda_2 - \lambda_1},$$
while if A is defective ($\lambda_1 = \lambda_2 =: \lambda$) then $s_m(\mathrm{tr}(A), \det(A)) = m\lambda^{m-1}$. Defectiveness is easy to detect: if $\mathrm{char}(K) = 2$, the matrix $A \in \mathrm{M}_2(K)$ is defective if and only if $\mathrm{tr}(A) = 0$, i.e., if and only if the characteristic polynomial is of the form $X^2 + \det(A)$ (with only eigenvalue $\sqrt{\det(A)} \in \overline{K}$). If $\mathrm{char}(K) \neq 2$, by the quadratic formula the matrix is defective if and only if $\mathrm{tr}(A)^2 - 4\det(A) = 0$ (with only eigenvalue $\mathrm{tr}(A)/2$). In any case, the matrix A is defective if and only if $\mathrm{tr}(A)^2 - 4\det(A) = 0$.

In case K is not a field, formula (A) still holds if λ_1, λ_2 are two eigenvalues in some overring $\overline{K}$ such that the characteristic polynomial of A equals $(X - \lambda_1)(X - \lambda_2)$ in $\overline{K}[X]$, the simplification in the defective case can always be done, and the simplification in the nondefective case can be done when $\lambda_2 - \lambda_1$ is a unit of $\overline{K}$.

In what follows we define the polynomials which will feature in the formula for the determinant. Informally speaking, given variables $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$, to build the polynomial $p_{r,k}(i_1, \ldots, i_m)$ we start with the product $x_1 \cdots x_r$ and then for each index i_j we substitute two consecutive x variables in the product, x_{i_j} and x_{i_j+1}, by the corresponding y variable y_{i_j} (so a y variable "weights" like two x variables), even cyclically: x_r and x_1 can be substituted together, but with the caveat that they are not substituted by y_r, but by y_k. The indices are taken so that the consecutive substitutions they imply are indeed possible. Then the polynomial $\pi(r, k)$ is the sum of all $p_{r,k}$ polynomials for all possible indices, the polynomial $\alpha(r, k)$ is the sum of those $p_{r,k}$ which do not have the variable y_k, and the polynomial $\beta(r, k)$ is the sum of those $p_{r,k}$ which do have the variable y_k .

Definitions 2.5. Given $r \in \mathbb{Z}$ we denote $[r] := \{1, \ldots, r\}$ if $r \geq 1$, $[r] := \emptyset$ otherwise. For a finite set $S \subseteq \mathbb{N}^*$, by $\binom{S}{m}_2$ with $m \in \mathbb{N}^*$ we denote the set of all m-combinations of the set satisfying $|s - t| \geq 2$ for all $s, t \in S$, and by $\binom{S}{m}_{2c}$ the subset which applies this rule also cyclically, i.e., the subset of $\binom{S}{m}_2$ which excludes those combinations including both $\min(S)$ and $\max(S)$. For example
$$\binom{[7]}{3}_{2c} = \{(1,3,5), (1,3,6), (1,4,6), (2,4,6), (2,4,7), (2,5,7), (3,5,7)\}.$$

We also denote $\binom{S}{0}_2 := \{0\}$ and $\binom{S}{0}_{2c} := \{0\}$ (even if S is empty). Given $k, r \in \mathbb{N}^*$ with $r \leq k+1$ and considering the ring $R := \mathbb{Z}[x_1, \ldots, x_k, x_{k+1}, y_1, \ldots, y_k]$, we denote

$$x'_i := x_i \text{ for } 1 \leq i \leq r,\ x'_{r+1} := x_1,$$
$$y'_i := y_i \text{ for } 1 \leq i < r,\ y'_r := y_k,$$

and define the monomial of R (computed inside the ring $\mathbb{Z}(x_1, \ldots, y_k)$)

$$p_{r,k}(i_1, \ldots, i_m) := x_1 \cdots x_r \frac{y'_{i_1} \cdots y'_{i_m}}{x'_{i_1} x'_{i_1+1} \cdots x'_{i_m} x'_{i_m+1}} \tag{2.6}$$

for $(i_1, \ldots, i_m) \in \binom{[r]}{m}_{2c}$ with $1 \leq m \leq \lfloor r/2 \rfloor$. With the same formula (2.6) and defining $x'_0 := 1, y'_0 := x_1$ we also extend the definition of $p_{r,k}(i_1, \ldots, i_m)$ to the case $i_1 = 0$, $(i_2, \ldots, i_m) \in \binom{[r]}{m-1}_{2c}$ (the second condition holding when $m > 1$). So we have

$$p_{r,k}(0) = x_1 \cdots x_r, \quad p_{r,k}(0, i) = p_{r,k}(i) \text{ for } i \in \binom{[r]}{m}_{2c}, m \geq 1.$$

In addition we define $p_{0,k}(0) := 1$.
For example,

$$p_{6,8}(3) = x_1 x_2 y_3 x_5 x_6,\ p_{6,8}(1,5) = y_1 x_3 x_4 y_5,$$
$$p_{6,8}(6) = x_2 x_3 x_4 x_5 y_8,\ p_{6,6}(3,6) = x_2 y_3 x_5 y_6,\ p_{7,6}(3,7) = x_2 y_3 x_5 x_6 y_6,$$
$$p_{3,4}(0) = x_1 x_2 x_3,\ p_{3,4}(0,3) = p_{3,4}(3) = x_2 y_4.$$

Now, for fixed $0 \leq r \leq k$ we denote in R the sum of all the $p_{r,k}$ by $\pi(r,k)$,

$$\pi(r,k) := \sum_{m=0}^{\lfloor r/2 \rfloor} \sum_{i \in \binom{[r]}{m}_{2c}} p_{r,k}(i), \tag{2.7}$$

the sum of those $p_{r,k}$ having degree 0 in y_k by $\alpha(r,k)$,

$$\alpha(r,k) := \sum_{m=0}^{\lfloor r/2 \rfloor} \sum_{i \in \binom{[r-1]}{m}_2} p_{r,k}(i), \tag{2.8}$$

and the sum of those $p_{r,k}$ having degree 1 in y_k by $\beta(r,k)$,

$$\beta(r,k) := \sum_{m=0}^{\lfloor (r-2)/2 \rfloor} \sum_{i \in \binom{[r-2]-\{1\}}{m}_2} p_{r,k}(i,r). \tag{2.9}$$

We extend the definition to $\beta(k+1,k)$ through formula (2.9).
Note that $\pi(0,k) = 1$, $\alpha(0,k) = 1$, $\beta(0,k) = 0 = \beta(1,k)$ and that, for $0 \le r \le k$, $\pi(r,k) = \alpha(r,k) + \beta(r,k)$. For example we have

$$\begin{aligned}
\pi(4,6) =& x_1x_2x_3x_4 + y_1x_3x_4 + x_1y_2x_4 + x_1x_2y_3 + x_2x_3y_6 + y_1y_3 + y_2y_6,\\
\alpha(4,6) =& x_1x_2x_3x_4 + y_1x_3x_4 + x_1y_2x_4 + x_1x_2y_3 + y_1y_3,\ \beta(4,6) = x_2x_3y_6 + y_2y_6,\\
\beta(6,5) =& x_2x_3x_4x_5y_5 + y_2x_4x_5y_5 + x_2y_3x_5y_5 + x_2x_3y_4y_5 + y_2y_4y_5.
\end{aligned}$$

Now, given elements $\overline{a} := (a_1, \ldots, a_k), \overline{d} := (d_1, \ldots, d_k) \in K^k$, we define $p_{r,k}^{\overline{a},\overline{d}}(i_1, \ldots, i_m)$ as the image of the evaluation of $p_{r,k}(i_1, \ldots, i_m)$ to K^k mapping $x_i \mapsto a_i$, $y_i \mapsto d_i$ for $1 \le i \le k$, and $x_{k+1} \mapsto a_1$. Analogously we define $\pi^{\overline{a},\overline{d}}(r,k)$, $\alpha^{\overline{a},\overline{d}}(r,k)$ and $\beta^{\overline{a},\overline{d}}(r,k)$.

Lemma 2.10. *Given $k \in \mathbb{N}^*$, in $\mathbb{Z}[x_1, \ldots, x_{k+1}, y_1, \ldots, y_k]$ we have*

$$\begin{aligned}
\alpha(r+1,k) =& x_{r+1}\alpha(r,k) + y_r\alpha(r-1,k) \text{ for } 1 \le r \le k-1,\\
\beta(r+1,k) =& x_r\beta(r,k) + y_{r-1}\beta(r-1,k) \text{ for } 2 \le r \le k.
\end{aligned}$$

Proof: We state the proof for the α polynomials, for the β polynomials is similar. For $r = 1$ we have $\alpha(2,k) = x_1x_2 + y_1$, $\alpha(1,k) = x_1$, $\alpha(0,k) = 1$, so indeed $\alpha(2,k) = x_2\alpha(1,k) + y_1\alpha(0,k)$. For $2 \le r \le k-1$ consider

$$\alpha(r+1,k) = \sum_{m=0}^{\lfloor (r+1)/2 \rfloor} \sum_{i \in \binom{[r]}{m}_2} p_{r+1,k}(i)$$

and fix some $p := p_{r+1,k}(i_1, \ldots, i_m)$ appearing as a term in the above expression of $\alpha(r+1,k)$, with indices rearranged so that $i_1 < \ldots < i_m$. Since $i_m < r+1$ (so y_k is not a factor of p), we have that either

- $i_m < r$, x_{r+1} is a factor of p and y_r is not, whence $p = x_{r+1}p_{r,k}(i_1, \ldots, i_m)$, or
- $i_m = r$, y_r is a factor of p and x_{r+1} is not, whence $p = y_rp_{r-1,k}(i_1, \ldots, i_{m-1})$ if $m \ge 2$ and $p = y_rp_{r-1,k}(0)$ if $m = 1$.

Denote

$$S_1(m) := \{(i_1, \dots, i_m) \in \binom{[r]}{m}_2 \mid i_1, \dots, i_m < r\},$$

$$S_2(m) := \{(i_1, \dots, i_{m-1}, r) \in \binom{[r]}{m}_2\}$$

and observe that $\binom{[r]}{m}_2$ is the disjoint union of S_1 and S_2 for $0 \le m \le \lfloor (r+1)/2 \rfloor$. We have $S_1(m) = \binom{[r-1]}{m}_2$ for $0 \le m \le \lfloor (r+1)/2 \rfloor$ and $S_2(m) = \{(i, r) \mid i \in \binom{[r-2]}{m-1}_2\}$ for $2 \le m \le \lfloor (r+1)/2 \rfloor$; in addition, since $p_{r+1,k}(0, r) = p_{r+1,k}(r) = y_r p_{r-1,k}(0)$, we can substitute $S_2(1)$ by $\{(i, r) \mid i \in \binom{[r-2]}{0}_2\}$. Therefore

$$\begin{aligned}
\alpha(r+1, k) = & \sum_{m=0}^{\lfloor r/2 \rfloor \, (*)} \sum_{i \in \binom{[r-1]}{m}_2} x_{r+1} p_{r,k}(i) + \sum_{m=1 \, (\star)}^{\lfloor (r+1)/2 \rfloor} \sum_{i \in \binom{[r-2]}{m-1}_2} y_r p_{r-1,k}(i) = \\
& = x_{r+1} \sum_{m=0}^{\lfloor r/2 \rfloor} \sum_{i \in \binom{[r-1]}{m}_2} p_{r,k}(i) \; + \; y_r \sum_{m=0}^{\lfloor (r-1)/2 \rfloor \, (\dagger)} \sum_{i \in \binom{[r-2]}{m}_2} p_{r-1,k}(i) = \\
& = x_{r+1} \alpha(r, k) + y_r \alpha(r-1, k),
\end{aligned}$$

with $(*)$ being true since $\lfloor (r+1)/2 \rfloor = \lfloor r/2 \rfloor$ when r is even while we cannot simultaneously have $m = \lfloor (r+1)/2 \rfloor$ and $i_m < r$ when r is odd, $(\star)$ since we cannot simultaneously have $m = 0$ and $i_m = r$, and $(\dagger)$ since $\lfloor (r+1)/2 \rfloor - 1 = \lfloor (r-1) \rfloor / 2$. ■

Theorem 2.11 (**Determinant of a tridiagonal k-Toeplitz matrix**). *Let K be any commutative unital ring. Given $k, n \in \mathbb{N}^*$ and $\overline{a}, \overline{b} := (b_1, \dots, b_k)$, $\overline{c} := (c_1, \dots, c_k) \in K^k$, put $d_i := -b_i c_i$ for $1 \le i \le k$ and $\overline{d} := (d_1, \dots, d_k)$. If $n \le k$ (i.e, if we are considering a general tridiagonal matrix) then*

$$\det(T_n(\overline{a}; \overline{b}; \overline{c})) = \alpha^{\overline{a}, \overline{d}}(n, k).$$

If $n > k$ write $n = mk + r$ with r the remainder of n modulo k, and denote $s(i) := s_i(\pi^{\overline{a}, \overline{d}}(k, k), d)$ for $i \in \mathbb{N}$; then

$$\begin{aligned}
& \det(T_n(\overline{a}; \overline{b}; \overline{c})) = \\
& = s(m)(\alpha^{\overline{a}, \overline{d}}(k, k) \alpha^{\overline{a}, \overline{d}}(r, k) + \alpha^{\overline{a}, \overline{d}}(k-1, k) \beta^{\overline{a}, \overline{d}}(r+1, k)) - d s(m-1) \alpha^{\overline{a}, \overline{d}}(r, k).
\end{aligned}$$

Proof: Fixed $k \in \mathbb{N}^*$ and $\overline{a} := a_1, \ldots, a_k, \overline{b} := b_1, \ldots, b_k, \overline{c} := c_1, \ldots, c_k \in K$, for $n \in \mathbb{N}^*$ denote $T_n := T_n(\overline{a}; \overline{b}; \overline{c})$ and $D(n) := \det(T_n)$, and denote also $D(0) := 1$. If $n = 1$ then $D(n) = \det((a_1)) = a_1$. Suppose $n \geq 2$ and write $T_n = (t_{ij})_{i,j=1}^n$. From Laplace expansion along the last column we see that $D(n) = t_{nn}D(n-1) - t_{n-1,n}D'$, with $D' = t_{n,n-1}D(n-2)$ by Laplace expansion along the last row of its associated submatrix (also true for $n = 2$ with $D(0) = 1$). If the residue of n modulo k is r, $n = km + r$, then $t_{nn} = a_r$, $t_{n-1,n} = b_{r-1}$, $t_{n,n-1} = c_{r-1}$, with $a_0 := a_k, b_0 := b_k, c_0 := c_k$. Therefore, putting $d_i := -b_i c_i$ for $0 \leq i \leq k$ and $d_{-1} := d_{k-1}$, we get for $D(n)$ the second-order linear difference equation with variable periodic coefficients

$$D(n) = a_r D(n-1) + d_{r-1} D(n-2) \text{ for } n \geq 2$$

or, written in matrix form,

$$\begin{pmatrix} D(n) \\ D(n-1) \end{pmatrix} = \begin{pmatrix} a_r & d_{r-1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} D(n-1) \\ D(n-2) \end{pmatrix}. \tag{1}$$

Define $A_r := \begin{pmatrix} a_r & d_{r-1} \\ 1 & 0 \end{pmatrix}$ for $0 \leq r \leq k$ (note $A_0 = A_k$), which has $\det(A_r) = -d_{r-1}$, and $A := A_k A_{k-1} \cdots A_1$. Observe that we also have

$$\begin{pmatrix} D(1) \\ D(0) \end{pmatrix} = \begin{pmatrix} a_1 \\ 1 \end{pmatrix} = \begin{pmatrix} a_1 & d_k \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = A_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Therefore, by recursion on (1), for $n = km + r$ we have

$$\begin{pmatrix} D(n) \\ D(n-1) \end{pmatrix} = A_r A_{r-1} \cdots A_1 \cdot A^m \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{2}$$

understanding $A_r \cdots A_1 = I$ when $r = 0$. Denote $A(r) := A_r \cdots A_1$ for $1 \leq r \leq k$ (note $A(k) = A$) and $\overline{d} := (d_1, \ldots, d_k)$. Let us show by induction on r that

$$A(r) = \begin{pmatrix} \alpha^{\overline{a},\overline{d}}(r,k) & \beta^{\overline{a},\overline{d}}(r+1,k) \\ \alpha^{\overline{a},\overline{d}}(r-1,k) & \beta^{\overline{a},\overline{d}}(r,k) \end{pmatrix}. \tag{3}$$

From Definitions 2.5, in the ring $\mathbb{Z}[x_1, \ldots, x_{k+1}, y_1, \ldots, y_k]$ we compute $\beta(2,k) = y_k$, $\pi(1,k) = x_1$, $\beta(1,k) = 0$, $\alpha(1,k) = x_1$, $\alpha(0,k) = 1$, so

$$\begin{pmatrix} \alpha^{\overline{a},\overline{d}}(1,k) & \beta^{\overline{a},\overline{d}}(2,k) \\ \alpha^{\overline{a},\overline{d}}(0,k) & \beta^{\overline{a},\overline{d}}(1,k) \end{pmatrix} = \begin{pmatrix} a_1 & d_k \\ 1 & 0 \end{pmatrix} = A_1 = A(1).$$

This shows the base case. Now assume that (3) is true for a fixed $1 \leq r \leq k-1$. Then

$$A(r+1) = A_{r+1}A(r) = \begin{pmatrix} a_{r+1} & d_r \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha^{\overline{a},\overline{d}}(r,k) & \beta^{\overline{a},\overline{d}}(r+1,k) \\ \alpha^{\overline{a},\overline{d}}(r-1,k) & \beta^{\overline{a},\overline{d}}(r,k) \end{pmatrix} =$$

$$= \begin{pmatrix} a_{r+1}\alpha^{\overline{a},\overline{d}}(r,k) + d_r\alpha^{\overline{a},\overline{d}}(r-1,k) & a_{r+1}\beta^{\overline{a},\overline{d}}(r+1,k) + d_r\beta^{\overline{a},\overline{d}}(r,k) \\ \alpha^{\overline{a},\overline{d}}(r,k) & \beta^{\overline{a},\overline{d}}(r+1,k) \end{pmatrix} =$$

$$= \begin{pmatrix} \alpha^{\overline{a},\overline{d}}(r+1,k) & \beta^{\overline{a},\overline{d}}(r+2,k) \\ \alpha^{\overline{a},\overline{d}}(r,k) & \beta^{\overline{a},\overline{d}}(r+1,k) \end{pmatrix}$$

by Lemma 2.10, as we needed to show.

Now, if $m = 0$ then by (2) we get $\begin{pmatrix} D(n) \\ D(n-1) \end{pmatrix} = A(r)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so $D(n) = \alpha^{\overline{a},\overline{d}}(r,k)$ by (1). If $m > 0$ then again by (2), setting $A(0) := I$, we get

$$\begin{pmatrix} D(n) \\ D(n-1) \end{pmatrix} = A(r)A^m \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{4),}$$

so $D(n)$ equals the $(1,1)$th element of $A(r)A^m$. Since by (3)

$$A = A(k) = \begin{pmatrix} \alpha^{\overline{a},\overline{d}}(k,k) & \beta^{\overline{a},\overline{d}}(k+1,k) \\ \alpha^{\overline{a},\overline{d}}(k-1,k) & \beta^{\overline{a},\overline{d}}(k,k) \end{pmatrix},$$

we get $\mathrm{tr}(A) = \alpha^{\overline{a},\overline{d}}(k,k) + \beta^{\overline{a},\overline{d}}(k,k) = \pi^{\overline{a},\overline{d}}(k,k)$. In addition we have $\det(A) = \prod_{r=1}^{k} \det(A_r) = (-1)^k d_1 \cdots d_k =: d$, thus by Lemma 2.3, if $s(i) := s_i(\pi^{\overline{a},\overline{d}}(k,k), d)$ then $A^m = s(m)A - ds(m-1)I$, so the first column of A^m is

$$\begin{pmatrix} s(m)\alpha^{\overline{a},\overline{d}}(k,k) - ds(m-1) \\ s(m)\alpha^{\overline{a},\overline{d}}(k-1,k) \end{pmatrix}. \tag{5}$$

By (3), if $r > 0$ then the first row of $A(r)$ is

$$\begin{pmatrix} \alpha^{\overline{a},\overline{d}}(r,k) & \beta^{\overline{a},\overline{d}}(r+1,k) \end{pmatrix}, \tag{6}$$

and the same holds for $A(0) = I$, since $\alpha^{\overline{a},\overline{d}}(0,k) = 1$, $\beta^{\overline{a},\overline{d}}(1,k) = 0$. Using (5),(6) in (4) we finally get

$$D(n) = s(m)(\alpha^{\overline{a},\overline{d}}(k,k)\alpha^{\overline{a},\overline{d}}(r,k) + \alpha^{\overline{a},\overline{d}}(k-1,k)\beta^{\overline{a},\overline{d}}(r+1,k)) - ds(m-1)\alpha^{\overline{a},\overline{d}}(r,k).$$

Observe that this formula gives $D(k) = \alpha^{\overline{a},\overline{d}}(k,k)$ (for $n = k$), since in this case $m = 1$, $r = 0$, $s(1) = 1, s(0) = 0$, $\alpha^{\overline{a},\overline{d}}(0,k) = 1, \beta^{\overline{a},\overline{d}}(1,k) = 0$, so the formula for $n < k$ actually works also for $n = k$. ■

Remark 2.12 (**Characteristic polynomial of a tridiagonal k-Toeplitz matrix**). Since Theorem 2.11 gives an explicit formula for the determinant of any tridiagonal k-Toeplitz matrix $T_n(\overline{a}, \overline{b}, \overline{c})$ over any commutative unital ring, it can also be used to explicitly compute the determinant of the tridiagonal k-Toeplitz matrix $T_n(\overline{X} - \overline{a}, \overline{b}, \overline{c})$, where $\overline{X} - \overline{a} := (X - a_1, \ldots, X - a_k) \in K[X]^k$, which gives the characteristic polynomial of $T_n(\overline{a}, \overline{b}, \overline{c})$.

The formula for the determinant in Theorem 2.11 allows also to compute any element of the inverse of a nonsingular tridiagonal k-Toeplitz matrix, since the submatrices giving rise to its cofactors turn out to be composed of diagonal blocks which are either triangular or again tridiagonal k-Toeplitz matrices.

Theorem 2.13 (**Inverse of a tridiagonal k-Toeplitz matrix**). *Let K be any commutative unital ring. Given $k \in \mathbb{N}^*$ and $\overline{a}, \overline{b} := (b_1, \ldots, b_k)$, $\overline{c} := (c_1, \ldots, c_k) \in K^k$, put $d_i := -b_i c_i$ for $1 \leq i \leq k$ and $\overline{d} := (d_1, \ldots, d_k)$. Given $n \in \mathbb{N}$, if $n \leq k$ denote $D_n(\overline{a}, \overline{d}) := \alpha^{\overline{a},\overline{d}}(n,k)$, else write $n = mk + r$ with r the remainder of n modulo k, put $d := (-1)^k d_1 \cdots d_k$, denote $s(i) := s_i(\pi^{\overline{a},\overline{d}}(k,k), d)$ for $i \in \mathbb{N}$ and*

$$D_n(\overline{a}, \overline{d}) :=$$
$$s(m)(\alpha^{\overline{a},\overline{d}}(k,k)\alpha^{\overline{a},\overline{d}}(r,k) + \alpha^{\overline{a},\overline{d}}(k-1,k)\beta^{\overline{a},\overline{d}}(r+1,k)) - ds(m-1)\alpha^{\overline{a},\overline{d}}(r,k).$$

Let S_k be the kth permutation group acting on K^k and for $j \in \mathbb{N}$ let $\sigma_j \in S_k$ be the jth cyclic permutation to the left, so that $\sigma_0(x_1, \ldots, x_k) = (x_1, \ldots, x_k)$, $\sigma_1(x_1, \ldots, x_k) = (x_2, \ldots, x_k, x_1)$, $\sigma_k = \sigma_0$, etc. Let $i\%k$ denote the residue of i modulo k and put $b_i := b_{(i-1)\%k+1}$, $c_i := c_{(i-1)\%k+1}$ for $i \in \mathbb{N}^*$. If $D_n(\overline{a}, \overline{d})$ is a unit of K then $T_n(\overline{a}; \overline{b}; \overline{c})$ is invertible and the (i,j)th element m_{ij} of its inverse is given by

$$(-1)^{i+j} \prod_{p=i}^{j-1} b_p \prod_{p=j}^{i-1} c_p \frac{D_{\min(i,j)-1}(\overline{a}; \overline{d}) D_{n-\max(i,j)}(\sigma_{\max(i,j)}(\overline{a}); \sigma_{\max(i,j)}(\overline{d}))}{D_n(\overline{a}; \overline{d})}.$$

Proof: Denote $T_n := T_n(\overline{a}; \overline{b}; \overline{c})$ and suppose $\det(T_n)$ is a unit of K. We compute its inverse through its adjugate matrix, so that the (r, s)th element of T_n^{-1}

is

$$m_{rs} := \frac{C_{sr}}{\det(T_n)} \qquad (1),$$

where $C_{sr} := (-1)^{r+s}\det(A_{sr})$ is the cofactor obtained from the submatrix A_{sr} of T_n formed by deleting the sth row and the rth column. Thus if $T_n = (t_{ij})_{i,j=1}^{n}$ and $A_{rs} = (a_{ij})_{i,j=1}^{n-1}$ we have

$$a_{ij} = \begin{cases} t_{ij}, & \text{if } i < r, j < s \\ t_{i+1,j}, & \text{if } i \geq r, j < s \\ t_{i,j+1}, & \text{if } i < r, j \geq s \\ t_{i+1,j+1}, & \text{if } i \geq r, j \geq s \end{cases} .$$

Since $t_{ij} = 0$ if $|j-i| \geq 2$, this implies that A_{rs} is a block upper triangular matrix when $r \leq s$ and a block lower triangular matrix when $r \geq s$, with three diagonal blocks. For ease of indexing, put $a_i := a_{(i-1)\%k+1}$ for $i \in \mathbb{N}^*$ and similarly with b_i, c_i. Then when $r \leq s$ we have

$$A_{rs} = \begin{array}{cl} \left(\begin{array}{c|ccc|c} \boldsymbol{A_1} & b_{r-1} & & & \\ \hline & \boldsymbol{c_r} & a_{r+1} & & \\ & & \boldsymbol{c_{r+1}} & \ddots & \\ & & & \ddots & b_s \\ \hline & & & & \boldsymbol{A_2} \end{array}\right) & \begin{array}{l} r\text{th row} \\ \\ \\ \\ \\ \end{array} \\ \begin{array}{r} \qquad\qquad\qquad s\text{th col} \end{array} & \end{array}$$

where A_1, A_2 are again tridiagonal k-Toeplitz matrices, concretely $A_1 = T_{r-1}(\overline{a};\overline{b};\overline{c})$, $A_2 = T_{n-s}(\sigma_s(\overline{a});\sigma_s(\overline{b});\sigma_s(\overline{c}))$. Since the middle diagonal block A_c is upper triangular with main diagonal composed from the lower diagonal of T_n, its determinant is $\det(A_c) = c_r c_{r+1}\cdots c_{s-1}$. Analogously, when $r \geq s$ we get $T_s(\overline{a};\overline{b};\overline{c})$, A_b, and $T_{n-r}(\sigma_r(\overline{a});\sigma_r(\overline{b});\sigma_r(\overline{c}))$ as the diagonal blocks of A_{rs}, with A_b lower triangular with main diagonal composed from the upper diagonal of T_n and determinant $\det(A_b) = b_s \cdots b_{r-1}$. Thus, since the determinant of a block-triangular matrix is the product of the determinants of its diagonal blocks, we get

$$\det(A_{rs}) = \begin{cases} c_r \cdots c_{s-1} \det(T_{r-1}(\overline{a};\overline{b};\overline{c}))\det(T_{n-s}(\sigma_s(\overline{a});\sigma_s(\overline{b});\sigma_s(\overline{c})), & \text{if } r \leq s \\ b_s \cdots b_{r-1} \det(T_{s-1}(\overline{a};\overline{b};\overline{c}))\det(T_{n-r}(\sigma_r(\overline{a});\sigma_r(\overline{b});\sigma_r(\overline{c}))) & \text{if } s \leq r \end{cases} .$$

The result now follows from (1) and the application of Theorem 2.11. ■

Remark 2.14. Observe that the product of determinants appearing in the computation of the element m_{ij} of the inverse (with $i < j$, say) is the same as that appearing for its reflected element m_{ji}, the only difference in their computations being that the factor $\prod_{p=i}^{j-1} b_p$ appearing for m_{ij} is substituted by the factor $\prod_{p=i}^{j-1} c_p$ for m_{ji}.

To conclude we provide an example of computation of the determinant, the characteristic polynomial, and an element of the inverse, of an specific tridiagonal k-Toeplitz matrix.

Example 2.15. Pick $n := 10$, $k := 3$, $a_1 := 1, a_2 := 0, a_3 := -1, b_1 := 2, b_2 := 5, b_3 := 1, c_1 := 3, c_2 := -2, c_3 := 1 \in \mathbb{Q}$ and consider $T_n(\overline{a};\overline{b};\overline{c})$. To find its determinant we compute

$$d_1 = -b_1c_1 = -6, d_2 = 10, d_3 = -1, d = (-1)^k d_1d_2d_3 = -60$$

$$\pi(k,k) = x_1x_2x_3 + y_1x_3 + x_1y_2 + x_2y_3,\ \pi^{\overline{a},\overline{d}}(k,k) = 16$$

$$n = 3\cdot k + 1 \Rightarrow m = 3, r = 1$$

$$s_m(x,y) = x^2 - y,\ s_{m-1}(x,y) = x$$

$$s(m) = s_m(\pi^{\overline{a},\overline{d}}(k,k), d) = 316,\ s(m-1) = s_{m-1}(\pi^{\overline{a},\overline{d}}(k,k), d) = 16$$

$$\alpha(k,k) = x_1x_2x_3 + y_1x_3 + x_1y_2,\ \alpha(k-1,k) = x_1x_2 + y_1,\ \alpha(r,k) = x_1$$

$$\alpha^{\overline{a},\overline{d}}(k,k) = 16,\ \alpha^{\overline{a},\overline{d}}(k-1,k) = -6,\ \alpha^{\overline{a},\overline{d}}(r,k) = 1$$

$$\beta(r+1,k) = y_3,\ \beta^{\overline{a},\overline{d}}(r+1,k) = -1.$$

Then $\det(T_n(\overline{a};\overline{b};\overline{c}))$ is

$$s(m)(\alpha^{\overline{a},\overline{d}}(k,k)\alpha^{\overline{a},\overline{d}}(r,k) + \alpha^{\overline{a},\overline{d}}(k-1,k)\beta^{\overline{a},\overline{d}}(r+1,k)) - ds(m-1)\alpha^{\overline{a},\overline{d}}(r,k) = 7912.$$

The characteristic polynomial of $T_n(\overline{a};\overline{b};\overline{c})$ is $det(T_n(\overline{X} - \overline{a};\overline{b};\overline{c}))$, so we compute

$$\alpha^{\overline{X}-\overline{a},\overline{d}}(k,k) = (X-1)X(X+1) - 6(X+1) + 10(X-1) = X^3 + 3X - 16$$

$$\alpha^{\overline{X}-\overline{a},\overline{d}}(k-1,k) = (X-1)X - 6 = X^2 - X - 6,\ \alpha^{\overline{X}-\overline{a},\overline{d}}(r,k) = X - 1$$

$$\pi^{\overline{X}-\overline{a},\overline{d}}(k,k) = X^3 + 3X - 16 - X = X^3 + 2X - 16,\ \beta^{\overline{X}-\overline{a},\overline{d}}(r+1,k) = -1$$

$$s(m) = s_m(\pi^{\overline{X}-\overline{a},\overline{d}}(k,k), d) = (X^3 + 2X - 16)^2 + 60 =$$
$$= X^6 + 4X^4 - 32X^3 + 4X^2 - 64X + 316$$

$$s(m-1) = s_{m-1}(\pi^{\overline{X}-\overline{a},\overline{d}}(k,k), d) = X^3 + 2X - 16$$

to get

$$X^{10} - X^9 + 6X^8 - 54X^7 + 66X^6 - 204X^5 + 1112X^4 - 1280X^3 + 1992X^2 - 8176X + 7912.$$

The $(4,5)$th element m_{45} of the inverse of $T_n(\overline{a};\overline{b};\overline{c})$ is

$$m_{45} = -\prod_{p=4}^{4} b_p \frac{D_3(\overline{a};\overline{d})D_{n-5}(\sigma_5(\overline{a});\sigma_5(\overline{d}))}{D_n(\overline{a};\overline{d})}.$$

We compute

$$b_4 = b_1 = 2,\ c_4 = c_1 = 3,\ d = -60$$
$$D_3(\overline{a},\overline{d}) = \alpha^{\overline{a},\overline{d}}(k,k) = 16,\ D_n(\overline{a},\overline{d}) = 7912$$
$$\sigma_5 = \sigma_2 \Rightarrow \sigma_5(a_1,a_2,a_3) = (a_3,a_1,a_2), \sigma_5(d_1,d_2,d_3) = (d_3,d_1,d_2)$$
$$n = 5 \Rightarrow m = 1, r = 2,\ s(1) = 1, s(0) = 0,\ \beta^{\sigma_5(\overline{a}),\sigma_5(\overline{d})}(k,k) = 10$$
$$\alpha^{\sigma_5(\overline{a}),\sigma_5(\overline{d})}(k,k) = 6,\ \alpha^{\sigma_5(\overline{a}),\sigma_5(\overline{d})}(k-1,k) = \alpha^{\sigma_5(\overline{a}),\sigma_5(\overline{d})}(r,k) = -2$$
$$D_5(\sigma_5(\overline{a}),\sigma_5(\overline{d})) = 1\cdot(6\cdot(-2)+(-2)\cdot 10) - (-60)\cdot 0\cdot(-2) = -32$$

to get $m_{45} = -2\dfrac{16\cdot(-32)}{7912} = 128/989$. In addition $m_{54} = \dfrac{c_4}{b_4} m_{45} = 192/989$.

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