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A NOTE ON EXPECTILES AND RELATED MEASURES

IDIR ARAB, TOMMASO LANDO AND PAULO EDUARDO OLIVEIRA

Abstract: We prove some comparison results between generalized quantiles of different degrees, including the ordinary quantiles and the increasingly popular expectiles. Based on the generalized quantiles, we extend the notion of odds functions to higher degrees, including the Omega ratio as a special case, and establish their convexity under mild conditions. Finally, we show that the stochastic order based on comparisons of expectiles is a suitable order of skewness.

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1. Introduction

Let X be a continuous random variable (RV) with cumulative probability distribution (CDF) F, survival function $\overline{F} = 1 - F$, and probability density f. Let $p \geq 1$ and assume that $\mathbb{E}(X^p) < \infty$.

As remarked, for example, by Newey and Powell (1987) or Chen (1996), the notion of quantile of order $\alpha \in [0, 1]$ of X may be generalized, up to a *degree* $p \geq 1$, as the value $x_p(\alpha)$ which solves the following minimization problem:

$$\min_x \Big(\alpha \mathbb{E}(X - x)_-^p + (1 - \alpha)\mathbb{E}(X - x)_+^p \Big), \tag{1}$$

where $t_+ = \max(t, 0)$ and $t_- = \max(-t, 0)$. This value, $x_p(\alpha)$, is known in the literature as L_p-*quantile* of order α (Chen, 1996). Hereafter, with a slight abuse of terminology, L_p-quantiles will be simply referred to as *generalized quantiles* of degree $p \geq 1$. Note that this should not be misunderstood with the wider definition of generalized quantile given by Bellini et al. (2014). For a more convenient representation, one may use the notations introduced by

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Fishburn (1980), based on fractionary integration:

$$F_p(x) = \frac{1}{\Gamma(p)} \int_{-\infty}^{x} (x-t)^{p-1}\, dF(t) \quad \text{and} \quad \overline{F}_p(x) = \frac{1}{\Gamma(p)} \int_{x}^{\infty} (t-x)^{p-1}\, dF(t), \tag{2}$$

to define the p-iterated integrals of F and $\overline{F}$, respectively, where $F = F_1$ and $\overline{F} = \overline{F}_1$. It is well known that $F_p(x) = \int_{-\infty}^{x} F_{p-1}(t)\, dt$ and $\overline{F}_p(x) = \int_{x}^{\infty} \overline{F}_{p-1}(t)\, dt$. It is now simple, differentiating (1), to verify that $x_p(\alpha)$ may be equivalently represented as the solution of the first order condition,

$$(1-\alpha)F_p(x) = \alpha \overline{F}_p(x).$$

It is clear from the above expression that the quantiles are obtained by setting $p = 1$, that is $x(\alpha) := x_1(\alpha) = F^{-1}(\alpha)$. On the other hand, choosing $p = 2$, we obtain the *expectiles*, namely $e(\alpha) := x_2(\alpha)$. Generalized quantiles are useful tools in statistics, as they may be employed to define regression models determined through possibly asymmetric loss functions (Newey and Powell, 1987), extending the scope of the quantile regression approach of Koenker and Bassett (1978). Moreover, such extensions have been used to define risk measures, in various fields related to finance, economics and insurance (see, for instance, Bellini et al. (2014) or Bellini and Di Bernardino (2017)). On this regard, another function that is often of interest, and which is closely related to the expectiles, is given by the *Omega ratio*, introduced by Keating and Shadwick (2002) as a measure of the performance of financial variables, and defined by

$$\Omega(x) = \frac{\overline{F}_2(x)}{F_2(x)} = \frac{\mathbb{E}(X-x)_+}{\mathbb{E}(X-x)_-}. \tag{3}$$

Owing to the growing interest in expectiles and Omega ratios, in spite of the lack of a clear characterization of their behaviour, some authors studied the coherence of such functions with respect to known comparisons criteria for distributions, discovering insightful relations with the field of stochastic orders (see Bellini (2012), Bellini et al. (2018), or Klar and Müller (2019)). In this paper, we prove some results concerning generalized quantiles and some related measures, such as Omega ratios, in terms of dispersion, geometric properties, and stochastic ordering.

Statisticians often argue that typically the expectiles (or generalized quantiles of higher degree p) are somewhat less *spread out*, or dispersed, around their *center*, when compared to the corresponding quantiles. Proving such a property would be especially useful, for instance, to understand the effect of moving from quantile regression to expectile regression, at a given level α, or to infer that expectiles measure risk in a more conservative way, that is, showing less variability with respect to the degree, compared to quantiles, as conjectured by Bellini et al. (2014). However, this may be true for some special distributions, but not in general: in fact, an example of a distribution for which quantiles and expectiles coincide was exhibited in Koenker (1993). In Section 3, under some mild conditions, we obtain a comparison result between generalized quantiles of different degrees. Similar conditions may be used to characterize the Omega ratio and, more generally, what we shall denote below as *generalized odds*, in terms of convexity properties.

As pointed out by Jones (1994), expectiles or, more generally, generalized quantiles, are the ordinary quantiles of the following CDF:

$$H_p(x) = \frac{F_p(x)}{F_p(x) + \overline{F}_p(x)}. \tag{4}$$

We shall refer to H_p as the *generalized quantile distribution.* This remark gives rise to the possibility of studying stochastic orders among the generalized quantile distributions that correspond to different RVs, say X and Y. A remarkable example, besides the trivial case $p = 1$, which yields the classic stochastic ordering problem, is represented by the choice $p = 2$, which corresponds to the *expectile order* of Bellini et al. (2018). Some authors investigated the relations, in terms of coherence, or isotonicity, between this ordering of expectiles (or the closely related omega ratios) and some known stochastic orders (Bellini (2012), Bellini et al. (2018), or Klar and Müller (2019)). In Section 5 we show that, in case of suitably standardized RVs X and Y, X dominates Y in the expectile order if and only if X is *more skewed* (to the right) than Y, in a sense that will be explained in the sequel. In other words, the expectile order (for standard RVs) is actually an order of skewness, being coherent with all the basic principles of a *skewness order* (see van Zwet (1964), Arnold and Groeneveld (1992), Oja (1981), or MacGillivray (1986) for some literature on this topic).

2. Preliminaries

Let X and Y be two RVs with CDFs F_X and F_Y. We shall recall some definitions of stochastic orders which will be useful in the sequel.

Definition 1 (Shaked and Shantikumar (2007))**.** We say that

(1) X dominates Y w.r.t. the convex (concave) order, denoted by $X \geq_{cx} Y$ ($X \geq_{cv} Y$), if $\mathbb{E}(g(X)) \geq \mathbb{E}(g(Y))$ for every convex (concave) function g;
(2) X dominates Y w.r.t. the convex transform order (denoted by $X \geq_c Y$) if $F_X^{-1} \circ F_Y$ is convex;
(3) X dominates Y w.r.t. the star order (denoted by $X \geq_* Y$) if $F_X^{-1} \circ F_Y$ is starshaped, that is, $\frac{F_X^{-1} \circ F_Y(x)}{x}$ is increasing.

The convex (concave) order expresses the notion of being more (less) dispersed, where clearly $X \geq_{cx} Y$ iff $Y \geq_{cv} X$. Differently, the convex transform order has been introduced by van Zwet (1964) to compare continuous RVs in terms of skewness. The star order is also a (weaker) order of skewness, which applies to the case of nonnegative RVs whose CDFs vanish at 0 (see Shaked and Shantikumar (2007), Oja (1981)). Coherently with the literature, stochastic orders may be denoted in terms of RVs or, equivalently, in terms of their distribution functions (meaning that $X \succ Y$ is considered equivalent to $F_X \succ F_Y$). We shall use these notations interchangeably.

We start with some useful lemmas. The following one gives a characterization of monotonicity in terms of sign changes.

Lemma 2 (Lemma 11 in Arab and Oliveira (2019))**.** *A real valued function g is increasing (decreasing) iff, for every $c \in \mathbb{R}$, $g(x) - c$ changes sign at most once when x traverses from $-\infty$ to $+\infty$, and, if the change occurs, it is in the order "$-, +$" ("$+, -$").*

Now, we can prove the following useful results. Recall that a function ϕ is said to be log-concave (log-convex) if $\ln \phi$ is concave (convex).

Lemma 3. *Let p and q be positive real numbers. If f is log-concave (log-convex), then, for every $p < q$, $\frac{F_p}{F_q}$ is decreasing (increasing) and $\frac{\overline{F}_p}{\overline{F}_q}$ is increasing (decreasing).*

Proof: Assume that f is log-concave. To show that $r_{p,q}(x) = \frac{F_p(x)}{F_q(x)}$ is decreasing, we apply Lemma 2 and look at the sign sequence of $r_{p,q}(x) - c$, which coincides with that of $s_{p,q}(x) = F_p(x) - cF_q(x)$. By an appropriate change of variable, (2) yields

$$s_{p,q}(x) = \int_0^\infty h(u) f(x-u)\, du,$$

where $h(u) = \frac{u^{p-1}}{\Gamma(p)}\left(1 - \frac{c\Gamma(p)}{\Gamma(q)}u^{q-p}\right)$. Taking into account that $p < q$, $h(u)$ has at most one sign change, and, if the change occurs, it is in the order "$+, -$". Since f is log-concave, it follows from Theorem 3.1 of Karlin (1968) that $s_{p,q}(x)$ has the same sign sequence as $h(x)$. Thus, Lemma 2 implies that $r_{p,q}(x)$ is decreasing. To prove that $\frac{\overline{F}_p(x)}{\overline{F}_q(x)}$ is increasing, again we look at $\frac{\overline{F}_p(x)}{\overline{F}_q(x)} - c$, which has the same sign changes as $k_{p,q}(x) = \overline{F}_p(x) - c\overline{F}_q(x)$. Composing $k_{p,q}(x)$ with $-x$, Proposition 6 of Arab et al. (2020) implies that the sign sequence of $k_{p,q}(x)$ is the reversed sign sequence of $\overline{k}_{p,q}(-x) := \overline{F}_p(-x) - c\overline{F}_q(-x)$. Again, (2) gives

$$\overline{k}_{p,q}(-x) = \int_0^\infty h(u)f(u-x)\,du.$$

The same arguments applied in the first part of the proof imply that $\overline{k}_{p,q}(x)$ changes sign at most once, and, if the change occurs, it is in the order "$+, -$". Taking into consideration that the sign sequence of $k_{p,q}$ is the reverse of that of $\overline{k}_{p,q}(x)$, so Lemma 2 implies the conclusion. The case when f is log-convex is proved analogously. ∎

In the particular case of integer degrees p and q, the previous result holds under weaker assumptions, as proved next.

Lemma 4. *Let p and q be positive integers.*

(1) *If $\frac{f}{1-F}$ is increasing (decreasing), then $\frac{\overline{F}_p}{\overline{F}_q}$ is increasing (decreasing) for every $p < q$.*
(2) *If $\frac{f}{F}$ is decreasing (increasing), then $\frac{F_p}{F_q}$ is decreasing (increasing) for every $p < q$.*

Proof: Assume that $\frac{f(x)}{1-F(x)}$ is increasing. We shall first prove by induction that $\frac{\overline{F}_p(x)}{\overline{F}_{p+1}(x)}$ is, for every $p \geq 1$, increasing. Let $r_p(x) = \frac{\overline{F}_p(x)}{\overline{F}_{p+1}(x)}$, and take $p = 1$. To prove that r_1 is increasing, using Lemma 2, it is enough to prove that $r_1(x) - c$ changes sign at most once for every positive constant c, and, if the change occurs it, is in the order "$-, +$". Note that $r_1(x) - c$ has the same sign change as $s(x) = \overline{F}_1(x) - c\overline{F}_2(x)$. Differentiating, we get $s'(x) = -f(x) + c\overline{F}(x)$ which has the same sign change as $c - \frac{f(x)}{\overline{F}(x)}$. Taking into account the assumption, it follows that s' changes sign at most once and, if the change occurs, it is in the order "$+, -$". Consequently, the sign change of $r_1(x) - c$ is at most "$-, +$", meaning that r_1 is increasing. Now, assuming that r_p is increasing, we need to prove that r_{p+1} is also increasing. Following the same technique used to prove that the sign sequence of $r_1(x)$ is at most "$-, +$", we find that also the sign sequence of $r_{p+1}(x) - c$ is at most "$-, +$", meaning that r_{p+1} is increasing. Therefore, $\frac{\overline{F}_p}{\overline{F}_{p+1}}$ is increasing for every positive integer p. Finally, to prove that $\frac{\overline{F}_p}{\overline{F}_q}$ is increasing for every positive integers $p < q$, it is enough to remark that

$$\frac{\overline{F}_p}{\overline{F}_q} = \prod_{j=0}^{q-p-1} \frac{\overline{F}_{p+j}}{\overline{F}_{p+1+j}},$$

which is a product of positive increasing functions. The proof of the second part of the lemma follows analogously. ■

3. Single-crossing results on generalized quantiles

We study the dispersion property mentioned in the introduction, proving that, under some mild conditions, generalized quantiles become more concentrated as the degree p increases. In particular, this clearly holds when comparing expectiles with quantiles. Such properties are a consequence of the following single-crossing results.

Lemma 5. *Assume that $0 < p < q$, $\frac{F_p}{F_q}$ is decreasing and $\frac{\overline{F}_p}{\overline{F}_q}$ is increasing. Then there exists some $\alpha_{p,q}$ such that:*

$$x_p(\alpha) < x_q(\alpha), \text{ for } \alpha < \alpha_{p,q}, \quad \text{and} \quad x_p(\alpha) > x_q(\alpha), \text{ for } \alpha > \alpha_{p,q}. \quad (5)$$

Proof: Denote by $K_{p,q}(x) = \frac{1}{H_q(x)} - \frac{1}{H_p(x)} = \frac{\overline{F}_q(x)}{F_q(x)} - \frac{\overline{F}_p(x)}{F_p(x)}$, which has the same sign change as $L_{p,q}(x) = \frac{F_p(x)}{F_q(x)} - \frac{\overline{F}_p(x)}{\overline{F}_q(x)}$. It follows from the monotonicity assumptions that $L_{p,q}$ is decreasing. Moreover, as $\lim_{x\to+\infty} \frac{F_p(x)}{F_q(x)} = 0$, and $\lim_{x\to-\infty} \frac{\overline{F}_p(x)}{\overline{F}_q(x)} = 0$, it follows that $\lim_{x\to-\infty} L_{p,q}(x) > 0$ and $\lim_{x\to+\infty} L_{p,q}(x) < 0$. Taking now into account that $L_{p,q}(x)$ is decreasing, the sign change of $L_{p,q}(x)$ is "$+, -$", therefore the sign change of $K_{p,q}(x)$ is "$+, -$", implying that $H_p(x) - H_q(x)$ has a "$+, -$" sign change, so the conclusion follows. ■

Now, the main result of this section easily follows from Lemmas 3, 4 and 5.

Theorem 6. *Assume one of the following conditions is satisfied:*

(1) $0 < p < q$ *and* f *be log-concave;*
(2) $p < q$ *are positive integers,* $\frac{f}{1-F}$ *is increasing and* $\frac{f}{F}$ *is decreasing.*

Then there exists some $\alpha_{p,q}$ *such that (5) holds.*

The log-concavity of the density, required by condition 1., is an important property in statistics (Marshal and Olkin, 2007, p.98), which is satisfied by several prominent models (uniform, power function with power parameter ≥ 1, normal, Gumbel, logistic, Laplace, exponential, gamma and Weibull with shape parameter $a \geq 1$, beta with shape parameters $a, b \geq 1$). The monotonicity assumptions of condition 2. are weaker than the log-concavity of f. In particular, assuming that $\frac{f}{1-F}$ is increasing, which is typically referred to as the *increasing hazard rate* condition, is a quite popular hypothesis in survival analysis, being equivalent to log-concavity of the survival function $\overline{F}$ (see, for example, (Marshal and Olkin, 2007, p.103)). Similarly, assuming that $\frac{f}{F}$ is decreasing, which is typically referred to as the *decreasing reversed hazard rate* condition, is equivalent to the log-concavity of the CDF F (Marshal and Olkin, 2007, p.178). It is hard to find instances of distributions that do not satisfy this property. Overall, conditions 1. and 2. are not restrictive, therefore Theorem 6 has a quite wide range of applicability.

With regard to the special case of comparisons between quantiles and expectiles, the following result is an immediate consequence of Theorem 6.

Corollary 7. *Let* X *be an RV such that* $\frac{f}{1-F}$ *is increasing and* $\frac{f}{F}$ *is decreasing. Denote by* μ *and* m *the mean and the median of* X*, respectively.*

(1) *If* $m \leq \mu$*, then there exists* $\alpha_0 < \frac{1}{2}$ *such that* $x(\alpha) < e(\alpha)$ *for* $\alpha < \alpha_0$ *and* $x(\alpha) > e(\alpha)$ *for* $\alpha > \alpha_0$*.*
(2) *If* $m > \mu$*, then there exists* $\alpha_1 > \frac{1}{2}$ *such that* $x(\alpha) < e(\alpha)$ *for* $\alpha < \alpha_1$ *and* $x(\alpha) > e(\alpha)$ *for* $\alpha > \alpha_1$*.*
(3) *If* $m = \mu$*, then* $x(\alpha) < e(\alpha)$ *for* $\alpha < \frac{1}{2}$ *and* $x(\alpha) > e(\alpha)$ *for* $\alpha > \frac{1}{2}$*.*

Proof: It follows from Theorem 6 that $x(\alpha)$ and $e(\alpha)$ cross once and the sign sequence of $x(\alpha) - e(\alpha)$ is, as α goes from 0 to 1, "$-, +$". taking into account that $x(\frac{1}{2}) = m$ and $e\left(\frac{1}{2}\right) = \mu$, the conclusion follows. ■

In the symmetric case, the above result ensures that, for $q > p$, the values of $x_q(\alpha)$ are less *variable*, or *spread out*, than those of $x_p(\alpha)$. In particular, this is stated in terms of stochastic orders.

Theorem 8. *Let F be a symmetric distribution with finite mean. Assume that $\frac{f}{1-F}$ is increasing and $\frac{f}{F}$ is decreasing, if $p, q \in \mathbb{N}$, or that f is log-concave, if $p, q \in \mathbb{R}$. Then, if $q > p$, $H_p \leq_{cv} H_q$ (equivalently, $H_q \leq_{cx} H_p$).*

Proof: Denote by $\mu_p = \int_{\mathbb{R}} x\, dH_p(x)$ and $m_p = x_p(\frac{1}{2}) = H_p^{-1}(\frac{1}{2})$ the mean and the median of H_p, respectively. If F is symmetric, with mean μ and median m, Proposition 1 of Chen (1996) implies that H_p is also symmetric w.r.t. the same center, namely $\mu_p = m_p = \mu = m$, for every $p \geq 1$. Taking into account Theorem 6, $H_q - H_p$ changes sing exactly once and the sign sequence is "$-, +$". Then, Theorem 3.A.44 of Shaked and Shantikumar (2007) implies the result. ■

4. Geometric properties of generalized odds functions

It is easy to see that the Omega ratio, given by (3), is the odds function corresponding to the generalized quantile CDF H_2, namely $\Omega = \frac{\overline{H}_2}{H_2} = \frac{\overline{F}_2}{F_2}$. It now becomes natural to look at Ω as a special case of a family of odds functions of degree $p \geq 0$, defined as follows.

Definition 9. Given a CDF F, the generalized odds function of degree $p \geq 0$ is given by

$$\Omega_p(x) = \frac{\overline{F}_{p+1}(x)}{F_{p+1}(x)} = \frac{\overline{H}_{p+1}(x)}{H_{p+1}(x)}, \tag{6}$$

where F_p and $\overline{F}_p$ are given by (2), and H_p and $\overline{H}_p$ by (4).

The Omega ratio (3) is obtained for $p = 1$, as $\Omega_1 := \Omega$, whereas, for $p = 0$, we obtain the classic odds function $\Omega_0 = \frac{\overline{F}}{F}$. Similarly, we may define *generalized reversed odds function* of degree $p \geq 0$ as $\widetilde{\Omega}_p = \frac{1}{\Omega_p}$. In this case, for $p = 0$, we obtain the *odds for failure*, which has been studied by Lando et al. (2020), whereas for $p = 1$ we obtain the reversed Omega ratio associated to a loss distribution, addressed by Sharma et al. (2017). In the literature, it is typically deduced that Ω, that is Ω_1, is a non-convex function, being the ratio of two convex functions. However, the convexity of the odds is an important property. In particular, the convexity of Ω_p, together with the fact that the function is decreasing, implies that the ratio between expected gains and losses has an increasing deceleration with respect to x. Below, we provide sufficient conditions for establishing convexity and log-convexity of the generalized odds function Ω_p, including the special cases Ω_0 and Ω, just based on the assumptions which are involved in the single-crossing results of Section 3. Similar results can be obtained for $\widetilde{\Omega}_p$.

Theorem 10.

(1) *If f is log-concave and p is real, then Ω_p is convex.*
(2) *If $\frac{f}{F}$ is decreasing and p is integer, then Ω_p is convex.*
(3) *If $\frac{f}{1-F}$ is decreasing and p is integer, then Ω_p is log-convex.*

Proof: (1) Differentiating Ω_p, we get

$$\begin{aligned} -\Omega_p'(x) &= \frac{\overline{F}_p(x)F_{p+1}(x)+F_p(x)\overline{F}_{p+1}(x)}{(F_{p+1}(x))^2} \\ &= \frac{\overline{F}_p(x)}{F_{p+1}(x)}+\frac{F_p(x)}{F_{p+1}(x)}\frac{\overline{F}_{p+1}(x)}{F_{p+1}(x)}. \end{aligned}$$

Taking into account that $\frac{f(x)}{F(x)}$ is decreasing, it follows from Lemma 3 that $\frac{F_p(x)}{F_{p+1}(x)}$ is also decreasing. Moreover, as $\frac{\overline{F}_p(x)}{F_{p+1}(x)}$ and $\frac{\overline{F}_{p+1}(x)}{F_{p+1}}$ are decreasing, $-\Omega_p{}'(x)$ is a sum of two decreasing functions, hence decreasing, meaning that Ω_p is convex.

(2) The proof follows from the previous arguments, using Lemma 4 instead of Lemma 3.
(3) Note that, if the failure rate $\frac{f(x)}{1-F(x)}$ is decreasing, then the reversed failure rate $\frac{f(x)}{F(x)}$ is also decreasing. Indeed, this follows from

$$\frac{f(x)}{F(x)}=\frac{f(x)}{\overline{F}(x)}\frac{\overline{F}(x)}{F(x)},$$

where the odds $\frac{\overline{F}(x)}{F(x)}$ is decreasing by construction. After some calculations, the derivative of $\ln\Omega_p$ is

$$\begin{aligned} \left(\ln\Omega_p\right)'(x) &= -\frac{F_{p+1}(x)}{\overline{F}_{p+1}(x)}\left(\frac{\overline{F}_p(x)}{F_{p+1}(x)}+\frac{\overline{F}_{p+1}(x)F_p(x)}{F_{p+1}(x)^2}\right) \\ &= -\frac{\overline{F}_p(x)}{\overline{F}_{p+1}(x)}-\frac{F_p(x)}{F_{p+1}(x)}. \end{aligned}$$

Since $\frac{f(x)}{1-F(x)}$ and $\frac{f(x)}{F(x)}$ are decreasing, Lemma 4 implies that $\frac{\overline{F}_p(x)}{\overline{F}_{p+1}(x)}$ and $\frac{F_p(x)}{F_{p+1}(x)}$ are decreasing, therefore $(\ln \Omega_p)'$ is increasing, meaning that $\ln(\Omega_p)$ is convex.

■

5. Relations to skewness

We recall a stochastic ordering related to expectiles that was recently introduced. Let us denote by $e_X(\alpha)$ and $e_Y(\alpha)$ the expectiles corresponding to the RVs X and Y, respectively.

Definition 11 (Bellini et al. (2018))**.** We say that X dominates Y in the expectile order, denoted by $X \geq_e Y$, if $e_X(\alpha) \geq e_Y(\alpha)$, for every $\alpha \in [0,1]$.

In this section, we consider suitably standardized RVs, and we relate the expectile order to the notion of skewness. Since skewness is a location-scale invariant concept, let X and Y be a pair of RVs such that $\mathbb{E}(X) = \mathbb{E}(Y) = \mu$ and $\mathbb{E}|X-\mu| = \mathbb{E}|Y-\mu|$. Henceforth, without loss of generality, for the sequel we shall let $\mu = 0$ and $\mathbb{E}|X| = 1$, like in Arnold and Groeneveld (1992). In doing so, we simplify the notations, without needing to rely on standardization, taking $\widetilde{X} = \frac{X-\mathbb{E}(X)}{\mathbb{E}|X-\mathbb{E}(X)|}$. Various stochastic orders have been introduced for comparing RVs in terms of skewness, see for instance van Zwet (1964), Arnold and Groeneveld (1992), Oja (1981), or MacGillivray (1986), among which, the convex transform order of van Zwet (1964) is the main (and strongest) one. We introduce a new ordering of skewness and show that it is actually equivalent to the expectile order among standardized variables.

Intuitively, we may say that X is *more skewed* (to the right) than Y if the left tail of X is *lighter* than that of Y and the right tail of X is *heavier* than that of Y. This justifies defining the following stochastic order.

Definition 12. Given the RVs X and Y, with CDFs F_X and F_Y, respectively, we say that X dominates Y in the s-order, denoted by $X \geq_s Y$, if

$$\int_{-\infty}^{x} F_X(t)\,dt \leq \int_{-\infty}^{x} F_Y(t)\,dt, \ \forall x \leq 0,$$

and

$$\int_{x}^{\infty} \overline{F}_X(t)\,dt \geq \int_{x}^{\infty} \overline{F}_Y(t)\,dt, \ \forall x \geq 0.$$

Expressing this relation in terms of known stochastic orders, $X \geq_s Y$ holds if both $-(X_-) \geq_{cv} -(Y_-)$ and $-(Y_+) \geq_{cv} -(X_+)$ hold, or equivalently, $X_+ \geq_{cx} Y_+$ and $Y_- \geq_{cx} X_-$ (note that the conditions $\mathbb{E}(X) = \mathbb{E}(Y) = 0$ and $\mathbb{E}|X| = \mathbb{E}|Y|$ imply $\mathbb{E}(X_+) = \mathbb{E}(Y_+)$ and $\mathbb{E}(X_-) = \mathbb{E}(Y_-)$). To see that the s-order is a suitable ordering of skewness, we remark that it satisfies some fundamental properties, as shown in the following theorem (in particular, the s-order is coherent with the convex transform order of van Zwet (1964)).

Theorem 13.

(1) *If X and Y are symmetric and $X \neq_d Y$, then X and Y are not comparable w.r.t. the s-order.*
(2) *If $Y = -X$, then either $X \geq_s Y$ or $Y \geq_s X$.*
(3) *If $F_X(0) \geq F_Y(0)$, $X \geq_c Y$ implies $X \geq_s Y$.*

Proof: (1) Assume that $X \geq_s Y$, and let $\Delta(x) = \int_{-\infty}^{x} F_X(t) - F_Y(t)\, dt$ and $\overline{\Delta}(x) = \int_{x}^{\infty} \overline{F}_X(t) - \overline{F}_Y(t)\, dt$. Since $\mathbb{E}(X_-) = \mathbb{E}(Y_-)$ that is $\Delta(0) = 0$, and $\Delta(x) \leq 0$, then $\Delta'(x) = F_X(x) - F_Y(x) \geq 0$ on some interval $[x_0, 0]$, where $x_0 < 0$. Taking into consideration that $F_X(0) = F_Y(0) = 0$ and the densities of X and Y are symmetric, it follows that $f_X - f_Y \leq 0$ for $x \in [x_0, 0]$, therefore $F_X(x) - F_Y(x) < 0$ for $x \in (0, -x_0]$, unless $X =_d Y$. Finally, as $\mathbb{E}(X_+) = \mathbb{E}(Y_+)$, that is $\overline{\Delta}(0) = 0$, and $\overline{\Delta}'(x) = F_X(x) - F_Y(x) < 0$, for $x \in (0, -x_0]$, which contradicts $X \geq_s Y$.

(2) This easily follows from the fact that the negative part of X coincides with the positive part of Y, and vice-versa.

(3) $X \geq_c Y$ obviously implies $X_+ \geq_c Y_+$. Since $F_X(0) \geq F_Y(0)$, the previous condition implies $X_+ \geq_* Y_+$, which in turn implies that $F_{X_+} - F_{Y_+}$ changes sign at most once, with sign sequence starting with "+". However, since $\mathbb{E}(X_+) = \mathbb{E}(Y_+)$, $F_{X_+} - F_{Y_+}$ changes sign exactly once, and the sign sequence is "$+, -$". Then, Theorem 3.A.44 of Shaked and Shantikumar (2007) gives $X_+ \geq_{cx} Y_+$ (which is equivalent to $-Y_+ \geq_{cv} -X_+$). Similarly, $-(X_-) \geq_c -(Y_-)$. However, since $F^{-1}_{-(X_-)} \circ F_{-(Y_-)}(x)$ is convex iff $\phi(x) = F^{-1}_{X_-} \circ F_{Y_-}(x) = -F^{-1}_{-(X_-)} \circ F_{-(Y_-)}(-x) = \psi(x)$ is concave ($\phi'(x) = \psi'(-x)$), the previous condition is equivalent to $Y_- \geq_c X_-$, implying that $F_{Y_-} - F_{X_-}$ changes sign exactly once, with sign sequence "$+, -$" (again, this is due to $\mathbb{E}(X_-) = \mathbb{E}(Y_-)$), which gives $Y_- \geq_{cx} X_-$ (which is equivalent to $-(X_-) \geq_{cv} -(Y_-)$).

■

The equivalence of the skewness order $\geq_s$ and the expectile order $\geq_e$ is established in the following theorem. This result may also be derived from Theorem 12 (see also Corolary 13) in Bellini et al. (2014), although the cited work does not deal with the notion of skewness.

Theorem 14. *Let* $\mathbb{E}(X) = \mathbb{E}(Y) = 0$*, then* $X \geq_e Y$ *iff* $X \geq_s Y$*.*

Proof: It is easily seen that $X \geq_e Y$ iff

$$H_X(t) = \frac{\mathbb{E}(X-t)_+ + t}{2\mathbb{E}(X-t)_+ + t} \leq \frac{\mathbb{E}(Y-t)_+ + t}{2\mathbb{E}(Y-t)_+ + t} = H_Y(t), \quad \forall t. \tag{7}$$

Composing both sides of the previous inequality by the decreasing function $\frac{1}{2x-1}$, (7) is equivalent to

$$\frac{\mathbb{E}(X-t)_+}{t} \geq \frac{\mathbb{E}(Y-t)_+}{t}, \forall t. \tag{8}$$

This condition holds iff $E(X-t)_+ \geq E(Y-t)_+$, for every $t > 0$ and $E(X-t)_+ \leq E(Y-t)_+$, for every $t < 0$, which can be equivalently expressed as $X_+ \geq_{cx} Y_+$ and $Y_- \geq_{cx} X_-$. ■

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Idir Arab
CMUC, Department of Mathematics, University of Coimbra, Portugal
E-mail address: idir.bhh@gmail.com

Tommaso Lando
SAEMQ, University of Bergamo (Italy) and VŠB-TU Ostrava (Czech Republic)
E-mail address: tommaso.lando@unibg.it

Paulo Eduardo Oliveira
CMUC, Department of Mathematics, University of Coimbra, Portugal
E-mail address: paulo@mat.uc.pt