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TRAVELLING WAVE AND SHOCK WAVE SOLUTIONS FOR INEXTENSIBLE STRING EQUATIONS

AYK TELCIYAN AND YASEMIN ŞENGÜL

ABSTRACT: In the present work we prove existence of travelling wave solutions for the motion of inextensible strings, examples of which are bullwhips, chains, flagella, suspension bridge and some galactic motion. We start by expressing the equation as a system of conservation law, and modifying it to obtain a hyperbolic system while also adding a dissipative regularization term. For this modified system, we show existence of travelling wave solutions. Then, by considering the limiting cases, we analyze various shock wave solutions for the original system, and hence the initial model, and the relations between them.

KEYWORDS: inextensible string equations, hyperbolic conservation law, entropy solution travelling wave solution, shock wave solution.
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1. Introduction

The free motion for a homogenous, inextensible string with unit length and density is governed by a system of equations consisting of the equation of motion and the inextensibility constraint which is given by

$$\begin{cases} \eta_{tt}(t,s) = (\sigma(t,s)\eta_s(t,s))_s, \\ |\eta_s| = 1, \end{cases} \tag{1.1}$$

where $\eta : [0,\infty) \times [0,1] \to \mathbb{R}^m$ is the unknown position vector for material point $s \in [0,1]$ at time $t \geq 0$. The auxiliary unknown scalar function σ, which is called *tension*, can be seen as a Lagrange multiplier satisfying the inextensibility constraint $|\eta_s| = 1$ as well as the equation

$$\sigma_{ss}(t,s) - |\eta_{ss}(t,s)|^2\sigma(t,s) + |\eta_{st}(t,s)|^2 = 0, \tag{1.2}$$

(which is derived in (Şengül and Vorotnikov, 2017, Sec. 2.4) in the presence of gravity, and below in Section 2 for (1.1)). The initial values for the governing

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equation can be stated as

$$\eta(0,s) = \alpha(s) \quad \text{and} \quad \eta_t(0,s) = \beta(s), \tag{1.3}$$

where $\alpha(s)$ is the initial position and $\beta(s)$ is the initial velocity. There are several possible boundary conditions to consider:

(1) Two fixed ends:

$$\eta(t,0) = \alpha(0) \quad \text{and} \quad \eta(t,1) = \alpha(1). \tag{1.4}$$

(2) Two free ends:

$$\sigma(t,0) = \sigma(t,1) = 0. \tag{1.5}$$

(3) Periodic boundary conditions:

$$\eta(t,0) = \eta(t,1) \text{ and } \sigma(t,0) = \sigma(t,1). \tag{1.6}$$

(4) Whip boundary conditions:

$$\eta(t,1) = 0 \quad \text{and} \quad \sigma(t,0) = 0. \tag{1.7}$$

Inextensible string equations are very well-known for a long time. Due to the technical difficulties arising during the analytis, there is not much literature on them. One can see that the pioneering works were mainly done on the 16th century (cf. Antman (2005)), which were focusing on the stationary equation to find the geometric shape of the curve satisfying these equations. It is known that famous mathematicians such as Galileo, Leibniz, Huygen and Johann Bernoulli had studied these equations in different contexts with various types of external forces.

Only few results about general well-posedness of inextensible string equation are known. One of the few existence results that we have, is given by Reeken. He approaches to problem by using chains. In his papers Reeken (1977, 1979a,b), he studies the infinite string with gravity when the initial values are near to trivial stable stationary solution. Serre Serre (1991) gives a relaxed model for inextensible strings, he discusses two possible approaches to the problem; the relax constraint and the chain as the limit of a stiff elastic string. He mentions that both shows a concentration phenomena either tension in time or energy in space.

PrestonPreston (2011) studies the motion of inextensible string with whip boundary conditions in the absence of gravity, he approximates the string with chains. He proves local existence and uniqueness in a weighted Sobolev space defined for the energy. In another article Preston (2012), he studies the geometric aspects of the space of arcs parameterized by unit speed in the

L^2-metric. He proves that the space of arcs is a submanifold of the space of all curves and the orthogonal projection exists but is not smooth, and as a consequence he gets a Riemannian exponential map that is continuous and even differentiable but not C^1. His result is the only known local existence result. Preston and Saxton in Preston and Saxton (2012) study the geodesics of the H^1 Riemannian metric on the space of inextensible curves. They use the results in Preston (2012) to show that the geodesic equation is C^∞ in a Banach topology which implies that there is a smooth Riemannian exponential map. We see another paper of Preston in collaboration with Bauer and Moller-Andersen Bauer et al. (2019), where they study the waving of the flags and their immersions. If we consider the whips as 2-dimensional objects, then flags are the 3-dimensional whips with similar boundary conditions. Furthermore, their way of approaching to problem is similar to Arnold's geometric studies of motion an incompressible fluid.

Şengül and Vorotnikov in Şengül and Vorotnikov (2017) rewrite the problem as a hyperbolic conservation law with discontinuous flux. They show the existence of the generalized Young measure solutions. Vorotnikov and Shi Shi and Vorotnikov (2019a) observe that the mean curvature flow is a gradient flow on a Riemannian structure with a degenerate geodesic distance which was shown by Michor and Mumford in Michor and Mumford (2007). They introduce a new related gradient flow with respect to non a degenerate distance. The new flow is obtained by orthogonal projection of the mean curvature on the tangent bundle of the infinite-dimensional submanifold and it can be seen as the formal gradient flow in a submanifold of the Wasserstein space of probability measures.In Shi and Vorotnikov (2019b), they study the gradient flow of the potential energy on a similar infinite-dimensional Riemannian manifold, akin to Preston (2012), which is the model for overdamped motion of a falling inextensible string. They show the exponential decay of the solution to the equilibrium after proving the existence of solutions.

In this work, we would like to investigate travelling and shock wave solutions for system (1.1). In Section 2 we give some preliminary information about the system including definitions of different notions of solutions for hyperbolic conservation laws. In Section 3 we transform (1.1) into a system of conservation law. In Section 4 we further transform our system into a hyperbolic system by modifying it with two parameters ϵ and δ, and prove existence of travelling wave solutions for this new system. In Section 5 we prove existence of shock wave solutions for (1.1) by investigating limits as ϵ and δ tend to zero.

2. Preliminaries

In this section, we discuss the equation for the tension and the conservation of energy for system (1.1). In the presence of gravity in the equation of motion, similar calculations have been done in Şengül and Vorotnikov (2017) and similar results were obtained. We also give preliminary information about hyperbolic conservation laws, which we refer to later in the manuscript.

2.1. Derivation of the equation for tension. Under sufficient regularity assumptions on η and σ, we first take the derivative on the right-hand side of the equation of motion in (1.1) and then multiply both sides by η_s to get $\eta_s \cdot \eta_{tt} = \eta_s \cdot \eta_s \sigma_s + \eta_s \cdot \eta_{ss}\sigma$. Since $\eta_s \cdot \eta_s = 1$ by the inextensibility constraint, we obtain $\eta_s \cdot \eta_{tt} = \sigma_s + \eta_s \cdot \eta_{ss}\, \sigma$. The last term on the right-hand side is 0 due to differentiating the constraint $|\eta_s|^2 = 1$ with respect to the spacial variable, which gives $\eta_s \cdot \eta_{ss} = 0$. Hence, we have $\eta_s \cdot \eta_{tt} = \sigma_s$. Taking the spatial derivative of this equality, and differentiating the inextensibility constraint twice with respect to time, which gives $\eta_{st} \cdot \eta_{st} + \eta_s \cdot \eta_{stt} = 0$, we obtain $\eta_{ss} \cdot \eta_{tt} - |\eta_{st}|^2 = \sigma_{ss}$. Rewriting this inequality by incorporating (1.1), we find (1.2) as required.

2.2. Conservation of Energy. Since we do not incorporate gravity in the equation of motion in (1.1), we only have the kinetic energy of the string defined by

$$K(t) = \frac{1}{2}\int_0^1 |\eta_t(t,s)|^2 ds. \tag{2.8}$$

Taking the time derivative of $K(t)$ and using (1.1) we obtain

$$\frac{d}{dt}K(t) = \int_0^1 \eta_t \cdot \big(\sigma\, \eta_s\big)_s ds.$$

Integration by parts gives

$$\frac{d}{dt}K(t) = \eta_t(t,1)\sigma(t,1)\eta_s(t,1) - \eta_t(t,0)\sigma(t,0)\eta_s(t,0) - \int_0^1 \sigma\, \eta_s \cdot \eta_{st} ds.$$

The integral term vanishes due to the derivative of the inextensibility constraint. The first two terms on the right-hand side also vanish when $s = 0$ or $s = 1$ is the fixed end, as well as in the whip boundary conditions case. Also, their difference is zero in the periodic case. Hence, we can conclude that the kinetic energy does not change by time. We can summarize this as follows.

Proposition 2.1. *Let (η, σ) be a regular solution of* (1.1) *satisfying any of the boundary conditions* (1.4) - (1.7)*. Then, the total energy, which is the kinetic energy, is conserved.*

2.3. Non-negativity of the tension. We do similar calculations as in Şengül and Vorotnikov (2017) to show non-negativity of the tension, the difference being the fact that gravity is not considered here.

Proposition 2.2. *Let (η, σ) be a solution to* (1.1)*,* (1.3) *with boundary conditions given as* (1.5)*,* (1.6) (1.7)*, or* (1.4) *with $|\alpha(0) - \alpha(1)| < 1$. Then, $\sigma \geq 0$ for all t.*

Proof: Assume that the minimum of $\sigma(t, \cdot)$ is negative for some t. From (1.2), we know that $-\sigma_{ss} + |\eta_{ss}|^2\sigma \geq 0$. By maximum principle, either σ is a negative constant or the maximum is achieved at $s = 0$ or $s = 1$. Clearly, the first possibility cannot hold with (1.5) and (1.7). With the other boundary conditions, it gives $|\eta_{ss}| \equiv 0$. This means that string is straight, and hence $|\eta(t, 0) - \eta(t, 1)| = 1$ holds. This clearly contradicts with (1.6), and with (1.4) it lead to $|\alpha(0) - \alpha(1)| = 1$. If, on the other hand, the second option holds, that is, if σ reaches its minimum on $s = 0$ or $s = 1$, then we must have $\sigma_s(t, 0) > 0$ or $\sigma_s(t, 1) < 0$, respectively. This immediately rules out the periodic case (1.6) so that it is only possible to have a minimum on fixed ends. However, due to the inextensibility constraint we have $\eta_s \cdot \eta_{ss} = 0$. Therefore, multiplying the equation of motion in (1.1) by η_s and using $|\eta_s|^2 = 1$, we obtain $\sigma_s = \eta_s {\cdot} \eta_{tt}$. If the minimum is achieved at 1, this gives $\sigma_s(t, 1) = 0$ with (1.7) or (1.4), which is a contradiction. Also, with (1.5) we have $\sigma(t, 1) = 0$ contradicting the initial assumption. If the minimum is achieved at 0, again $\sigma_s(t, 1) = 0$ with (1.4), giving a contradiction. Also, with (1.7) and (1.5), we have $\sigma(t, 0) = 0$, which contradicts the initial assumption. ■

2.4. Hyperbolic conservation laws. In this subsection, we give some necessary definitions, such as that of conservation law and entropy condition, by mostly consulting to Evans (2010).

Definition 2.3. *A conversation law is a first-order differential equation of the form*

$$\mathbf{u}_t + \mathbf{F}(\mathbf{u})_x = \mathbf{0}, \tag{2.9}$$

where $\mathbf{u} : [0,\infty) \times \mathbb{R} \to \mathbb{R}^m$ *is the unknown,* $\mathbf{u} = \mathbf{u}(t,x)$*,* $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^m$ *is the flux function (which is an* $m \times n$ *matrix for* $x \in \mathbb{R}^n$*).*

Definition 2.4. *System* (2.9) *is called strictly hyperbolic if for each* $z \in \mathbb{R}^m$ *the eigenvalues of* $D\mathbf{F}(z)$ *are real and distinct.*

It is well-known that solutions of (2.9) are not smooth in general. One can seek for weak solutions (in the sense of ditributions) and these solutions can contain shock-waves.

Definition 2.5. *A weak solution* $\mathbf{u}(x,t)$ *of* (2.9) *is called a shock wave solution if it has the form*

$$\mathbf{u}(x,t) = \begin{cases} \mathbf{u}_l, & \text{if } x < ct, \\ \mathbf{u}_r, & \text{if } x > ct. \end{cases} \tag{2.10}$$

In this case u_r*,* u_l *and* c *are related by the Rankine-Hugoniot equation*

$$\mathbf{F}(\mathbf{u}_r) - \mathbf{F}(\mathbf{u}_l) = c(\mathbf{u}_r - \mathbf{u}_l).$$

Conversely, if the Rankie-Hugoniot equation holds, then the function $\mathbf{u}(x,t)$ *defined by* (2.10) *is indeed a solution of* (2.9).

We know that weak solutions of hyperbolic conservation laws are not necessarily unique. The so-called *entropy condition* which is motivated by the second law of thermodynamics for gas dynamics (cf. Conlon and Liu (1981)) is used to guarantee uniqueness. It is possible to describe shock wave solutions quite accurately when $\mathbf{u}_r$ is close to $\mathbf{u}_l$.

Definition 2.6. *Given a fixed state* $\mathbf{z}_0 \in \mathbb{R}^m$*, we define the shock set*

$$S(\mathbf{z}_0) = \{\mathbf{z} \in \mathbb{R}^m | \mathbf{F}(\mathbf{z}) - \mathbf{F}(\mathbf{z}_0) = c(\mathbf{z} - \mathbf{z}_0) \text{ for a scalar } c = c(\mathbf{z},\mathbf{z}_0)\}.$$

In some small neighbourhood of $\mathbf{z}_0$, $S(\mathbf{z}_0)$ consists of the union of m smooth curves $S_i(\mathbf{z}_0)$, $i = 1, 2, \ldots, m$.

Liu's entropy condition (Liu (1976), also Conlon (1980) and Evans (2010)): Suppose that $\mathbf{u}_r \in S_i(\mathbf{u}_l)$ for some $1 \le i \le m$. The shock is said to satisfy *Liu's entropy condition* if for all $\mathbf{u} \in S_i(\mathbf{u}_l)$ the inequality

$$c(\mathbf{u}_l, \mathbf{u}_r) \le c(\mathbf{u}_l, \mathbf{u}), \tag{2.11}$$

holds.

Definition 2.7. *We say that* $\mathbf{u}(t,x)$ *is a weak solution of* (2.9) *provided the equality*

$$\int_0^\infty \int_{-\infty}^\infty \mathbf{u}\cdot\mathbf{w}_t + \mathbf{F}(\mathbf{u})\cdot\mathbf{w}_x dxdt = 0 \tag{2.12}$$

holds, where $\mathbf{w} : [0,\infty)\times\mathbb{R}\to\mathbb{R}^m$ *is smooth with compact support. This identity, which is derived by supposing* $\mathbf{u}$ *is a smooth solution makes sense if* $\mathbf{u}$ *is merely bounded. Therefore,* $\mathbf{u}\in L^\infty([0,\infty)\times\mathbb{R};\mathbb{R}^m)$.

3. Transformed system

In this section, as a result of some changes of variables, we transform system (1.1) into the form of a system of conservation law. In order to do so, we assume that $\sigma\geq 0$ and let $\kappa=\sigma\eta_s$, so that we have $\sigma=|\kappa|$ and $\eta_s=\frac{\kappa}{|\kappa|}$. As a result (1.1) becomes

$$\begin{cases} \eta_{tt}=\kappa_s, \\ \eta_s=\dfrac{\kappa}{|\kappa|}. \end{cases} \tag{3.13}$$

Defining $v=\eta_t$, we can write

$$\begin{cases} v_t=\kappa_s, \\ v_s=\left(\dfrac{\kappa}{|\kappa|}\right)_t. \end{cases}$$

In order to write this system in the form of (2.9), we swap the time and spatial variables to get

$$\begin{cases} \kappa_t - v_s = 0, \\ v_t - \left(\dfrac{\kappa}{|\kappa|}\right)_s = 0. \end{cases} \tag{3.14}$$

Defining $\beta := (\kappa, v)\in\mathbb{R}^m\times\mathbb{R}^m$, we can write (3.14) as

$$\beta_t + \mathbf{F}(\beta)_s = 0$$

where $\mathbf{F}(z)=\left(-z_2, -\frac{z_1}{|z_1|}\right)$, for any $z=(z_1,z_2)$, so that

$$D\mathbf{F}=\begin{pmatrix} 0 & -1 \\ -\dfrac{d}{dz_1}\dfrac{z_1}{|z_1|} & 0 \end{pmatrix}.$$

Clearly, $D\mathbf{F}$ does not have distinct and positive eigenvalues, and hence the system is not strictly hyperbolic. Note also that the term $\dfrac{\kappa}{|\kappa|}$ is not differentiable when $\sigma = 0$. In the next section, we modify (3.14) to obtain a system of hyperbolic conservation law.

4. Travelling Wave Solutions

In this section, we find the explicit travelling wave solutions of the transformed system (3.14). Therefore, we will assume that $s \in \mathbb{R}$.

4.1. Obtaining a hyperbolic conservation law. We modify the system (3.14) by adding a term including the parameter $\delta > 0$ so that we have a hyperbolic conservation law. In order to indicate dependance on the parameter, we denote the variables as $\beta_\delta = (\kappa_\delta, v_\delta)$ and get

$$\begin{cases} \partial_t \kappa_\delta - \partial_s v_\delta = 0, \\ \partial_t v_\delta - \partial_s \left(\dfrac{\kappa_\delta}{\sqrt{\delta + |\kappa_\delta|^2}} \right) = 0. \end{cases} \tag{4.15}$$

For this system we have

$$D\mathbf{F} = \begin{pmatrix} 0 & -1 \\ -\dfrac{\delta}{(\delta + |\kappa_\delta|^2)^{3/2}} & 0 \end{pmatrix},$$

and hence the eigenvalues of the above matrix can be found as $\lambda_{1,2} = \mp \sqrt[4]{\dfrac{\kappa_\delta^2}{\delta + |\kappa_\delta|^2}}$. We can conclude that this system has real and distinct eigenvalues for $\kappa_\delta > 0$, and hence is a strictly hyperbolic conservation system by Definition 2.4.

4.2. Travelling wave solutions. In this section, we find the travelling waves solutions of a new hyperbolic system explicitly. In order to obtain this new system, we perturb (4.15) by adding to the second equation a dissipative regularization term with a parameter $\epsilon > 0$ in front. Also, we denote our solution as $\beta_{\delta,\epsilon}$ to indicate dependance on both parameters. As a result we have

$$
\begin{cases}
\partial_t \kappa_{\delta,\epsilon} - \partial_s v_{\delta,\epsilon} = 0, \\
\partial_t v_{\delta,\epsilon} - \partial_s \left(\dfrac{\kappa_{\delta,\epsilon}}{\sqrt{\delta + |\kappa_{\delta,\epsilon}|^2}} \right) = \epsilon \partial_{ss} v_{\delta,\epsilon}.
\end{cases}
\tag{4.16}
$$

The idea of adding the term $\epsilon\partial_{ss}v_{\delta,\epsilon}$ is to provide a small viscosity effect, which, theoretically, should smear out sharp shocks. Note that we added the regularizing term to the second equation only, rather than the whole system. Physically this makes sense since the first equation in system (4.16) (as well as the previous equivalent systems) is only the compatibility equation while the second equation is the Newton's law. Also, since ϵ and δ are small numbers, there is no change in the physical nature of the inextensible string. Now, let us define the travelling wave solution.

Definition 4.1. *Let $u(s,t)$ be a solution of a partial differential equation. A particular solution U of the form*

$$U(s,t) = V(s - ct)$$

is called a travelling wave solution, where c is wave speed, and V is the wave profile.

We denote the travelling wave profile corresponding to our solution $\beta_{\delta,\epsilon}(s,t)$ as $\mu_{\delta,\epsilon}(s - ct/\epsilon)$. Also, we change $\kappa_{\delta,\epsilon}$ and $v_{\delta,\epsilon}$ into their wave profiles $\xi_{\delta,\epsilon}$ and $\gamma_{\delta,\epsilon}$, respectively, so that $\mu_{\delta,\epsilon}(s - ct/\epsilon) = (\xi_{\delta,\epsilon}, \gamma_{\delta,\epsilon})$. Notice that doing this, our system of partial differential equations become an ordinary differential equations system. Defining $a = (s - ct)/\epsilon$, we would like to rewrite the system (4.16) as an ordinary differential equation system for this new variable. In order to do so, we should find derivatives of $\xi_{\delta,\epsilon}$ and $\gamma_{\delta,\epsilon}$, which are one-variable, vector-valued functions in $\mathbb{R}^m$. Therefore, their derivatives (Jacobians) are vectors in $\mathbb{R}^m$. That is, denoting differentiation with respect to a as $\dot{} = \frac{d}{da}$, we have $\dot{\xi}_{\delta,\epsilon} = (\dot{\xi}^1_{\delta,\epsilon}, \dot{\xi}^2_{\delta,\epsilon}, \dots, \dot{\xi}^m_{\delta,\epsilon})$, and similarly for $\dot{\gamma}_{\delta,\epsilon}$. Here, each $\xi^j_{\delta,\epsilon}$ for $j = 1, \dots, m$, is a real-valued function of a single variable. As a result, (4.16) becomes the system of vector equations given by

$$
\begin{cases}
\dot{\gamma}_{\delta,\epsilon} + c\dot{\xi}_{\delta,\epsilon} = 0 \\
\ddot{\gamma}_{\delta,\epsilon} + c\dot{\gamma}_{\delta,\epsilon} + \left(\dfrac{\delta}{(\delta + |\xi_{\delta,\epsilon}|^2)^{3/2}} \right) \dot{\xi}_{\delta,\epsilon} = 0.
\end{cases}
\tag{4.17}
$$

Theorem 4.2. *Assume that $\epsilon, \delta > 0$ are given, and c is a constant. Then, there exists solutions $\mu_{\delta,\epsilon} \in C^\infty(\mathbb{R};\mathbb{R}^m)$ to the system* (4.17).

Proof: Using the first equation in (4.17), we can write the second equation as the second order, non-linear differential equation given by

$$\ddot{\xi}_{\delta,\epsilon} - \left(\frac{\delta}{c(\delta + |\xi_{\delta,\epsilon}|^2)^{3/2}} - c \right) \dot{\xi}_{\delta,\epsilon} = 0. \tag{4.18}$$

We consider (4.19) for each component of $\xi_{\delta,\epsilon}$ separately while keeping the same notation for simplicity. Once the solution is found componentwise, we are going to comment on the vector solution. Denoting a single component of $\xi_{\delta,\epsilon}$ by z, we can rewrite this equation as

$$\ddot{z} - \frac{d}{da} G(z) = 0, \tag{4.19}$$

where $G : \mathbb{R} \to \mathbb{R}$ is given by

$$G(z) = \int^z \left(\frac{\delta}{c(\delta + |w|^2)^{3/2}} - c \right) dw. \tag{4.20}$$

Therefore, equivalently, we have

$$\dot{z} - G(z) = \text{constant} =: K_1. \tag{4.21}$$

Now, solving (4.21) with $G(z)$ as in (4.20) gives

$$\frac{c}{1 - \delta c^4} \left[\frac{\delta c^2}{2} \ln z^2 + \frac{\sqrt{\delta}}{2} \ln \left| \frac{\sqrt{\delta + z^2} - \sqrt{\delta}}{\sqrt{\delta + z^2} + \sqrt{\delta}} \right| - \frac{1}{c^2} \ln \left| 1 - c^2 \sqrt{\delta + z^2} \right| \right] = K_1\, a + K_2,$$

where K_2 is also an integration constant. A similar implicit expression for both components of the vector $\xi_{\delta,\epsilon}$ can be found, which together give the vector solution of (4.19) as

$$\frac{c}{1 - \delta c^4} \left[\frac{\delta c^2}{2} \ln(\xi_{\delta,\epsilon})^2 + \frac{\sqrt{\delta}}{2} \ln \left| \frac{\sqrt{\delta + (\xi_{\delta,\epsilon})^2} - \sqrt{\delta}}{\sqrt{\delta + (\xi_{\delta,\epsilon})^2} + \sqrt{\delta}} \right| - \frac{1}{c^2} \ln \left| 1 - c^2 \sqrt{\delta + (\xi_{\delta,\epsilon})^2} \right| \right] = \mathcal{K}_1\, a + \mathcal{K}_2, \tag{4.22}$$

where $\mathcal{K}_1$ and $\mathcal{K}_2$ denote the corresponding constant vectors. Similarly, one can find an implicit expression for $\gamma_{\delta,\epsilon}$ by using the first equation in (4.17) so that $\mu_{\delta,\epsilon} = (\xi_{\delta,\epsilon}, \gamma_{\delta,\epsilon})$ is obtained, which completes the proof. ■

Remark 4.3. *Note from* (4.22) *that, for system* (4.17)*, the parameter* δ *appears explicitly in the solution, while* ϵ *dependance is via the variable* a*.*

Theorem 4.4. *The regularized system* (4.16) *admits travelling wave solutions* $\mu_{\delta,\epsilon}$ *for any fixed* $\epsilon, \delta > 0$ *satisfying the asymptotic boundary conditions given as*

$$\lim_{a\to-\infty} \mu_{\delta,\epsilon} = (\xi^l_{\delta,\epsilon}, \gamma^l_{\delta,\epsilon}), \quad \lim_{a\to\infty} \mu_{\delta,\epsilon} = (\xi^r_{\delta,\epsilon}, \gamma^r_{\delta,\epsilon}), \quad \lim_{a\to\mp\infty} \dot{\mu}_{\delta,\epsilon} = 0, \tag{4.23}$$

if and only if Liu's entropy criterion (2.11) *is satisfied.*

Proof: Assume that system (4.16) has travelling wave solutions $\mu_{\delta,\epsilon}$. Integrating (4.18) with respect to a on $\mathbb{R}$ we obtain

$$\dot{\xi}_{\delta,\epsilon}(a)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \left(\frac{\delta}{c(\delta + |\xi_{\delta,\epsilon}|^2)^{3/2}} - c \right) \dot{\xi}_{\delta,\epsilon}\, da = 0.$$

The first term vanishes due to the last condition in (4.23). Hence, we are left with the integral term only. Taking the integral for the last part gives

$$\int_{-\infty}^{\infty} \frac{\delta}{c(\delta + |\xi_{\delta,\epsilon}|^2)^{3/2}} \dot{\xi}_{\delta,\epsilon}(a)\, da - c \left(\xi_{\delta,\epsilon}\Big|_{-\infty}^{\infty} \right) = 0.$$

Writing $\dot{\xi}_{\delta,\epsilon}(a)\, da = d\xi_{\delta,\epsilon}$ and changing the integral limits accordingly gives

$$\int_{\xi^l_{\delta,\epsilon}}^{\xi^r_{\delta,\epsilon}} \frac{\delta}{c(\delta + |z|^2)^{3/2}}\, dz - c(\xi^r_{\delta,\epsilon} - \xi^l_{\delta,\epsilon}) = 0.$$

Finally, taking the integral, we obtain

$$\frac{\xi^r_{\delta,\epsilon}}{c(\delta + |\xi^r_{\delta,\epsilon}|^2)^{1/2}} - \frac{\xi^l_{\delta,\epsilon}}{c(\delta + |\xi^l_{\delta,\epsilon}|^2)^{1/2}} - c(\xi^r_{\delta,\epsilon} - \xi^l_{\delta,\epsilon}) = 0.$$

This is equivalent to say

$$c^2 = \frac{\frac{\xi^r_{\delta,\epsilon}}{(\delta + |\xi^r_{\delta,\epsilon}|^2)^{1/2}} - \frac{\xi^l_{\delta,\epsilon}}{(\delta + |\xi^l_{\delta,\epsilon}|^2)^{1/2}}}{\xi^r_{\delta,\epsilon} - \xi^l_{\delta,\epsilon}}. \tag{4.24}$$

Suppose now that $\xi^l_{\delta,\epsilon} < \xi^r_{\delta,\epsilon}$. Let us define

$$\frac{\frac{z}{(\delta + |z|^2)^{1/2}} - \frac{\xi^l_{\delta,\epsilon}}{(\delta + |\xi^l_{\delta,\epsilon}|^2)^{1/2}}}{c} - c(z - \xi^l_{\delta,\epsilon}) =: g(z).$$

Clearly, $g(\xi^l_{\delta,\epsilon}) = 0$ and $g(\xi^r_{\delta,\epsilon}) = 0$ by (4.24). Therefore, in order that the system (4.16) has a traveling wave solution with (4.23) we require

$$g(z) > 0 \quad \text{for} \quad \xi^l_{\delta,\epsilon} < z < \xi^r_{\delta,\epsilon}.$$

But this is precisely Liu's entropy criterion (2.11). A similar calculation also works for $\xi^l_{\delta,\epsilon} > \xi^r_{\delta,\epsilon}$. ∎

As a result of Theorem 4.2 we obtain the following conclusion.

Corollary 4.5. *Liu's entropy criterion* (2.11) *is satisfied for system* (4.16).

Having obtained traveling wave solutions, we would like to pass to the limit in both parameters δ and ϵ in order to relate them with other notions of solutions. For this purpose, we consider the system

$$\begin{cases} \partial_t \kappa_\epsilon - \partial_s v_\epsilon = 0, \\ \partial_t v_\epsilon - \partial_s \left(\dfrac{\kappa_\epsilon}{|\kappa_\epsilon|} \right) = \epsilon \partial_{ss} v_\epsilon. \end{cases} \tag{4.25}$$

Theorem 4.6. *System* (4.25) *admits travelling wave solutions* $\mu_\epsilon = (\xi_\epsilon, \gamma_\epsilon)$. *Moreover, these solutions can be obtained as the limit as* $\delta \to 0$ *in Theorem 4.2.*

Proof: We start by rewriting the system (4.25) in terms of traveling wave profiles ξ_ϵ and γ_ϵ for κ_ϵ and v_ϵ, respectively. We have the following system

$$\begin{cases} \dot\gamma_\epsilon + c\dot\xi_\epsilon = 0, \\ \ddot\gamma_\epsilon + c\dot\gamma_\epsilon = 0. \end{cases}$$

The second equation can be easily solved to obtain $\gamma_\epsilon(a) = b_1 + b_2 e^{-ca}$, for constants b_1 and b_2. Using this, we solve the first equation to get $\xi_\epsilon(a) = b_3 - \frac{b_2}{c} e^{-ca}$, for the additional constant b_3. With $b_2 = \frac{1}{c}$ and $b_3 = \frac{1}{c^2}$ we see that travelling wave solutions in Theorem 4.2 converge to ξ_ϵ as δ goes to 0 for $K_1 = 1$ and $K_2 = 0$. Similarly for γ_ϵ. ∎

5. Shock Wave Solutions

In this section, using the explicit travelling wave solutions and their limits, we show the existence of the shock wave solutions.

It is well-known that a shock wave satisfies Liu's entropy condition (2.11) if and only if it is a limit as $\epsilon \to 0$, of traveling wave solutions of the viscosity equation

$$\mathbf{u}_t + \mathbf{F}(\mathbf{u})_x = \epsilon \mathbf{u}_{xx}, \tag{5.26}$$

associated with (2.9) (see e.g., Conley and Smoller (1971), Smoller and Conley (1972)). Thus, as explained in Conlon and Liu (1981), the entropy satisfying solutions of (2.9) are the physically relevant ones. Note that the form of (5.26) is different from our modified system (4.16), namely, in (4.16) the viscosity term is only added to the second equation in the system.

Clearly, the limit as $\epsilon \to 0$ of our solution $\beta_{\delta,\epsilon}$ for system (4.16) gives a shock wave connecting the states β^l_δ and β^r_δ. Considering the travelling wave profiles $\mu_{\delta,\epsilon}$ together with (4.23) the Rankine-Hugoniot condition in Definition 2.5 becomes

$$\frac{d\mu_{\delta,\epsilon}}{da} = \mathbf{F}(\mu_{\delta,\epsilon}) - \mathbf{F}(\mu^l_{\delta,\epsilon}) - c(\mu_{\delta,\epsilon} - \mu^l_{\delta,\epsilon}). \tag{5.27}$$

Now, consider the left state $\mu^l_{\delta,\epsilon}$ as being given. Then, if there is a trajectory of the vector field (5.27) joining the critical point $\mu^l_{\delta,\epsilon}$ to the critical point $\mu^r_{\delta,\epsilon}$, then we have a shock wave solution to (4.15). However, we already obtained travelling wave solutions. Therefore, we have the following result.

Theorem 5.1. *System* (4.15) *admits shock wave solutions* β_δ*.*

Proof: By the theory developed by Conley and Smoller in Conley and Smoller (1971) and Smoller and Conley (1972). ■

We would like to use Conlon's Conlon (1980) results in order to show that the system (3.14) has a shock wave solution which can be obtained as the limit $\epsilon \to 0$ of the travelling wave solutions of the viscous system (4.25) obtained in Theorem 4.6.

Corollary 5.2. *The solution* β_ϵ *of system* (4.25) *converges as* $\epsilon \to 0$ *to the shock wave solution* β *of system* (3.14)*.*

Proof: Firstly, we introduce the following constant $\zeta(M, m) > 0$ that depends only m and M, where $m := |\lambda_1 - \lambda_2|$, $\lambda_{1,2}$ are the eigenvalues of $\mathbf{F}$, and M is the following bound in

$$\sup_{|\beta_\epsilon|\leq 1} |\mathbf{F}| \leq M.$$

Note that we know existence of such M due to convexity of

$$\mathbf{F} = \begin{pmatrix} 0 & -1 \\ -\dfrac{1}{(|\kappa_\epsilon|^2)^{1/2}} & 0 \end{pmatrix},$$

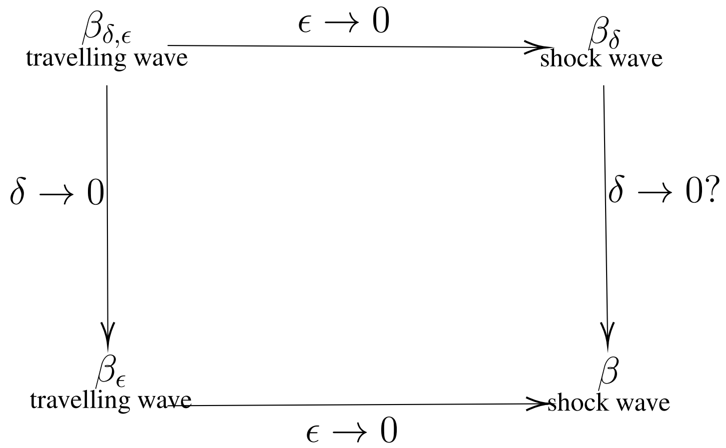

FIGURE 1. Relations between solutions

which is obtained from the viscous system (4.25). We follow the reasoning given in (Conlon, 1980, Page 15-17). We modify the results for our case by slightly changing some assumptions. However, none of the conclusions change.

Since the transformed system (3.14) is the limit of the viscous system (4.25) as $\epsilon \to 0$, we immediately satisfy the shock condition (E) in Conlon (1980) . Therefore, the shock wave solution of (4.16), whose definition was given in (2.10), satisfies condition (E). This implies that there is a trajectory of vector fields satisfying the Rankine-Hugoniot condition which joins β^l_ϵ to β^r_ϵ. Hence, by (Conlon, 1980, Theorem 4.2), we know that there exists a shock wave solution to system (3.14) with $\beta^l_\epsilon = (\kappa^l_\epsilon, v^l_\epsilon)$ and β^r_ϵ satisfying the entropy condition (E). Here β^l_ϵ is a constant whereas in the original statement of the theorem it was taken 0 for simplicity. ■

5.1. Relations between shock wave solutions. We would like to understand the relation between the shock wave solution β_δ of system (4.15) and the shock wave solution β of system (3.14) (see Figure 1). From the beginning, system (4.15) was designed in such a way that not only it is a hyperbolic conservation law, but also it converges pointwise to the system (3.14) as $\delta \to 0$. This suggests that the shock profile β_δ should converge to the shock profile β as $\delta \to 0$. Since the explicit forms of these shock waves are not known, the tool we use in order to see the relation between them will be the corresponding equations of motion for both systems.

Recall that we obtained (3.13) from (1.1) by putting $\sigma = |\kappa|$ and $\eta_s = \frac{\kappa}{|\kappa|}$. Now, we would like to write the corresponding equation of motion and the inextensibility constraint by using (4.15). Since v stands for η_t we can immediately write

$$\begin{cases} \kappa_s = \eta_{tt}, \\ |\eta_s| = \dfrac{|\kappa|}{\sqrt{\delta + |\kappa|^2}}, \end{cases} \tag{5.28}$$

where, for convenience, we dropped indication of δ on η and κ, and cancelled out one t derivative on both sides of the second equality before taking the modulus. Here, the first equation is the equation of motion with $\kappa = \sigma\eta_s$, whereas the second one represents the inextensibility constraint for the modified system (4.15). Using $\kappa = \sigma\eta_s$ in the second equality in (5.28) we obtain

$$|\eta_s|^2 = 1 - \frac{\delta}{\sigma^2}.$$

This means, (4.15) can be rewritten as

$$\begin{cases} \eta_{tt} = (\sigma\eta_s)_s, \\ |\eta_s|^2 = 1 - \frac{\delta}{\sigma^2}. \end{cases} \tag{5.29}$$

Multiplying the first equation by η_s replacing $|\eta_s|^2$ by using the constraint in (5.29), and using the fact that $\eta_s \cdot \eta_{ss} = \delta\sigma^{-3}\sigma_s$ which is obtained from the same relation, we obtain

$$\eta_{tt} \cdot \eta_s = \sigma_s$$

as in the case of (1.1). Taking another derivative with respect to the space variable we find $\eta_{stt} \cdot \eta_s + \eta_{tt} \cdot \eta_{ss} = \sigma_{ss}$. Now, differentiating the second equality in (5.29) with respect to t and multiplying the resulting relation with η_{st} we obtain

$$|\eta_{st}|^2 + \eta_s \cdot \eta_{stt} = \delta\big(- 3\sigma^{-4}\sigma_t^2 + \sigma^{-3}\sigma_{tt}\big).$$

Replacing the term $\eta_s \cdot \eta_{stt}$ by $\sigma_{ss} - \eta_{stt} \cdot \eta_s$ and using the equation of motion once again to replace $\eta_{tt} \cdot \eta_{ss}$ we obtain

$$\sigma_{ss} - |\eta_{ss}|^2\sigma + |\eta_{st}|^2 = \frac{\delta}{\sigma^2}\left(\frac{\sigma\sigma_s^2 - 3\sigma_t^2 + \sigma\sigma_{tt}}{\sigma^2}\right). \tag{5.30}$$

Comparing (1.2) and (5.30) we see that the only difference is the right-hand side. Moreover, since we would like to see the relation between the two shock wave solutions of (1.1) and (5.29) as $\delta \to 0$, we see that in this case

$\delta/\sigma^2 \to 0$ must hold. Therefore, in (5.30) the second ratio must be bounded. Indeed, considering the case of $\sigma(s)$ that is independent of time, although still spatially nonuniform, the right-hand side becomes $\delta\sigma_s^2/\sigma^3$. In this case, Hanna and Santangelo Hanna and Santangelo (2013) observed that

$$\sigma^2 = \frac{2B^3}{3}s,$$

for a constant B, under the assumption that $\sigma(0) = 0$. This gives $\sigma_s^2 + \sigma_{ss}\sigma = 0$. Hence, equation (5.30) reduces to

$$\left(1 + \frac{\delta}{\sigma^2}\right)\sigma_{ss} - |\eta_{ss}|^2\sigma + |\eta_{st}|^2 = 0.$$

Since, we know that $\delta/\sigma^2 \to 0$ we would obtain the same equation for σ. Although this is a special case to consider due to σ assumed to be time-independent, it suggests that the same conclusion might hold also in the general case.

6. Positivity of σ revisited

Since $|\eta_s|^2 > 0$ must hold, the second equality in (5.29) implies that $\sigma > \sqrt{\delta} > 0$ so that σ stays positive for the approximating system. Under some hypotheses of compatible boundary data, this might also hold for the limiting system (1.1). This is the first time that positivity of the tension is shown for hyperbolic systems that are derived from inextensible string equations.

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Ayk Telciyan
CMUC, Department of Mathematics ,University of Coimbra, 3001-501 Coimbra, Portugal
E-mail address: ayktelciyan@mat.uc.pt

Yasemin Şengül
Faculty of Engineering and Natural Sciences, Sabanci University, Tuzla 34956, Istanbul, Turkey
E-mail address: yaseminsengul@sabanciuniv.edu