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ON FUNCTORS PRESERVING PROJECTIVE RESOLUTIONS

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Abstract: It is important for applications of Homological Algebra in Representation Theory to have control over the behaviour of (minimal) projective resolutions under various functors. In this article we describe three broad families of functors that preserve such resolutions. We will use these results in our work on Representation Theory of Schur algebras.

Keywords: projective resolution, stratifying ideal, graded module, graded algebra, relative homological algebra.
AMS Subject Classification (1991): 18E10, 16W50, 18G25, 18G10.

Introduction

It is well known that finite dimensional modules over finite dimensional algebras admit minimal projective resolutions. The explicit knowledge of a minimal projective resolution for a given module permits to reduce computation of Ext-groups involving this module to an, albeit sometimes complicated, linear algebra problem.

The present article was conceived in the sequence of our research of Ext-groups between Weyl modules for the general linear group GL_n or, equivalently, of Ext-groups between Weyl modules for Schur algebras. The determination of $\mathrm{Ext}^i_{GL_n}(M,N)$, where M, N are Weyl modules, is an open problem in the representation theory of GL_n and of other algebraic groups, see [8], [9] for the cases of SL_2 and GL_2. We hope that the construction of suitable projective resolutions of Weyl modules over Schur algebras can shed some light on this problem.

In our work to explicitly build these projective resolutions, we found that it is important to be able to pass between different abelian categories with suitably chosen functors. These functors should preserve minimal projective

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resolutions, not necessarily for all objects, but at least for the objects we are interested in.

In this article we present our results concerning three of these functors. Let Γ be a monoid. The first two functors are defined on the category of Γ-graded modules over a Γ-graded algebra. The first functor is the forgetful functor that erases the grading information. The second functor is defined as a twisted product $- \ltimes_\Gamma N$ for a B-module N, where B is a Γ-algebra (the explicit definitions are given in Section 3). In Sections 2 and 3 we determine sufficient conditions for which these functors preserve minimal projective resolutions for all objects.

In the last section we study the functor $A/I \otimes_A -$, where A is an algebra and I is an ideal. This functor usually does not preserve (minimal) projective resolutions for all objects, but in favorable circumstances it preserves (minimal) projective resolutions of A/I-modules considered as A-modules. In this case we say that I is a *stratifying* ideal. The equivalent condition for I to be stratifying is that $\operatorname{Tor}_k^A(A/I, A/I) = 0$ for $k \geq 1$.

If R is a subring of A which does not necessarily lie inside the center of A, one can define relative torsion groups $\operatorname{Tor}_k^{(A,R)}(A/I, A/I)$ (see [7]). In [11] we found a combinatorial criterion for $\operatorname{Tor}_k^{(A,R)}(A/I, A/I) = 0$, $k \geq 1$. In the last section of the article we establish a sufficient condition on the triple (A, R, I) under which $\operatorname{Tor}_k^{(A,R)}(A/I, A/I) = 0$ for $k \geq 1$ implies the same for $\operatorname{Tor}_k^A(A/I, A/I)$. Thanks to this, one can use the combinatorial criterion from [11] to prove that I is a stratifying ideal in A.

For the convenience of the reader, in the first section of the article, we prove some results concerning minimal projective resolutions in general abelian categories. Despite the fact that these results are sometimes accepted as true in this general context, we were not able to find explicit proofs for them in the literature. So we chose to provide full details in their treatment.

1. Superfluous subobjects and minimal projective covers in abelian categories

In this section we work in a very abstract setting, collecting definitions and proving results in general abelian categories. These results and concepts will be then applied to concrete categories in the following sections of the article.

Let $\mathcal{C}$ be an abelian category. We follow MacLane [6], and define a *subobject* of $Y \in \mathcal{C}$ as an equivalence class of monomorphisms $\psi\colon X \to Y$, where

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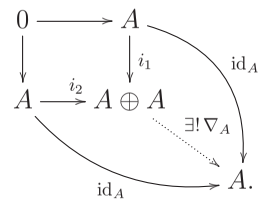
$\psi \sim \psi' \colon X' \to Y$ if there is an isomorphism $\rho \colon X \to X'$ such that $\psi' \circ \rho = \psi$. We use upper case letters to denote objects and italic uppercase letters to denote subobjects. To indicate that $\mathcal{X}$ is a subobject of Y we write $\mathcal{X} \subset Y$. If $\mathcal{X}$ and $\mathcal{Z}$ are subobjects of Y represented by monomorphisms $\psi_X \colon X \to Y$ and $\psi_Z \colon Z \to Y$ we say that $\mathcal{Z}$ is contained in $\mathcal{X}$ and write $\mathcal{Z} \leq \mathcal{X}$ if there is $\phi \colon Z \to X$ such that $\psi_X \circ \phi = \psi_Z$. Given an object Y we denote by $\mathcal{Y}$ the maximal subobject of Y, i.e. the subobject given by the equivalence class of id_Y.

Dually a *quotient* of Y is an equivalence class of epimorphisms $\tau \colon Y \to Z$, where $\tau \sim \tau' \colon Y \to Z'$ if and only if there is an isomorphism $\rho \colon Z \to Z'$ such that $\rho \circ \tau = \tau'$.

We adopt the convention that for an arrow $\phi \colon X \to Y$ the kernel $\ker \phi$ and the image $\operatorname{Im} \phi$ of ϕ are subobjects of X and Y, respectively. Similarly the cokernel $\operatorname{Coker} \phi$ of ϕ is a quotient of Y. This is a legitimate point of view as explained in [6, VIII.1].

As usual we use i_k to denote the canonical embeddings associated with direct sums. For an object A we denote by ∇_A the codiagonal map, which is determined by the universal property of the direct sum



Given two subobjects $\mathcal{X}_1$ and $\mathcal{X}_2$ of $Y \in \mathcal{C}$, we define the sum $\mathcal{X}_1 + \mathcal{X}_2$ as the image of $\nabla_Y \circ (\psi_1 \oplus \psi_2)$ in Y, where, for $i = 1, 2$, the map ψ_i is a representative of $\mathcal{X}_i$.

For a morphism $\phi \colon X \to Y$ and a subobject $\mathcal{N}$ of X represented by a monomorphism $\psi_N \colon N \to X$, we define the subobject $\phi(\mathcal{N})$ to be $\operatorname{Im}(\phi \circ \psi_N)$.

Now, if $\mathcal{M}$ is a subobject of Y represented by a monomorphism $\psi_M \colon M \to Y$, we define $\phi^{-1}(\mathcal{M}) \subset X$ as the image of the left vertical arrow in the pull-back diagram

$$\begin{array}{ccc} \widetilde{M} & \longrightarrow & M \\ \downarrow & & \downarrow{\scriptstyle \psi_M} \\ X & \xrightarrow{\phi} & Y. \end{array} \tag{1}$$

Suppose $\mathcal{X} \subset Y$. We say that $\mathcal{X}$ is a *superfluous subobject* of Y if for any subobject $\mathcal{T}$ of Y the equality $\mathcal{X} + \mathcal{T} = \mathcal{Y}$ implies $\mathcal{T} = \mathcal{Y}$. We will write $\mathcal{X} \Subset Y$ in this case.

We will define minimal projective covers using superfluous subobjects. As in this article we study preservability of projective covers under various functors, it is convenient to have at hand various elementary properties of superfluous subobjects, which will be given in Proposition 1.3. These properties are well-known and easy to prove in the case $\mathcal{C}$ is the category of modules over a ring. To lift these properties to arbitrary abelian categories, we will use the Freyd-Mitchell embedding:

Theorem 1.1. *Let $\mathcal{B}$ be a small abelian category. Then there exists a ring R such that there is a full and faithful exact functor $F\colon \mathcal{B} \to R$-Mod.*

Notice that the above result refers only to small abelian categories. As we work with arbitrary abelian categories, we will use the following fact proved on page 85 of [3].

Theorem 1.2. *Let $\mathcal{C}$ be an abelian category and X a set of objects in $\mathcal{C}$. Then there is a full subcategory $\mathcal{B}(X)$ of $\mathcal{C}$ such that*

i) $X \subset \mathrm{Ob}\mathcal{B}(X)$;
ii) $\mathcal{B}(X)$ is a small abelian category;
iii) $\mathcal{B}(X)$ is stable under finite limits and colimits.

We say that a functor F between abelian categories is exact if one of the two equivalent conditions holds:

i) F preserves exact sequences;
ii) F preserves finite limits and finite colimits.

Below we list standard properties of fully faithful exact functors, which we will use without further reference. If $F\colon \mathcal{B} \to \mathcal{C}$ is a full and faithful exact functor between abelian categories, then

- $F(0) \cong 0$;
- F preserves and reflects monomorphisms and epimorphisms;
- F preserves and reflects kernels, cokernels, and, thus, also images.

Given $\mathcal{X} \subset Y$ in $\mathcal{B}$, we define $F(\mathcal{X}) \subset F(Y)$ as the equivalence class of $F(\psi_X)$, where $\psi_X\colon X \to Y$ is a representative of $\mathcal{X}$. Then there hold

- If $\mathcal{X}_1$, $\mathcal{X}_2 \subset Y$, then $F(\mathcal{X}_1 + \mathcal{X}_2) = F(\mathcal{X}_1) + F(\mathcal{X}_2)$;
- If $\mathcal{X}_1$ and $\mathcal{X}_2$ are two different subobjects in Y then $F(\mathcal{X}_1) \neq F(\mathcal{X}_2)$;

In the next proposition we relate relative with classical torsion groups.

Proposition 4.2. *Let A be a free R-algebra, I an ideal of A, and M an R-free left A-module. Then $\mathrm{Tor}_k^{(A,R)}(A/I, M) \cong \mathrm{Tor}_k^A(A/I, M)$, for all k.*

Proof: We consider the bar resolution $\beta(A, R, M)$ of M. Every module in this resolution is of the form $A \otimes A^{\otimes k} \otimes M$, with $k \geq 0$, where all the tensor products are taken over R. Since M and A are free R-modules, $A^{\otimes k} \otimes M$ is a free R-module. Hence $A \otimes A^{\otimes k} \otimes M$ is a free A-module. This shows that $\beta(A, R, M)$ is a projective resolution of M in the category of left A-modules. Now, both tor groups $\mathrm{Tor}_k^{(A,R)}(A/I, M)$ and $\mathrm{Tor}_k^A(A/I, M)$ can be computed using the complex $A/I \otimes_A \beta(A, R, M)$. This proves the result. ∎

Theorem 4.3. *Let $M \in A/I$-Mod be an R-free left module. Then, in the conditions of the previous two propositions, the functor $A/I \otimes_A -$ sends every projective resolution of M in A-Mod to a projective resolution of M in A/I-Mod. If the initial resolution of M in A-Mod is minimal, then the final resolution in A/I-Mod is also a minimal projective resolution of M .*

Proof: Let $P_\bullet \twoheadrightarrow M$ be a projective resolution of M in A-Mod. By Propositions 4.1 and 4.2, $\mathrm{Tor}_k^A(A/I, M) \cong 0$, for $k \geq 1$. Therefore, since $\mathrm{Tor}_0^A(A/I, M) = A/I \otimes_A M \cong M$, the complex $A/I \otimes_A P_\bullet \twoheadrightarrow M$ is exact. As every P_k is a projective A-module, it follows that $A/I \otimes_A P_k$ is an A/I-projective module.

To prove that the minimality is preserved, consider in A-Mod the minimal projective resolution $P_\bullet \twoheadrightarrow M$, with differentials $d_k\colon P_{k+1} \to P_k$ for $k \geq -1$ (for simplicity, we write $P_{-1} = M$). This can be decomposed into short exact sequences

$$0 \to \ker \alpha_k \xrightarrow{\beta_k} P_{k+1} \xrightarrow{\alpha_k} \ker \alpha_{k-1} \to 0,$$

where $d_k = \beta_{k-1}\alpha_k$. Write F for the functor $A/I \otimes_A -$. Then it is easy to see that $\ker(F(d_k)) = \ker(F(\alpha_k)) = \mathrm{Im}(F(\beta_k)) = \pi(\mathrm{Im}(\beta_k)) = \pi(\ker(\alpha_k)) = \pi(\ker(d_k))$, where $\pi\colon P_{k+1} \mapsto F(P_{k+1})$ is the epimorphism given by $x \mapsto 1 \otimes x$. Since $\ker(d_k) \Subset P_{k+1}$, the result follows from Proposition 1.3 (i).

∎

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the differentials and the splitting maps are the usual ones and can be found in [7].

Let N be a right A-module and M a left A-module. Given an (A,R)-projective resolution $P_\bullet$ of M, we define the *relative tor groups* $\mathrm{Tor}_k^{(A,R)}(N,M) = H_k(N\otimes_A P_\bullet)$, all $k\geq 0$. It follows from Theorem IX.8.5 in [7], that the groups $\mathrm{Tor}_k^{(A,R)}(N,M)$ are independent of the choice of the (A,R)-projective resolution of M and, in particular, can be computed using the bar resolution of M.

As we mentioned in the introduction, in [11] we obtained an efficient combinatorial criterion for a triple (A,R,I) to have the property $\mathrm{Tor}_k^{(A,R)}(A/I,A/I)\cong 0$ for $k\geq 1$. In the next series of propositions we derive various consequences of this property under additional conditions on (A,R,I) culminating in Theorem 4.3. This gives a criterion for the functor $A/I\otimes -$ to preserve (minimal) projective resolutions.

Proposition 4.1. *Let A be an R-algebra and I an ideal of A such that, for $k\geq 1$, $\mathrm{Tor}_k^{(A,R)}(A/I,A/I)\cong 0$. Suppose that A and A/I are projective as right R-modules. Then $\mathrm{Tor}_k^{(A,R)}(A/I,M)\cong 0$, for $k\geq 1$ and any $M\in A/I$-Mod.*

Proof: Since $\mathrm{Tor}_k^{(A,R)}(A/I,A/I)\cong 0$ for $k\geq 1$, we get that the complex

$$A/I\otimes_A \beta(A,R,A/I)\twoheadrightarrow A/I\otimes_A A/I$$

is exact. Moreover, the differentials in this complex are homomorphisms of A/I-bimodules. Further, the first term of this complex is $A/I\otimes_A A/I\cong A/I$ and every other term is of the form $A/I\otimes_A A^{\otimes(k+1)}\otimes A/I\cong A/I\otimes A^{\otimes k}\otimes A/I$, where all the unnamed tensor products are over R and $k\geq 0$. Now, since A/I is a projective right R-module, we get that $A/I\otimes A$ is a projective right A-module. This fact together with the fact that A is a projective right R-module, implies that $A/I\otimes A$ is a projective right R-module. Continuing, we get that $A/I\otimes A^{\otimes k}$ are projective right R-modules, for all $k\geq 0$. Thus $A/I\otimes A^{\otimes k}\otimes A/I$ is a right A/I-projective module. Therefore the exact complex $A/I\otimes_A\beta(A,R,A/I)\twoheadrightarrow A/I\otimes_A A/I$ splits in the category of right A/I-modules. Hence $A/I\otimes_A\beta(A,R,A/I)\otimes_{A/I}M\twoheadrightarrow A/I\otimes_A A/I\otimes_{A/I}M$ is an exact complex. But it is isomorphic to $A/I\otimes_A\beta(A,R,M)\twoheadrightarrow A/I\otimes_A M\cong M$, and therefore it computes the torsion groups $\mathrm{Tor}_k^{(A,R)}(A/I,M)$. We get then that $\mathrm{Tor}_k^{(A,R)}(A/I,M)\cong 0$, $k\geq 1$. ∎

- F preserves images and pre-images of subobjects.

We can now state and prove the properties of superfluous subobjects that we mentioned before.

Proposition 1.3. *Let $\mathcal{C}$ be an abelian category. Then the following statements hold:*

(i) Suppose that $\mathcal{N}\Subset M$ and $\phi\colon M\to M'$ is an arrow in $\mathcal{C}$. Then $\phi(\mathcal{N})\Subset M'$.
(ii) Let $\mathcal{N}_k\Subset M$ in $\mathcal{C}$, for k in some finite index set I. Then $\sum_{k\in I}\mathcal{N}_k\Subset M$.
(iii) Let $\mathcal{N}_k\subset M_k$, $k\in I$, be a finite family of subobject-object pairs in $\mathcal{C}$. Then the following two assertions are equivalent:
(a) $\mathcal{N}_k\Subset M_k$, for all $k\in I$;
(b) $\sum_{k\in I} i_k(\mathcal{N}_k)\Subset\bigoplus_{k\in I} M_k$.

Proof: (i) Let $\mathcal{T}\subset M'$ be such that $\phi(\mathcal{N})+\mathcal{T}=\mathcal{M}'$. Our first aim is to show that $\mathcal{N}+\phi^{-1}(\mathcal{T})=\mathcal{M}$. Let $\psi_T\colon T\to M'$ and $\psi_N\colon N\to M$ be representatives of $\mathcal{T}$ and $\mathcal{N}$, respectively. Denote by $\mathcal{B}$ the category $\mathcal{B}(T,N,M,M')$, whose existence is asserted by Theorem 1.2. Since $\mathcal{B}$ is stable under finite limits and colimits, it is enough to show that $\mathcal{N}+\phi^{-1}(\mathcal{T})=\mathcal{M}$ in $\mathcal{B}$.

Let R be a ring, such that there is a full and faithful exact functor $F\colon\mathcal{B}\to R$-Mod. Applying F to $\phi(\mathcal{N})+\mathcal{T}=\mathcal{M}'$ we get $(F\phi)(F\mathcal{N})+F(\mathcal{T})=F(M')$. Now, for every $m\in F(M)$ there are $n\in F(\mathcal{N})$ and $t\in F(\mathcal{T})$ such that $(F\phi)(m)=(F\phi)(n)+t$. In particular, $(F\phi)(m-n)=t$ belongs to $F(\mathcal{T})$. Therefore, $m-n\in(F\phi)^{-1}(F\mathcal{T})$. This shows that $m=n+(m-n)$ is an element of $F(\mathcal{N})+(F\phi)^{-1}(F\mathcal{T})$. Since m was an arbitrary element of FM, we get that $F(\mathcal{N})+(F\phi)^{-1}(F\mathcal{T})=F(M)$. Since F is exact this implies $\mathcal{N}+\phi^{-1}(\mathcal{T})=\mathcal{M}$. As $\mathcal{N}\Subset M$, we get $\phi^{-1}(\mathcal{T})=\mathcal{M}$. In particular, $\mathcal{N}\leq\phi^{-1}(\mathcal{T})$. Hence $\phi(\mathcal{N})\leq\phi\phi^{-1}(\mathcal{T})\leq\mathcal{T}$. Therefore $\mathcal{T}=\phi(\mathcal{N})+\mathcal{T}=\mathcal{M}'$.

(ii) It is enough to prove the statement in case the cardinality of I is 2. Let $\mathcal{T}\subset M$ be such that $\mathcal{N}_1+\mathcal{N}_2+\mathcal{T}=\mathcal{M}$. Since $\mathcal{N}_1\Subset M$, we get $\mathcal{N}_2+\mathcal{T}=\mathcal{M}$. Now $\mathcal{N}_2\Subset M$, implies $\mathcal{T}=\mathcal{M}$.

(iii) Suppose $\mathcal{N}_k\Subset M_k$, for all $k\in I$. Then, by (i) we get $i_k(\mathcal{N}_k)\Subset\bigoplus_{j\in I}M_j$. Therefore $\sum_{k\in I}i_k(\mathcal{N}_k)\Subset\bigoplus_{k\in I}M_k$, by statement (ii) of the proposition.

Conversely, suppose $\sum_{k\in I}i_k(\mathcal{N}_k)\Subset\bigoplus_{k\in I}M_k$. Fix $\ell\in I$. We will show that $\mathcal{N}_\ell\Subset M_\ell$. Suppose $\mathcal{T}\subset M_\ell$ is such that $\mathcal{N}_\ell+\mathcal{T}=\mathcal{M}_\ell$.

Define $\mathcal{S} = i_\ell(\mathcal{T}) + \sum_{k\neq\ell} i_k(\mathcal{M}_k)$. Then $\sum_{k\in I} i_k(\mathcal{N}_k) + \mathcal{S} = \sum_{k\in I} i_k(\mathcal{M}_k)$ is the top subobject of $\bigoplus_{k\in I} M_k$. Since $\sum_{k\in I} i_k(\mathcal{N}_k)$ is a superfluous subobject of $\bigoplus_{k\in I} M_k$, we get $\mathcal{S} = \sum_{k\in I} i_k(\mathcal{M}_k)$. Applying the ℓth canonical projection $p_\ell\colon \bigoplus_{k\in I} M_k \to M_\ell$, we get $\mathcal{T} = \mathcal{M}_\ell$. This shows that $\mathcal{N}_\ell \Subset M_\ell$. ∎

It should be noted that the properties stated in Proposition 1.3(ii) and (iii) cannot be extended to infinite sums. To give counter-examples we need the notion of radical. Given a ring R and an R-module M we can define $\mathrm{Rad}(M)$ as the sum of all superfluous subobjects in M. This definition is equivalent to the usual one via the intersection of maximal subobjects by [1, Proposition 9.13].

Now consider the case $R = \mathbb{Z}$ and $M = \mathbb{Q}/\mathbb{Z}$. By [1, Exercise 9.2], we have $\mathrm{Rad}(\mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$. This shows that $\mathrm{Rad}(\mathbb{Q}/\mathbb{Z})$ is not a superfluous submodule of $\mathbb{Q}/\mathbb{Z}$ despite being a sum of superfluous submodules.

Next let R be a ring and M an R-module such that $\mathrm{Rad}(M)$ is not superfluous in M. Denote by S the set of all superfluous submodules in M. The set S is infinite as otherwise we would get a contradiction to Proposition 1.3(iii). Consider the submodule $\widetilde{N} := \bigoplus_{N\in S} N$ of $\widetilde{M} := \bigoplus_{N\in S} M$. Then $\widetilde{N}$ is an infinite direct sum of superfluous submodules. We will show that $\widetilde{N}$ is not superfluous.

Since $\mathrm{Rad}(M)$ is not a superfluous subobject of M, there is a submodule $T \subset M$ such that $T \neq M$ and $T + \mathrm{Rad}(M) = M$. We define

$$\widetilde{T} = \left\{ (m_N)_{N\in S} \in \widetilde{M} \,\middle|\, \sum_{N\in S} m_N \in T \right\}.$$

We will show that $\widetilde{N} + \widetilde{T} = \widetilde{M}$ and $\widetilde{T} \neq \widetilde{M}$. For the second assertion, note that if we take $m \in M \setminus T$ and $N \in S$ then $i_N(m)$ is not an element of $\widetilde{T}$.

Now we will show that $\widetilde{T} + \widetilde{N} = \widetilde{M}$. For this it is enough to check that for every $m \in M$ and $N \in S$ the element $i_N(m)$ belongs to $\widetilde{T} + \widetilde{N}$. Since $T + \mathrm{Rad}(M) = M$ and $\mathrm{Rad}(M)$ is the sum of all superfluous subobjects we can write m as a linear combination $m = t + \sum_{N'\in S'} m_{N'}$, $\quad t \in T, \quad m_{N'} \in N'$, where S' is a finite subset of S. Now $i_{N'}(m_{N'}) \in \widetilde{N}$ for every $N' \in S'$. Further the elements $i_N(t)$, $i_N(m_{N'}) - i_{N'}(m_{N'})$ of $\widetilde{M}$ lie in $\widetilde{T}$. Therefore

$$i_N(m) = \left(i_N(t) + \sum_{N'\in S'} (i_N(m_{N'}) - i_{N'}(m_{N'})) \right) + \sum_{N'\in S'} i_{N'}(m_{N'})$$

epimorphism

$$\phi\colon \bigoplus_{j=1}^{k} M_2 \otimes Rx_j \to M_2 \ltimes_\Gamma N$$

of Γ-graded A-modules. We have $\phi\left(\bigoplus_{j=1}^{k} M_1 \otimes Rx_j\right) = M_1 \ltimes_\Gamma N$. Thus, by Proposition 1.3(i), $M_1 \ltimes_\Gamma N \Subset M_2 \ltimes_\Gamma N$ in A-Γ-gr. Therefore $T = M_2 \ltimes_\Gamma N$. ∎

The following corollary is an immediate consequence of Propositions 3.2 and 3.3.

Corollary 3.4. *Let $\phi\colon P \to M$ be projective cover in A-Γ-gr. Suppose that N is a B-module which is flat and finitely generated over R, and such that all ${}_\beta N$ are projective B-modules. Then $\phi \ltimes_\Gamma N\colon P \ltimes_\Gamma N \to M \ltimes_\Gamma N$ is a projective cover in $A \ltimes_\Gamma B$-Γ-gr.*

For future reference, we bring together in the next theorem the results proved in this section.

Theorem 3.5. *Let A be a Γ-graded algebra and B a Γ-algebra. Suppose that N is a B-module which is flat and finitely generated over R, and such that for all $\beta \in \Gamma$ the B-modules ${}_\beta N$ are projective. Then the functor $- \ltimes_\Gamma N\colon A$-$\Gamma$-gr $\to A \ltimes_\Gamma B$-Γ-gr preserves minimal projective resolutions.*

4. Relative stratifying ideals and projective resolutions

In this section we adopt a different setting. Once more R denotes a commutative ring with identity, but A is simply an associative $R-$algebra. Given an ideal I of A, we are interested in determining conditions for the functor $A/I \otimes_A -\colon A$-Mod $\to A/I$-Mod (or, equivalently, the functor $N \mapsto N/IN$) to preserve minimal projective resolutions. For this we will use relative homological algebra. So we start with a brief overview of this topic.

We say that an A-module P is (A,R)-*projective* if for every epimorphism $f\colon M \to N$ which is split as an epimorphism of R-modules, the homomorphism $\mathrm{Hom}_A(P, f)$ is surjective.

Given an A-module M and an exact complex $P_\bullet \twoheadrightarrow M$, we say that $P_\bullet$ is an (A,R)-*projective resolution of* M if every P_k is an (A,R)-projective module and the complex $P_\bullet \twoheadrightarrow M$ is split as a complex of R-modules. Every A-module M admits a canonical (A,R)-projective resolution, $\beta_\bullet(A,R,M) \twoheadrightarrow M$, known as *bar* resolution. Recall that $\beta_k(A,R,M) = A^{\otimes_R(k+1)} \otimes_R M$, and

Proposition 3.2. *Let A be a Γ-graded R-algebra and B a Γ-algebra. Suppose that N is a B-module with the property that all the B-modules ${}_\beta N$, $\beta \in \Gamma$, are projective. Then, for any projective Γ-graded A-module P, the Γ-graded $A \ltimes_\Gamma B$-module $P \ltimes_\Gamma N$ is projective.*

Proof: First we consider the case $P = A[\beta]$, for some $\beta \in \Gamma$. We claim that $A[\beta] \ltimes_\Gamma N \cong (A \ltimes_\Gamma {}_\beta N)[\beta]$ as Γ-graded $A \ltimes_\Gamma B$-modules, and so it is projective. For this note that, for any $a' \in A_\alpha$, $a \in A$, $b \in B$, and $x \in N$, the $A \ltimes_\Gamma B$-action on $A[\beta] \ltimes_\Gamma N$ gives $(a \otimes b)(a' \otimes x) = aa' \otimes b^{\alpha\beta} x$. On the other hand, in $(A \ltimes_\Gamma {}_\beta N)[\beta]$ there holds $(a \otimes b)(a' \otimes x) = aa' \otimes (b^\alpha)^\beta x = aa' \otimes b^{\alpha\beta} x$. Therefore the identity map gives the desired isomorphism.

Now let P be an arbitrary projective Γ-graded A-module. Then P is a direct summand of $\bigoplus_{\beta \in I} A[\beta]$, for some family I of elements in Γ. Since the functor $- \ltimes_\Gamma N$ preserves direct sums, we get that $P \ltimes_\Gamma N$ is a direct summand of

$$\Big(\bigoplus_{\beta \in I} A[\beta] \Big) \ltimes_\Gamma N \cong \bigoplus_{\beta \in I} \Big(A[\beta] \ltimes_\Gamma N \Big),$$

which is projective. Therefore $P \ltimes_\Gamma N$ is a projective Γ-graded $A \ltimes_\Gamma B$-module. ∎

Let N be a B-module which is flat as an R-module and such that all ${}_\beta N$ are projective B-modules. Then Proposition 3.2 shows that the functor $- \ltimes_\Gamma N$ preserves projective resolutions. Note that it does not map in general a minimal projective resolution into a minimal projective resolution.

Proposition 3.3. *Suppose $M_1 \Subset M_2$ in A-Γ-gr. Then, for any B-module N which is finitely generated over R, we have $M_1 \ltimes_\Gamma N \Subset M_2 \ltimes_\Gamma N$ in $A \ltimes_\Gamma B$-Γ-gr.*

Proof: Let T be a Γ-graded $A \ltimes_\Gamma B$-submodule of $M_2 \ltimes_\Gamma N$ such that

$$M_1 \ltimes_\Gamma N + T = M_2 \ltimes_\Gamma N. \tag{7}$$

Every Γ-graded $A \ltimes_\Gamma B$-module can be considered as a Γ-graded A-module via the canonical homomorphism $A \to A \ltimes_\Gamma B$. Therefore (7) also holds in the category of Γ-graded A-modules. Let $\{x_1, \ldots, x_k\}$ be a generating set of N over R. Since $M_1 \Subset M_2$ in A-Γ-gr, we get from Proposition 1.3(iii), that $\bigoplus_{j=1}^k M_1 \otimes Rx_j \Subset \bigoplus_{j=1}^k M_2 \otimes Rx_j$ in A-Γ-gr. Consider the canonical

is a sum of an element in $\widetilde{T}$ and of an element in $\widetilde{N}$. This shows $\widetilde{T} + \widetilde{N} = \widetilde{M}$ and thus $\widetilde{N}$ is not a superfluous submodule of $\widetilde{M}$ despite being a direct sum of superfluous subobjects in the corresponding components.

For completeness of the exposition we recall the definitions of projective object and projective resolution. An object P in $\mathcal{C}$ is called *projective* if for every epimorphism $\phi\colon X \to Y$ in $\mathcal{C}$ the map $\mathcal{C}(P, f)\colon \mathcal{C}(P, X) \to \mathcal{C}(P, Y)$ is an epimorphism. A *projective cover* of Y in $\mathcal{C}$ is a projective object P together with an epimorphism $\pi\colon P \to Y$ such that $\ker \pi \Subset P$. If a projective cover of Y exists it is unique up to isomorphism (cf. [12, Theorem 5.1]).

A *projective resolution* of an object $M \in \mathcal{C}$ is an exact complex $P_\bullet = (P_k, k \geq -1)$ with differentials $d_k\colon P_{k+1} \to P_k$, $P_{-1} = M$, and P_k projective. This resolution is called *minimal* if, additionally, $\ker(d_k) \Subset P_{k+1}$ for all $k \geq -1$.

It follows from the definition, that if $P_\bullet$ is a minimal projective resolution of M, then $d_{-1}\colon P_0 \to M$ and $d_k\colon P_{k+1} \to \ker(d_{k-1})$ for $k \geq 0$ are projective covers. Since a projective cover is unique up to isomorphism, we see by an induction argument, that a minimal projective resolution of M is unique up to isomorphism.

2. Graded algebras and modules

Let R be a commutative ring with identity, Γ a monoid with neutral element e, and A a Γ-*graded associative R-algebra*, that is we have an R-module decomposition

$$A = \bigoplus_{\gamma \in \Gamma} A_\gamma,$$

satisfying $A_{\gamma_1} A_{\gamma_2} \subset A_{\gamma_1 \gamma_2}$ for all $\gamma_1, \gamma_2 \in \Gamma$. We also assume that the identity e_A of A is an element of A_e.

A left A-module M is Γ-*graded* if $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$, where each M_γ is an R-submodule of M, and $A_{\gamma_1} M_{\gamma_2} \subset M_{\gamma_1 \gamma_2}$ for all $\gamma_1, \gamma_2 \in \Gamma$. We will work with the category A-Γ-gr of left Γ-graded A-modules and A-module homomorphisms respecting the grading: a map of A-modules $f\colon M_1 \to M_2$ is in A-Γ-gr if $f(M_{1,\gamma}) \subset M_{2,\gamma}$ for all $\gamma \in \Gamma$. This category A-Γ-gr is abelian (see for example Proposition 3.1 in [12]).

Given a Γ-graded A-module M, the support of M is

$$\operatorname{supp}(M) = \{\, \gamma \in \Gamma \mid M_\gamma \ncong 0 \,\}.$$

We call a monoid Γ equipped with an order $<$ an *ordered monoid* if for any three elements α, β, $\gamma \in \Gamma$ such that $\alpha < \beta$ we have that also $\alpha\gamma < \beta\gamma$ and $\gamma\alpha < \gamma\beta$. Notice that we require that the multiplication with γ preserves strict inequalities. This is a stronger condition than to require just preservability of non-strict inequalities. Terminological conventions on this matter can vary from article to article.

A poset $(\mathcal{P}, <)$ is called *well-founded* if every strictly decreasing sequence of elements in $\mathcal{P}$ is finite. With this definition we can speak about well-founded ordered monoids.

The following theorem is a reformulation of Proposition 5.2 in [10].

Theorem 2.1. *Let Γ be a well-founded ordered monoid such that e is the least element of Γ. Then every Γ-graded A-module has a projective cover if and only if every A_e-module has a projective cover.*

Consider the general monoid Γ. For every $\beta \in \Gamma$ and every Γ-graded A-module M, we define the Γ-graded A-module $M[\beta]$ to be M as a left A-module with the homogeneous components given by $M[\beta]_\gamma = \bigoplus_{\alpha\beta=\gamma} M_\alpha$. Note that each component M_α of M appears exactly once in the decomposition $M[\beta] = \bigoplus_{\gamma\in\Gamma} M[\beta]_\gamma$, since M_α is a direct summand of $M[\beta]_{\alpha\beta}$ and not a direct summand of any $M[\beta]_\gamma$ for $\gamma \neq \alpha\beta$.

It is proved in [10, Proposition 4.3], that the set $\{\, A[\beta] \mid \beta \in \Gamma\}$ is a set of projective generators for the category of Γ-graded A-modules. This fact will be used in the proof of the following proposition.

Proposition 2.2. *Let Γ be a monoid and A a Γ-graded algebra. If P is a projective Γ-graded A-module, then P is projective as an A-module.*

Proof: Since P is a projective Γ-graded A-module it is a direct summand of $\bigoplus_{\beta\in\mathcal{B}} A[\beta]$, for some family $\mathcal{B}$ of elements in Γ. As $A[\beta] \cong A$ as an A-module we get that P is a direct summand of the free A-module $\bigoplus_{\beta\in\mathcal{B}} A$. Hence P is a projective A-module. ∎

Proposition 2.2 implies that if $P_\bullet$ is a projective resolution of a Γ-graded A-module M then $P_\bullet$ is a projective resolution of M considered as an A-module. If $P_\bullet$ is minimal as Γ-graded projective resolution, it is not always true that it is minimal if considered without grading. Our next step will be to determine conditions under which minimality is preserved upon forgetting the grading.

Note that the maps $A \to A \ltimes_\Gamma B$ and $B \to A \ltimes_\Gamma B$ given by

$$a \mapsto a \otimes 1_B \qquad\qquad b \mapsto 1_A \otimes b$$

are homomorphisms of algebras, being the first one a homomorphism of Γ-graded algebras.

Let N be a B-module and $M = \bigoplus_{\gamma\in\Gamma} M_\gamma$ a Γ-graded A-module. We define an $(A \ltimes_\Gamma B)$-module structure on $M \otimes N$ as follows

$$a_{\gamma_1} \otimes b \otimes m_{\gamma_2} \otimes x \mapsto a_{\gamma_1} m_{\gamma_2} \otimes b^{\gamma_2} x,$$

for all $a_{\gamma_1} \in A_{\gamma_1}$, $b \in B$, $m_{\gamma_2} \in M_{\gamma_2}$ and $x \in N$. We denote this module by $M \ltimes_\Gamma N$. This is a Γ-graded module, with $(M \ltimes_\Gamma N)_\gamma = M_\gamma \otimes N$.

Let $\varphi\colon M_1 \to M_2$ be a homomorphism of Γ-graded A-modules and $\psi\colon N_1 \to N_2$ a homomorphism of B-modules. We write $\varphi \ltimes_\Gamma \psi$ for the map

$$M_1 \ltimes_\Gamma N_1 \to M_2 \ltimes_\Gamma N_2$$
$$m \otimes x \mapsto \varphi(m) \otimes \psi(x).$$

Clearly $\varphi \ltimes_\Gamma \psi$ is a homomorphism of Γ-graded $A \ltimes_\Gamma B$-modules. It follows that the correspondence

$$(M, N) \mapsto M \ltimes_\Gamma N$$
$$(\varphi, \psi) \mapsto \varphi \ltimes_\Gamma \psi$$

gives a bifunctor from the categories A-Γ-gr and B-Mod to the category $(A \ltimes_\Gamma B)$-Γ-gr. In particular, for each B-module N, we have the functor $- \ltimes_\Gamma N$ from the category A-Γ-gr to the category $(A \ltimes_\Gamma B)$-Γ-gr. This functor is exact if and only if N is a flat R-module. Note that $- \ltimes_\Gamma N$ preserves arbitrary direct sums. In fact, the R-isomorphism

$$\phi\colon \Big(\bigoplus_{i\in I} M_i\Big) \ltimes_\Gamma N \to \bigoplus_{i\in I} (M_i \ltimes_\Gamma N)$$
$$(m_i)_{i\in I} \otimes x \mapsto (m_i \otimes x)_{i\in I}.$$

is also an isomorphism of Γ-graded $A \ltimes_\Gamma B$-modules.

Next we will establish a sufficient condition for the functor $- \ltimes_\Gamma N$ to preserve projective objects. For every $\beta \in \Gamma$, we denote by ${}_\beta N$ the B-module with the same underlying abelian group as N but with the B-action defined by

$$b \cdot_\beta x = b^\beta x, \tag{6}$$

for every $x \in N$ and $b \in B$.

there are a ring homomorphism $\phi\colon S \to A$ and an S-bimodule isomorphism $\gamma\colon A_1 \otimes_S A_2 \to A$ such that

$$\gamma(\phi_1(s)\otimes 1) = \gamma(1\otimes\phi_2(s)) = \phi(s), \quad \gamma(a_1\otimes a_2) = \gamma(a_1\otimes 1)\gamma(1\otimes a_2)$$

$$\gamma(a_1a_1'\otimes 1) = \gamma(a_1\otimes 1)\gamma(a_1'\otimes 1), \quad \gamma(1\otimes a_2a_2') = \gamma(1\otimes a_2)\gamma(1\otimes a_2'). \tag{5}$$

If A is a twisted product of A_1 and A_2 over S, one can define a *twisting homomorphism* of abelian groups

$$T\colon A_2\otimes_S A_1 \to A_1\otimes_S A_2$$
$$a_2\otimes a_1 \mapsto \gamma^{-1}(\gamma(1\otimes a_2)\gamma(a_1\otimes 1)).$$

Note that it is then possible to reconstruct A from ϕ_1, ϕ_2 and T. The name *twisted product* is justified by the existence of the map T.

Twisted products of algebras over fields where studied in [4]. A more general approach, that can be applied to monoids in arbitrary monoidal categories, was considered in [2].

Next we study a twisted product involving a Γ-graded algebra. As usual, all the unnamed tensor products are considered over R.

We say that an R-algebra B is a Γ-*algebra* if there is a right action of Γ on B

$$r\colon B\times\Gamma \to B$$
$$(b,\gamma)\mapsto b^\gamma,$$

such that for each $\gamma\in\Gamma$ the map $b\mapsto b^\gamma$ is an algebra homomorphism.

Let A be a Γ-graded R-algebra and B a Γ-algebra. We define a binary operation m on $A\otimes B$ by

$$(a\otimes b)(a'\otimes b') = aa'\otimes b^\beta b',\ \text{for } a\in A_\alpha,\ a'\in A_\beta,\ b,b'\in B.$$

The R-module $A\otimes B$ when considered together with the binary operation m will be denoted by $A\ltimes_\Gamma B$. It is routine to check that the following proposition holds.

Proposition 3.1. *The pair $(A\ltimes_\Gamma B, m)$ is an R-algebra with identity $1_A\otimes 1_B$. It is a twisted product of A and B (over R), where ϕ_A, ϕ_B, and $\phi_{A\ltimes_\Gamma B}$ are the unity maps, and $\gamma\colon A\otimes B\to A\ltimes_\Gamma B$ is the identity map. Moreover $A\ltimes_\Gamma B$ is Γ-graded, with the grading given by $(A\ltimes_\Gamma B)_\gamma = A_\gamma\otimes B$, $\gamma\in\Gamma$.*

Proposition 2.3. *Let Γ be a well-founded ordered monoid with least element e, and A a Γ-graded algebra. Suppose M is a Γ-graded A-module with finite support and N is a superfluous subobject of M in the category A-Γ-gr. Then N is a superfluous subobject of M in the category A-Mod.*

Proof: We will prove the proposition by induction on the cardinality of the support of M. If $|\operatorname{supp}(M)| = 1$ there is nothing to prove, as any A-submodule of M is automatically Γ-graded. Suppose the result holds for all M with $|\operatorname{sup}(M)|\le n-1$. Consider M with $|\operatorname{supp}(M)| = n$. Let T be an A-submodule of M such that $N+T=M$. We have to show that $T=M$. Let γ be a maximal element of $\operatorname{supp}(M)$. Define M' as the Γ-graded A-submodule of M generated by $\bigoplus_{\beta\neq\gamma} M_\beta$, that is $M'_\beta = M_\beta$ if $\beta\neq\gamma$, and $M'_\gamma = \bigoplus_{\beta<\gamma}\bigoplus_{\alpha\beta=\gamma} A_\alpha M_\beta$. Note that in the last sum, we can take $\beta<\gamma$ and not just $\beta\neq\gamma$, as the existence of α such that $\alpha\beta=\gamma$ implies that $\gamma>\beta$. Also, since γ is a maximal element of $\operatorname{supp}(M)$, both M'_γ and M_γ are Γ-graded A-submodules of M. We will prove first that

$$M'_\gamma \subset T. \tag{2}$$

By Proposition 1.3(i), we have that

$$(N+M_\gamma)/M_\gamma \Subset M/M_\gamma \tag{3}$$

in the category A-Γ-gr. Since the support of M/M_γ has cardinality $n-1$, by the induction hypothesis we get that (3) also holds in A-Mod. Obviously

$$(N+M_\gamma)/M_\gamma + (T+M_\gamma)/M_\gamma = M/M_\gamma\,.$$

Therefore

$$(T+M_\gamma)/M_\gamma = M/M_\gamma\,. \tag{4}$$

It is now easy to show (2). In fact, we only have to check that $A_\alpha M_\beta\subset T$ for every $\beta<\gamma$ and α, such that $\alpha\beta=\gamma$. Note that, for such α we have $\alpha\neq e$ and so $\alpha>e$. Let $y\in M_\beta$ and $a\in A_\alpha$. It follows from (4) that there is $z\in M_\gamma$ such that $y+z\in T$. Thus also $a(y+z)\in T$. Since $\alpha\gamma>\gamma$ and γ is a maximal element of $\operatorname{supp}(M)$, we get that $az=0$. Thus $ay\in T$, i.e. $A_\alpha M_\beta\subset T$. Now define

$$\overline{M} = M\Big/M'_\gamma\,, \qquad \overline{N} = \big(N+M'_\gamma\big)\Big/M'_\gamma \subset \overline{M}, \qquad \overline{T} = T\Big/M'_\gamma \subset \overline{M}.$$

Note that $\overline{M}$ is the internal direct sum of $\overline{M}_\gamma$ and $\bigoplus_{\beta\neq\gamma}\overline{M}_\beta$ in the category of Γ-graded A-modules. Our next step is to prove that $\overline{N}\Subset\overline{M}$ in the

category A-Mod. By Proposition 1.3(i) we have that $\overline{N} \Subset \overline{M}$ in the category A-Γ-gr. Since $\overline{N}$ is Γ-graded, we get that $\overline{N}_\gamma$ is a Γ-graded A-submodule of $\overline{M}_\gamma$ and $\bigoplus_{\beta\neq\gamma}\overline{N}_\beta$ is a Γ-graded A-submodule of $\bigoplus_{\beta\neq\gamma}\overline{M}_\beta$. Moreover $(\bigoplus_{\beta\neq\gamma}\overline{N}_\beta)\bigoplus\overline{N}_\gamma = \overline{N}$. Therefore from Proposition 1.3(iii), we have that $\overline{N}_\gamma \Subset \overline{M}_\gamma$ and $\bigoplus_{\beta\neq\gamma}\overline{N}_\beta \Subset \bigoplus_{\beta\neq\gamma}\overline{M}_\beta$ in the category A-Γ-gr. Since the cardinalities of the supports of $\overline{M}_\gamma$ and of $\bigoplus_{\beta\neq\gamma}\overline{M}_\beta$ are less than n, the induction assumption gives that $\overline{N}_\gamma \Subset \overline{M}_\gamma$ and $\bigoplus_{\beta\neq\gamma}\overline{N}_\beta \Subset \bigoplus_{\beta\neq\gamma}\overline{M}_\beta$ in A-Mod. Applying Proposition 1.3(iii) with $\mathcal{C}$ the category A-Mod, we conclude that $\overline{N} \Subset \overline{M}$ in A-Mod. Thus $\overline{N}+\overline{T} = \overline{M}$ implies that $\overline{T} = \overline{M}$. Therefore $T = M$. ∎

We will use the result of Proposition 2.3 only in the case M is a projective module. It is natural to wonder if the condition of finiteness on $|\operatorname{supp}(M)|$ is redundant. The next example shows that this is not the case.

Let R be a commutative ring with identity, with the property that its Jacobson radical $J = J(R)$ is not left T-nilpotent. For example, we can take $R = \mathbb{Z}_{(p)}$, the localization of $\mathbb{Z}$ at the ideal (p), for some prime p. Consider R as an $\mathbb{N}$-graded ring with $R_0 = R$ and all other homogeneous components equal to zero. Define the $\mathbb{N}$-graded R-module F by $F_n = {}_RR$. Clearly F is a projective $\mathbb{N}$-graded R-module. If we consider F as an R-module, by Proposition 9.19 in [1], we have $N := \operatorname{Rad}(F) = \bigoplus_{n\in\mathbb{N}} J$. In particular, N is an $\mathbb{N}$-graded submodule of F. It is easy to check that $N \Subset F$ in R-$\mathbb{N}$-gr. Indeed, if $T = \bigoplus_{n\in\mathbb{N}} T_n$ is another $\mathbb{N}$-graded R-submodule of F such that $N+T = F$, then for each component n we have $N_n+T_n = R$, i.e. $J+T_n = R$. Since $J \Subset R$ in R-Mod, this implies $T_n = R$ and $T = F$.

By Proposition 17.10 and Lemma 28.3 in [1], we know that $N = JF$, and that JF is a superfluous subobject of F in R-Mod if and only if J is a left T-nilpotent ideal. Hence, under our assumption on the ring R, N is not a superfluous submodule of F in R-Mod. This shows that forgetting the grading can render a superfluous subobject to become non-superfluous.

Proposition 2.4. *Let Γ be a monoid, A a Γ-graded algebra with finite support, and M a Γ-graded A-module with finite support. Then there exists a projective resolution $P_\bullet$ of M in A-Γ-gr such that each P_k has finite support.*

Proof: It is enough to show that for every M with finite support there is a projective Γ-graded A-module P with finite support and an epimorphism $f\colon P \to M$. Then using this fact for $\ker(f)$, we get $P_2 \to \ker(f)$, and

hence the first two steps of a projective resolution of M with finite support. Repeating this process recursively we obtain a projective resolution of the required type.

Since $\{\, A[\beta] \mid \beta \in \Gamma\}$ is a set of projective generators of A-Γ-gr, there is $P = \bigoplus_{\beta\in\Gamma} A[\beta]^{\kappa_\beta}$, where κ_β are cardinals, and an epimorphism of Γ-graded A-modules $f\colon P \to M$. Suppose $\beta \notin \operatorname{supp}(M)$. Then the restriction of f to each summand $A[\beta]$ in P is zero. In fact, the module $A[\beta]$ is generated as an A-module by the element $e \in A_e \subset A[\beta]_\beta$, and its image under f is in $M_\beta = 0$. Therefore, without loss of generality, we can assume that $\kappa_\beta = 0$ for all $\beta \notin \operatorname{supp}(M)$. Since $|\operatorname{supp}(A[\beta])| \leq |\operatorname{supp}(A)|$ and A has finite support, we get that $|\operatorname{supp}(P)| \leq \sum_{\beta\in\operatorname{supp}(M)} |\operatorname{supp}(A[\beta])| \leq |\operatorname{supp}(M)| \cdot |\operatorname{supp}(A)|$ is finite. ∎

We can now state and prove the main result of this section.

Theorem 2.5. *Let Γ be a well-founded ordered monoid with least element e, and A a Γ-graded algebra with finite support. Given a Γ-graded A-module M with finite support, let $P_\bullet$ be a minimal projective resolution of M in A-Γ-gr. Then $P_\bullet$ is a minimal projective resolution of M in the category A-Mod. In other words, the grading forgetting functor from A-Γ-gr to A-Mod preserves minimal projective resolutions of Γ-graded A-modules with finite support.*

Proof: We know that $P_\bullet$ is a projective resolution of M in the category A-Mod by Proposition 2.2. Thus we have only to check that it is minimal.

By Proposition 2.4 there is a projective resolution $\overline{P}_\bullet$ of M in A-Γ-gr such that all $\overline{P}_k$, $k \geq 0$, have finite support. Since $P_\bullet$ is a minimal projective resolution (by applying for example Theorem 5.1 in [12]) there is an embedding of $P_\bullet$ into $\overline{P}_\bullet$. Thus each P_k, $k \geq 0$, has finite support. Since the resolution $P_\bullet$ is minimal in A-Γ-gr, all the maps $d_{-1}\colon P_0 \to M$ and $d_k\colon P_{k+1} \to P_k$ for $k \geq 0$ have superfluous kernels in A-Γ-gr. From Proposition 2.3, these kernels are also superfluous in A-Mod. ∎

3. Twisted products

We start this section with an overview of the concept of twisted product of rings. Then we specialise to twisted products of Γ-graded algebras and modules, and study under which conditions the functor $- \ltimes_\Gamma N$, defined below, preserves minimal projective resolutions.

Given rings S, A_1, and A_2 , suppose we have ring homomorphisms $\phi_i\colon S \to A_i$, for $i = 1, 2$. We say that A is a *twisted product* of A_1 and A_2 over S if