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# WHAT MAKES A STONE TOPOLOGICAL ALGEBRA PROFINITE

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Abstract: This paper is a contribution to understanding what properties should a topological algebra on a Stone space satisfy to be profinite. We reformulate and simplify proofs for some known properties using syntactic congruences. We also clarify the role of various alternative ways of describing syntactic congruences, namely by finite sets of terms and by compact sets of continuous self mappings of the algebra.

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## 1. Introduction

Profinite algebras, that is, inverse limits of inverse systems of finite algebras, appear naturally in several contexts. There is an extensive theory of profinite groups, which appear as Galois groups but are also studied as a generalization of finite groups [17, 25]. One may also view $p$-adic number theory as an early study of special profinite algebraic structures [26]. In the context of general algebras, profinite topologies seem to have first appeared in [11]. A fruitful line of development came about with the discovery that formal equalities between elements of free profinite algebras may be used to describe pseudovarieties [23, 10], which are classes of finite algebras of a fixed type closed under taking homomorphic images, subalgebras, and finite direct products. Since pseudovarieties play an important role in algebraic theories developed for computer science,

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profinite algebras are also a useful tool in that context. In particular, profinite semigroups have been extensively used (see, for instance, [2, 28, 8, 4, 24, 5, 6]).

As the underlying topological structure of a profinite algebra is compact and 0-dimensional, that is, a Stone space, profinite algebras may also be viewed as dual spaces of Boolean algebras. For finitely generated relatively free profinite algebras, the corresponding Boolean algebras have special significance [2, Theorem 3.6.1] and are particularly relevant in the applications of finite semigroup theory to the theory of formal languages, where they appear as Boolean algebras of regular languages. The dual role of the algebraic operations in relatively free profinite algebras has also been investigated [19, 20, 18].

For certain classes of (topological) algebras, it turns out that being a Stone space is sufficient to guarantee profiniteness. Special cases were considered in [22] but the essential ingredient lies in the fact that syntactic congruences are determined by finitely many terms [1, 15] using an idea of Hunter [21] that may be traced back to Numakura [22]. For such classes of algebras, profiniteness is thus a purely topological property, although this is not true in general.

Recently, several characterizations of profiniteness in a Stone topological algebra $A$ have been obtained in [27]. They are formulated in terms of topological properties of the translation monoid of the algebra $A$, which is a submonoid of the monoid of continuous transformations of $A$, which is itself a topological monoid under the compact-open topology.

We explore further the role of syntactic congruences in the characterization of profiniteness. This leads to an extension to topological algebras over topological signatures of Gehrke's sufficient condition for a quotient of a profinite algebra to be profinite, with a simplified proof. The proofs of Schneider and Zumbrägel's characterizations of profiniteness in this language are also somewhat simplified. The key ingredient is quite simple: a Stone topological algebra is profinite if and only if the syntactic congruence of every clopen subset is clopen.

As already mentioned above, the existence of descriptions of the syntactic congruences of clopen subsets by a finite number of terms is a sufficient condition for profiniteness. In such terms, all but one variable are evaluated to arbitrary values in the algebra, which means that potentially infinitely many polynomials in one variable are used. In a profinite algebra, each such syntactic congruence may in fact be described by finitely many linear polynomials in one variable, which may depend on the congruence. We explore more generally the property of a syntactic congruence being determined by a compact set of continuous self mappings of the algebra, a property that, for a clopen subset of a locally



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compact algebra is equivalent to the congruence being clopen. This leads to several further characterizations of profiniteness for a Stone topological algebra in terms of how their syntactic congruences may be described.

## 2. Topological algebras

We say that an equivalence relation on a topological space $X$ is *closed* (respectively *open* or *clopen*) if it is a closed (respectively open or clopen) subset of the product space $X \times X$. It is easy to verify that an equivalence relation is open if and only if it is clopen, if and only if its classes are open, if and only if its classes are clopen (see, for instance, [6, Exercise 3.40]). For a closed equivalence relation, the classes are closed, but the converse fails in general [6, Exercise 3.39]. Given an equivalence relation $\theta$ on a topological space $X$, the quotient set $X/\theta$ is endowed with the largest topology that renders continuous the natural mapping $X \to X/\theta$. In particular, a set of $\theta$-classes is closed (respectively, open) if and only if so is its union in $X$.

Unlike some literature on topology, we require the Hausdorff separation property for a space to be locally compact or compact.

The following observation may be considered as an exercise in topology, based on [12, Chapter I, §10.4, Proposition 8].

**Proposition 2.1.** *Let $X$ be a compact space and $\theta$ an equivalence relation on $X$. Then the quotient space $X/\theta$ is compact if and only if $\theta$ is closed.*

Following [27], by a *signature* we mean a sequence $\Omega = (\Omega_n)_{n\in\mathbb{N}}$ of sets. We say that it is a *topological signature* if each set $\Omega_n$ is endowed with a topology.

An $\Omega$-*algebra* is a pair $(A, E)$, where $A$ is a set and $E = (E_n^A)_{n\in\mathbb{N}}$ is a sequence of *evaluation* mappings $E_n^A : \Omega_n \times A^n \to A$. In case $A$ is a topological space and $\Omega$ is a topological signature, we say that the $\Omega$-algebra is a *topological* $\Omega$-*algebra* if each mapping $E_n^A$ is continuous. For an $\Omega$-algebra $(A, E)$ and $w \in \Omega_n$, we let $w_A : A^n \to A$ be the operation defined by

$$w_A(a_1, \ldots, a_n) = E_n^A(w, a_1, \ldots, a_n).$$

Reference to the sequences $E$ and $\Omega$, which should be understood from the context, will usually be omitted and so we simply say that $A$ is an algebra.

We view finite algebras as discrete topological algebras. Note that the requirement that the evaluation mappings be continuous may still be nontrivial. For instance, if we take the signature $\Omega$ with one binary operation symbol and $\Omega_1$ the one point ($\infty$) compactification of $\mathbb{N}$, we may consider finite semigroups

as $\Omega$-algebras by interpreting the binary operation symbol as the semigroup multiplication, each unary operation symbol $n \in \mathbb{N}$ as the $n!$ power and $\infty$ as the unique idempotent power. The evaluation mappings are continuous, and so we obtain a topological $\Omega$-algebra. However, it is sometimes useful to consider an ("unnatural") interpretation of the operation symbol $\infty$ that makes the evaluation mapping $E_1$ discontinuous. This idea underlies the recent paper [7].

For a class $\mathcal{K}$ of topological algebras, we say that a topological algebra $A$ is *residually* $\mathcal{K}$ if, for every pair $a, a'$ of distinct elements of $A$, there is a continuous homomorphism $\varphi : A \to B$ into a member $B$ of $\mathcal{K}$ such that $\varphi(a) \neq \varphi(b)$.

A topological algebra $A$ is said to be *profinite* if it is residually finite and $A$ is compact. Equivalently, $A$ is an inverse limit of finite discrete algebras (see, for instance, [5]).

A continuous mapping from a topological space $X$ to a topological algebra $A$ is said to be a *generating mapping* if its image generates (algebraically) a dense subalgebra of $A$. In case $X$ is a subset of $A$, we say that the topological algebra $A$ is *generated* by $X$ if the inclusion $X \hookrightarrow A$ is a generating mapping. We also say that $A$ is $X$*-generated* if there is a generating mapping $X \to A$ and that it is *finitely generated* if it is $X$-generated for some finite set $X$.

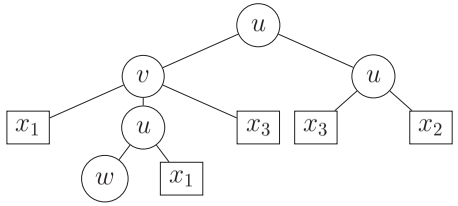
By a *congruence* on an $\Omega$-algebra $A$ we mean an equivalence relation $\theta$ that is compatible with the $\Omega$ operations. The set $A/\theta$ of all congruence classes $a/\theta$ inherits a natural structure of $\Omega$-algebra: for $w \in \Omega_n$, $w_{A/\theta}(a_1/\theta, \ldots, a_n/\theta) = w_A(a_1, \ldots, a_n)/\theta$. In case $A$ is a topological $\Omega$-algebra, the quotient algebra $A/\theta$ is a topological $\Omega$-algebra for the quotient topology.

For a set $X$, let $T_\Omega(X)$ be the $\Omega$*-term algebra*, that is, the absolutely free $\Omega$-algebra. The elements of $T_\Omega(X)$ are usually viewed in computer science as trees whose leaves are labeled by elements of $X$ or of $\Omega_0$ and each non-leaf node is labeled by an element of some $\Omega_n$, in which case the node has exactly $n$ sons.

For instance, for $u \in \Omega_2$, $v \in \Omega_3$, and $w \in \Omega_0$, the term

$$u(v(x_1, u(w, x_1), x_3), u(x_3, x_2))$$

is represented by the labeled tree pictured below.






(8) *for every clopen subset $L \subseteq A$, the congruence $\sigma_L$ is $F$-determined by some finite subset $F$ of $M(A)$;*
(9) *for every clopen subset $L \subseteq A$, the congruence $\sigma_L$ is $F$-determined by some finite subset $F$ of $\mathcal{C}(A)$;*
(10) *for every clopen subset $L \subseteq A$, the congruence $\sigma_L$ is $C$-determined by some compact subset $C$ of $\mathcal{C}(A)$.*

*Proof*: By Theorems 4.1, 4.3 and 4.4, we have the equivalences (2) $\Leftrightarrow$ (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). In fact, in the proof of Theorem 4.1, we showed that (1) $\Rightarrow$ (6) $\Rightarrow$ (2), whence we also have (1) $\Leftrightarrow$ (6)

The implication (5) $\Rightarrow$ (4) is obvious. To establish the reverse implication (4) $\Rightarrow$ (5) it is enough to prove that $\overline{M(A)}$ is a 0-dimensional topological monoid, which holds if so is $\mathcal{C}(A)$. Now, given a clopen subset $L \subseteq A$, we have

$$\mathcal{C}(A) \setminus [K, L] = \bigcup_{a \in K} [a, A \setminus L],$$

which shows that $[K, L]$ is closed for every subset $K$ of $A$. Thus, if $K, L \subseteq A$ are clopen, then so is $[K, L]$. Hence, $\mathcal{C}(A)$ is 0-dimensional and it was already observed after Proposition 5.4 that $\mathcal{C}(A)$ is a topological monoid.

The equivalence of (1), (7), (8), and (10) follows from Theorems 5.16 and 4.1. To conclude the proof, it remains to observe that the implications (8) $\Rightarrow$ (9) and (9) $\Rightarrow$ (10) are trivial. ∎

Some of the results of Section 5 suggest looking at locally compact residually discrete algebras as a generalization of profinite algebras and, more generally at locally compact 0-dimensional algebras as a generalization of Stone topological algebras. However, it is not clear where such a study might lead, perhaps for lack of interesting examples. Examples 5.12 and 5.17 show that much of the good behavior observed in the compact case breaks down for locally compact algebras.

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Now we take $L = \{(n,n) \mid n \in \mathbb{N}\}$, which is clopen in $A \times B$ because

$$L = \bigcup_{n\in\mathbb{N}} \{(n,n)\} \quad \text{and} \quad (A \times B) \setminus L = \bigcup_{n\in\mathbb{N}} \{n\} \times (B \setminus \{n\}).$$

We claim that one class of the syntactic congruence $\sigma_L$ is $A \times \{\infty\}$. All elements in $A \times \{\infty\}$ are $\sigma_L$-related as it is not possible to multiply them by any element and obtain a result in $L$. To show that elements from $A \times \{\infty\}$ are not $\sigma_L$-related with other elements, consider pairs $(i,j)$ and $(k,\infty)$ with $i,j,k \in \mathbb{N}$. Then $(i+j,i)\cdot(i,j) = (i+j,i+j) \in L$ but $(i+j,i)\cdot(k,\infty) = (\max\{i+j,k\},\infty) \notin L$. Hence, $(i,j)$ and $(k,\infty)$ are not $\sigma_L$-equivalent. Finally, we claim that $A \times \{\infty\}$ is not open, which establishes that $\sigma_L$ is not clopen. To show that $A \times \{\infty\}$ is not open, recall first that in the product space $A \times B$, a base of the topology consists of open subsets $O \times O'$ with $O$ and $O'$ open subsets respectively of $A$ and $B$. However, if we consider $(k,\infty)$ in such $O \times O'$, then there is also some element $(k,\ell)$ in $O \times O'$ with the same first coordinate and $\ell \in \mathbb{N}$.

This example has another feature that it is worth noting. The algebra $A \times B$ is residually finite. Hence, while the condition that the syntactic congruence of a clopen subset of a locally compact 0-dimensional algebra is always clopen implies that the algebra is residually discrete, the converse fails even under the stronger assumption of residual finiteness. This is in contrast with the case of compact algebras, for which the two conditions are equivalent by Theorem 4.1. ∎

## 6. Summary of results and conclusion

In conclusion, we have the following result building on the various characterizations of profiniteness in Stone topological algebras presented in this paper.

**Theorem 6.1.** *The following conditions are equivalent for a Stone topological algebra $A$:*

(1) *$A$ is profinite;*
(2) *for every clopen subset $L \subseteq A$, $\sigma_L$ is a clopen congruence;*
(3) *$M(A)$ is equicontinuous;*
(4) *$M(A)$ is relatively compact in $\mathcal{C}(A)$;*
(5) *the closure of $M(A)$ in $\mathcal{C}(A)$ is a profinite submonoid;*
(6) *for every clopen subset $L \subseteq A$, there exists a continuous homomorphism $\varphi : A \to B$ onto a finite algebra $B$ such that $L = \varphi^{-1}(\varphi(L))$;*
(7) *for every clopen subset $L \subseteq A$, the congruence $\sigma_L$ is determined by some finite set of terms;*



If the tree representing the term $t$ has exactly one occurrence of $x \in X$ as a leaf label, then we say that $t$ is *linear in $x$*. For instance, the term above is linear only in $x_2$.

Let $\varphi : X \to A$ be any mapping. Since the term algebra $T_\Omega(X)$ is the absolutely free algebra over the set $X$, there is a unique homomorphism $\hat{\varphi} : T_\Omega(X) \to A$ such that $\hat{\varphi} \circ \iota = \varphi$, where $\iota$ is the inclusion mapping of $X$ in $T_\Omega(X)$. In case $X = \{x_1, \ldots, x_m\}$, we also denote $\hat{\varphi}(t)$ by $t_A\big(\varphi(x_1), \ldots, \varphi(x_m)\big)$ for $t \in T_\Omega(X)$. In this case, given elements $a_j \in A$ $(j \neq i)$, a term $t \in T_\Omega(X)$ determines a *polynomial transformation* of $A$ given by

$$a \mapsto t_A(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_m).$$

Note that, when $A$ is a topological algebra, such transformations are continuous. The *translation monoid* $M(A)$ consists of all such polynomial transformations given by terms that are linear in the distinguished variable $x_i$.

More generally, for two topological spaces $X$ and $Y$, let $\mathcal{C}(X,Y)$ be the set of continuous mappings from $X$ to $Y$; it is endowed with the compact-open topology, for which a subbase consists of all sets of the form

$$[K,U] = \big\{f \in \mathcal{C}(X,Y) : f(K) \subseteq U\big\},$$

where $K \subseteq X$ and $U \subseteq Y$ are respectively compact and open. Subsets of $\mathcal{C}(X,Y)$ are endowed with the induced topology. We also write $\mathcal{C}(X)$ for $\mathcal{C}(X,X)$. Note that, in case $A$ is a topological algebra, $M(A)$ is a submonoid of the monoid $\mathcal{C}(A)$.

Note that, if $X$ is compact 0-dimensional (that is, a *Stone space*), then we may restrict the choice of both $K$ and $U$ to be clopen subsets of $X$ in the subbasic open sets $[K,U]$ of the compact-open topology of $\mathcal{C}(X)$ (cf. [13, Chapter X, §3.4, Remark 2]).

## 3. Syntactic congruences

We say that an equivalence relation $\theta$ on a set $A$ *saturates* a subset $L$ of $A$ if $L$ is a union of $\theta$-classes. The following purely algebraic result is well known. We provide a proof for the sake of completeness.

**Lemma 3.1.** *For an algebra $A$ and a subset $L$ of $A$, the set of all pairs $(a,a') \in A \times A$ such that*

$$\forall f \in M(A)\ \big(f(a) \in L \iff f(a') \in L\big)$$

*is the largest congruence on $A$ saturating $L$.*

*Proof*: Let $\theta$ denote the set of pairs in the statement of the lemma. Note that it is an equivalence relation on $A$. Let $w \in \Omega_n$ and $(a_i, a_i') \in \theta$ for $i = 1, \ldots, n$ and suppose that $f \in M(A)$ is such that $f\big(w_A(a_1, \ldots, a_n)\big) \in L$. Considering the polynomial transformation $g$ given by $g(x) = f\big(w_A(x, a_2, \ldots, a_n)\big)$, we deduce from $(a_1, a_1') \in \theta$ that $f\big(w_A(a_1', a_2, \ldots, a_n)\big) \in L$. Proceeding similarly on each of the remaining components, we see that $f\big(w_A(a_1', \ldots, a_n')\big) \in L$. Hence, $\theta$ is a congruence on $A$. Taking for $f$ the identity transformation of $A$ we conclude that $\theta$ saturates $L$.

Now, suppose that $\rho$ is a congruence on $A$ saturating $L$ and let $(a, a') \in \rho$ and $f \in M(A)$. Because polynomial transformations of $A$ preserve $\rho$-equivalence, we have $\big(f(a), f(a')\big) \in \rho$. Since $\rho$ saturates $L$, it follows that $f(a) \in L$ if and only if $f(a') \in L$, which shows that $(a, a') \in \theta$. Hence, $\rho$ is contained in $\theta$, thereby completing the proof of the lemma. ∎

The congruence of the lemma is called the *syntactic congruence* of $L$ on $A$ and it is denoted $\sigma_L^A$.

**Remark 3.2.** Let $L$ be a subset of an algebra $A$ and let $\alpha_L$ be the equivalence relation whose classes are $L$ and $A \setminus L$. Then we may reformulate the definition of the syntactic congruence by the following formula:

$$\sigma_L^A = \bigcap_{f \in M(A)} (f \times f)^{-1}(\alpha_L) = \bigcap_{f \in M(A)} \alpha_{f^{-1}(L)}. \tag{3.1}$$

**Corollary 3.3.** *Let $L$ be a clopen subset of a topological algebra $A$. Then the syntactic congruence $\sigma_L^A$ is closed.*

*Proof*: Consider the equivalence relation $\alpha_L$ of Remark 3.2, so that formula (3.1) holds. Since $\alpha_L$ is a clopen subset of $A \times A$ and each mapping $f \in M(A)$ is continuous, so that $f \times f \in \mathcal{C}(A \times A)$, it follows that $\sigma_L^A$ is an intersection of clopen sets of the form $(f \times f)^{-1}(\alpha_L)$ and, therefore, it is closed. ∎

The following result is another simple application of Lemma 3.1.

**Proposition 3.4.** *Let $\varphi : A \to B$ be an onto homomorphism between two $\Omega$-algebras and let $L$ be a subset of $B$. Then we have*

$$(\varphi \times \varphi)^{-1}\sigma_L^B = \sigma^A_{\varphi^{-1}(L)},$$

*so that $\varphi$ induces an isomorphism $A/\sigma^A_{\varphi^{-1}(L)} \to B/\sigma_L^B$.*



Since both $\overline{F_T}$ and $M(A)$ determine $\sigma_L$, so does $\overline{F}_T$. We claim that $\overline{F}_T$ is compact. Indeed, for each $t \in T$, we have

$$\overline{F}_t = \overline{F} \cap t_A^\sharp(A^k).$$

But note that $t_A^\sharp(A^k)$ is compact, because $A^k$ is compact and $t_A^\sharp$ is continuous by Proposition 5.4(3). Therefore, $\overline{F}_t$, being an intersection of a closed with a compact subset, is itself compact. It follows that $\overline{F}_T = \bigcup_{t \in T} \overline{F}_t$ is compact, as claimed. ∎

Combining Theorem 5.9 and Propositions 5.1 and 5.15, we obtain the following main result of this section.

**Theorem 5.16.** *The following conditions are equivalent for a compact algebra $A$ and a subset $L \subseteq A$:*

(1) *$\sigma_L$ is clopen;*
(2) *$L$ is clopen and $\sigma_L$ is compactly determined;*
(3) *$L$ is clopen and $\sigma_L$ is finitely determined;*
(4) *$L$ is clopen and $\sigma_L$ is finitely determined by a set of terms;*
(5) *the quotient algebra $A/\sigma_L$ is finite and discrete.*

The following example shows that it is not possible to extend Proposition 5.15 for the case of locally compact algebras. In fact, we exhibit a locally compact semigroup and a clopen subset whose syntactic congruence is not clopen. This syntactic congruence is not compactly determined by Theorem 5.9, while it is determined by a finite set of terms as in fact every syntactic congruence of a semigroup has this property.

**Example 5.17.** We consider the topological semigroup obtained as the direct product of the following locally compact semigroups $A$ and $B$, so that it is locally compact. Let $A = \mathbb{N}$ be the discrete semigroup of natural numbers with maximum as operation, and let $B$ be the one point compactification of the usual additive semigroup of natural numbers $\mathbb{N}$. While the description of the locally compact semigroup $A$ is clear, we describe the compact semigroup $B$ in more detail: we have $B = \mathbb{N} \cup \{\infty\}$, where $\infty$ is the new point for which we put $\infty + n = n + \infty = \infty + \infty = \infty$, for $n \in \mathbb{N}$. We also recall that the open sets in $B$ are all subsets of $\mathbb{N}$ together with the subsets of the form $\{\infty\} \cup (\mathbb{N} \setminus F)$, where $F$ is a finite subset of $\mathbb{N}$.

It follows that

$$\sigma_L \subseteq \bigcap_{i=1,\dots,r;\ b,b'\in A} \big((s_i)^\sharp_A(b,b',v)\times(s_i)^\sharp_A(b,b',v)\big)^{-1}(\alpha_L)$$
$$\subseteq \big(t^\sharp_A(v)\times t^\sharp_A(v)\big)^{-1}(\alpha_L),$$

which shows that $\sigma_L$ is also determined by a finite set of terms that are linear in $x_1$ by simply taking, for a finite set of terms determining $\sigma_L$, the union of the sets of terms constructed above for each term in the given set. ∎

The next lemma examines how the operation $F \mapsto F_T$ behaves with respect to topological closure when $T$ is a finite set of terms.

**Lemma 5.14.** *Let $A$ be a compact algebra, $F$ be a subset of $\mathcal{C}(A)$ and $T$ be a finite set of terms in $T_\Omega(\{x_1,\dots,x_{k+1}\})$. Then, $\overline{F_T}$ is contained in $\overline{F}_T$.*

*Proof*: Let $f \in \overline{F_T}$. Then, we can write $f$ as a limit

$$f = \lim_{i\in I} f_i,$$

where $\{f_i\}_{i\in I}$ is a net in $F_T$. For each $i \in I$, choose $t_i \in T$ and $v_i \in A^k$ such that $f_i = (t_i)^\sharp_A(v_i)$. Since $T$ is finite and $A^k$ is compact, we may extract a subnet $\{f_{i_j}\}_{j\in J}$ such that $\{t_{i_j}\}_{j\in J}$ takes a constant value $t \in T$ and $\{v_{i_j}\}_{j\in J}$ converges to $v$ in $A^k$. By Proposition 5.4(3), $t^\sharp_A$ is continuous and it follows that

$$f = \lim_{j\in J} f_{i_j} = \lim_{j\in J} t^\sharp_A(v_{i_j}) = t^\sharp_A(v).$$

Hence, $f$ lies in $\overline{F}_T$. ∎

We are now ready to achieve the goal announced at the beginning of this subsection.

**Proposition 5.15.** *Let $A$ be a compact algebra, and $L$ be a clopen subset of $A$. If $\sigma_L$ is determined by a finite set of terms, then it is compactly determined.*

*Proof*: By Lemma 5.13, there is a finite subset $T$ of $T_\Omega(\{x_1,\dots,x_{k+1}\})$ consisting of terms linear in $x_1$ and a subset $F$ of $M(A)$ such that $F_T$ determines $\sigma_L$. Then, by Proposition 5.7, $\sigma_L$ is also determined by $\overline{F_T}$. By Lemma 5.14, we have

$$\overline{F_T} \subseteq \overline{F}_T \subseteq M(A).$$



*Proof*: By the general correspondence theorem of universal algebra [14, Theorem 6.20], $(\varphi\times\varphi)^{-1}\sigma^B_L$ is a congruence on $A$. It saturates $\varphi^{-1}(L)$ since, if $(a,a') \in (\varphi\times\varphi)^{-1}\sigma^B_L$ and $a \in \varphi^{-1}(L)$, then $(\varphi(a),\varphi(a')) \in \sigma^B_L$ and $\varphi(a)$ belongs to $L$ and, therefore, so does $\varphi(a')$. This shows that $(\varphi\times\varphi)^{-1}\sigma^B_L \subseteq \sigma^A_{\varphi^{-1}(L)}$.

Conversely, suppose that $(a,a') \in \sigma^A_{\varphi^{-1}(L)}$ and $t(x_1,\dots,x_m)$ is a term linear in $x_1$ such that $t_B(\varphi(a),b_2,\dots,b_m) \in L$ with the $b_j$ in $B$. For each $j \in \{2,\dots,m\}$, let $a_j \in A$ be such that $b_j = \varphi(a_j)$. Then, we must have $t_A(a,a_2,\dots,a_m) \in \varphi^{-1}(L)$, which yields $t_A(a',a_2,\dots,a_m) \in \varphi^{-1}(L)$, which in turn entails $t_B(\varphi(a'),b_2,\dots,b_m) \in L$. This establishes the reverse inclusion $\sigma^A_{\varphi^{-1}(L)} \subseteq (\varphi\times\varphi)^{-1}\sigma^B_L$. ∎

## 4. Profiniteness

Since "Stone algebra" already has a different meaning in the literature, we call a *Stone topological algebra* a topological algebra whose underlying topological space is a Stone space.

To give an application of Proposition 3.4, we first need some connections of syntactic congruences with profiniteness. Note that, as a result of the characterization of syntactic congruences in Lemma 3.1, the syntactic congruence of a clopen subset of a topological algebra is always closed.

**Theorem 4.1.** *A Stone topological algebra $A$ is profinite if and only if, for every clopen subset $L$ of $A$, the syntactic congruence $\sigma^A_L$ is clopen.*

*Proof*: ($\Rightarrow$) It follows from residual finiteness and compactness that there is a continuous homomorphism $\varphi : A \to B$ onto a finite algebra $B$ such that $L = \varphi^{-1}(\varphi(L))$ [3, Lemma 4.1]. The kernel* of $\varphi$ is thus a clopen congruence on $A$ that saturates $L$. By Lemma 3.1, $\sigma^A_L$ contains the kernel of $\varphi$. Hence, the classes of $\sigma^A_L$ are also clopen, as they are unions of classes of the kernel of $\varphi$.

($\Leftarrow$) We need to show that $A$ is residually finite. Given distinct points $a,a' \in A$, since $A$ is a Stone space, there is a clopen subset $L \subseteq A$ such that $a \in L$ and $a' \notin L$. Since $\sigma^A_L$ is clopen, the canonical homomorphism $A \to A/\sigma^A_L$ is a continuous homomorphism onto a finite (discrete) algebra that separates the points $a$ and $a'$, as $\sigma^A_L$ saturates $L$ by Lemma 3.1. ∎

We are now ready to prove the following result. The special case where the signature is discrete was first proved in [18, Theorem 4.3] using duality theory.

*By the *kernel* of a mapping $\varphi : S \to T$ we mean the equivalence relation on $S$ given by $\{(s_1,s_2) \in S^2 : \varphi(s_1) = \varphi(s_2)\}$.

**Theorem 4.2.** *Let $A$ be a profinite algebra and let $\theta$ be a closed congruence on $A$ such that $A/\theta$ is 0-dimensional. Then the quotient $A/\theta$ is profinite.*

*Proof*: Let $B = A/\theta$ and let $\varphi : A \to B$ be the canonical homomorphism. By assumption and Proposition 2.1, $B$ is a Stone topological algebra. We apply the criterion for $B$ to be profinite of Theorem 4.1. So, let $L$ be a clopen subset of $B$. By Proposition 3.4, we obtain the equality

$$(\varphi \times \varphi)^{-1}\sigma_L^B = \sigma^A_{\varphi^{-1}(L)}.$$

Since $\varphi$ is continuous, the set $\varphi^{-1}(L)$ is clopen. Hence, by Theorem 4.1, $\sigma^A_{\varphi^{-1}(L)}$ is a clopen congruence, and so $A/\sigma^A_{\varphi^{-1}(L)}$ is finite. By Proposition 3.4, the algebra $B/\sigma_L^B$ is finite. Since $\sigma_L^B$ is a closed congruence by Corollary 3.3, it follows that $\sigma_L^B$ is clopen. ∎

A compact space $X$ has a natural uniform structure in the sense of [12, Chapter II] with *base* consisting of the open neighborhoods of the diagonal $\{(x, x) : x \in X\}$, that is, its *entourages* are the subsets of $X \times X$ that contain such neighborhoods. This is the only uniform structure that determines the topology of $X$. In case $X$ is a Stone space, one may take as basic members the sets of the form $\bigcup_{i=1}^n L_i \times L_i$ where the $L_i$ constitute a clopen partition of $X$; we call the union $\bigcup_{i=1}^n L_i \times L_i$ the *entourage determined* by the partition $L_1, \ldots, L_n$. Suppose $A$ is a profinite algebra and $L_1, \ldots, L_n$ is a clopen partition of $A$. By Theorem 4.1, the set $\theta = \bigcap_{i=1}^n \sigma^A_{L_i}$ is a clopen congruence on $A$ that saturates each set $L_i$. Hence, to obtain a subbase of the uniform structure of $A$ one may consider only the clopen partitions defined by clopen congruences.

Let $X$ be a uniform space and let $\mathcal{F} \subseteq \mathcal{C}(X)$. We say that $\mathcal{F}$ is *equicontinuous* if for every entourage $\alpha$ of $X$ and every $x \in X$, there is an open subset $U \subseteq X$ such that $x \in U$ and, for every $f \in \mathcal{F}$, $f(U) \times f(U) \subseteq \alpha$. We say that $\mathcal{F}$ is *uniformly equicontinuous* if for every entourage $\alpha$ of $X$, there is an entourage $\beta$ of $X$ such that, for every $f \in \mathcal{F}$, $(f \times f)(\beta) \subseteq \alpha$. In case $X$ is compact, the two properties are equivalent [13, Chapter X, §2.1, Corollary 2].

Here is another characterization of profiniteness for Stone topological algebras. It is taken from [27, Theorem 4.4]. The proof presented here is basically the same as that in [27] although it is slightly simplified thanks to the usage of syntactic congruences.

**Theorem 4.3.** *A Stone topological algebra $A$ is profinite if and only if $M(A)$ is equicontinuous.*



To establish Claim (2), it suffices to observe that the argument at the beginning of the proof of Proposition 5.3 shows that a finitely determined syntactic congruence has finite index, which is not the case of $\sigma_L$ by (1). ∎

**5.3. Compact determination versus finite determination by terms.** We next show that, for compact algebras, a syntactic congruence is compactly determined if and only if it is determined by a finite set of terms, in the sense introduced at the beginning of Subsection 5.1.

We start with some notation. Let $A$ be an algebra, $F$ be a subset of $\mathcal{C}(A)$, and $S$ be a subset of $\mathcal{C}(A^{k+1}, A)$ Given $s \in S$, in view of Proposition 5.4, we obtain a function $s^\sharp \in \mathcal{C}(A^k, \mathcal{C}(A))$. We define:

$$F_S = \{f \in F : \exists s \in S\ \exists v \in A^k, f = s^\sharp(v)\}.$$

We also abbreviate $F_{\{s\}}$ by $F_s$. Note that

$$F_S = \bigcup_{s\in S} F_s = F \cap \left(\bigcup_{s\in S} s^\sharp(A^k)\right) \subseteq \mathcal{C}(A).$$

In case $T$ is subset of $T_\Omega(\{x_1, \ldots, x_{k+1}\})$, we also write $F_T$ for $F_S$, where $S = \{t_A : t \in T\}$.

**Lemma 5.13.** *Let $L$ be a subset of an algebra $A$. Then $\sigma_L$ is determined by a finite set of terms if and only if it is determined by some set of the form $F_T$, where $T$ is a finite subset of $T_\Omega(\{x_1, \ldots, x_{k+1}\})$ for some $k \geqslant 0$, consisting of terms linear in $x_1$, and $F \subseteq M(A)$.*

*Proof*: The if part of the statement of the lemma is trivial. For the converse, we first associate with each term $t \in T_\Omega(\{x_1, \ldots x_{k+1}\})$ a term $s \in T_\Omega(X)$ with $X = \{y_1, \ldots, y_r, x_2, \ldots, x_{k+1}\}$ by replacing each occurrence of $x_1$ by a distinct $y_i$. Let

$$s_i = s_{T_\Omega(X)}(y^{(i-1)}, x, z^{(r-i)}, x_2 \ldots, x_{k+1}) \in T_\Omega(x, y, z, x_2, \ldots, x_{k+1}),$$

where $u^{(\ell)}$ stands for $\ell$ components equal to $u$. Then each term $s_i$ is linear in $x$ and the following formulas hold for $a, a' \in A$ and $v \in A^k$:

$$\begin{aligned}
(s_1)^\sharp_A(a, a', v)(a') &= t^\sharp_A(v)(a')\\
(s_r)^\sharp_A(a, a', v)(a) &= t^\sharp_A(v)(a)\\
(s_i)^\sharp_A(a, a', v)(a) &= (s_{i+1})^\sharp_A(a, a', v)(a') \quad (i = 1, \ldots, r-1).
\end{aligned}$$

as the evaluation and diagonal mappings are continuous, again respectively by Proposition 5.4(2) and Lemma 5.6, and $\alpha_L$ is clopen, we get

$$(f' \times f')(x) = \lim_{j \in J}(f_{i_j} \times f_{i_j})(x) \notin \alpha_L.$$

This shows that:

$$x \in \bigcup_{f \in F'} (f \times f)^{-1}(A^2 \setminus \alpha_L) = A^2 \setminus \bigcap_{f \in F'} (f \times f)^{-1}(\alpha_L).$$

Thus, the reverse inclusion in (5.1) is proved and $\sigma_L$ is $F'$-determined. ∎

In case $A$ is a compact algebra and $L$ is a clopen subset of $A$, one may use the ideas in the proof of Proposition 5.1 to show that every minimal compact subset of $\mathcal{C}(A)$ determining $\sigma_L$ is finite. Combining with Proposition 5.11, it follows that every compact subset of $\mathcal{C}(A)$ determining $\sigma_L$ contains a finite such set. The following example shows that Proposition 5.11 cannot be improved in the same direction for arbitrary locally compact algebras.

**Example 5.12.** Consider the additive monoid $\mathbb{N}$ of natural numbers under the discrete topology. Let $L$ be an infinite subset of $\mathbb{N}$ containing no infinite arithmetic progression, for instance the set of all powers of 2 or the set of all primes. We claim that:

(1) $\sigma_L$ is the equality relation;
(2) $\sigma_L$ is not finitely determined.

As $\mathbb{N}$ is locally compact, being discrete, and $\sigma_L$ is clopen, for the same reason, Theorem 5.9 yields that $\sigma_L$ is compactly determined. Hence, by (2), $\sigma_L$ is an example of a compactly determined syntactic congruence of a clopen subset of a locally compact algebra that is not finitely determined.

To prove Claim (1), since $\mathbb{N}$ is a commutative monoid, a pair $(m, n)$ of natural numbers belongs to $\sigma_L$ if and only if

$$\forall x \in \mathbb{N}\ (m + x \in L \iff n + x \in L). \tag{5.2}$$

Suppose that Property (5.2) holds with $m < n$. Since $L$ is infinite, there exists $k$ such that $m + k$ belongs to $L$ and whence so does $n + k$. Using (5.2), we deduce that all the elements in the arithmetic progression starting with $m + k$ with period $n - m$ lie in $L$, which contradicts the assumption on the set $L$. Hence no two distinct elements of $\mathbb{N}$ can be $\sigma_L$-equivalent, thereby proving (1).



*Proof*: ($\Rightarrow$) Let $L_1, \ldots, L_n$ be a clopen partition of $A$ and let $\theta = \bigcap_{i=1}^n \sigma_{L_i}^A$. Then, by Lemma 3.1, for every $f \in M(A)$, we have

$$(f \times f)(\theta) \subseteq \bigcup_{i=1}^n L_i \times L_i.$$

($\Leftarrow$) We apply the criterion of Theorem 4.1. So, let $L$ be a clopen subset of $A$ and consider the entourage $\alpha_L$ determined by the clopen partition $L, A \setminus L$. By (uniform) equicontinuity, there exists a clopen partition $L_1, \ldots, L_n$ of $A$, determining an entourage $\beta$ of $X$, such that, for every $f \in M(A)$, the inclusion $(f \times f)(\beta) \subseteq \alpha_L$ holds or, equivalently, $\beta \subseteq \bigcap_{f \in M(A)} (f \times f)^{-1}(\alpha_L)$. By Remark 3.2, we have

$$\bigcap_{f \in M(A)} (f \times f)^{-1}(\alpha_L) = \sigma_L^A.$$

Hence, we have $\beta \subseteq \sigma_L^A$. It follows that each class of $\sigma_L^A$ is a union of some of the $L_i$ and, therefore it is clopen. Hence, $\sigma_L^A$ is a clopen congruence and $A$ is profinite by Theorem 4.1. ∎

A subset of a topological space $X$ is said to be *relatively compact* if it is contained in a compact subset of $X$. The following is a reformulation of Theorem 4.3. As observed in [27], the equivalence between the criteria of Theorems 4.3 and 4.4 is an immediate application of the Arzelà-Ascoli theorem of functional analysis [13, Chapter X, §2.5, Corollary 3].

**Theorem 4.4** ([27])**.** *A Stone topological algebra $A$ is profinite if and only if $M(A)$ is relatively compact in $\mathcal{C}(A)$.*

## 5. Determination of syntactic congruences

In this section, we examine several ways of describing syntactic congruences.

**5.1. Determination by terms and functions.** We say that a syntactic congruence $\sigma_L$ of an algebra $A$ is *finitely determined by terms* if there exists a finite subset $F$ of $T_\Omega(\{x_1, \ldots, x_m\})$ such that $\sigma_L$ consists of all pairs $(a, a') \in A^2$ for which

$$\forall t \in F\ \forall b_2, \ldots, b_m \in A\ \big(t_A(a, b_2, \ldots, b_m) \in L \iff t_A(a', b_2, \ldots, b_m) \in L\big).$$

We then also say that the set of terms $F$ *determines* $\sigma_L$. In Lemma 5.13 below, it is observed that if $\sigma_L$ is determined by a finite set of terms, then it is also determined by such a set in which every term is linear in the first variable.

For instance, semigroups, monoids, rings, and distributive lattices all have syntactic congruences determined by finite sets of terms for all subsets, even in a uniform way in the sense that the same finite set of terms works for all syntactic congruences on all algebras of the chosen type.

Generalizing a result of Numakura [22] (for the cases of semigroups and distributive lattices), the first author [1] has shown that a sufficient condition for a Stone topological algebra to be profinite is that its syntactic congruences of clopen subsets are finitely determined (see also [9, 15]).

There is a stronger form of determination of a syntactic congruence, which is suggested by Lemma 3.1. We say that the syntactic congruence $\sigma_L$ of a subset $L$ of an algebra $A$ is *$S$-determined* if $S$ is a set of functions $A \to A$ such that, for all $a, a' \in A$, $(a, a') \in \sigma_L$ holds if and only if

$$\forall f \in S \ \big(f(a) \in L \iff f(a') \in L\big).$$

For instance, $\sigma_L$ is $M(A)$-determined by Lemma 3.1. As in Remark 3.2, note that $\sigma_L$ is $S$-determined if and only if

$$\sigma_L = \bigcap_{f \in S} (f \times f)^{-1}(\alpha_L) = \bigcap_{f \in S} \alpha_{f^{-1}(L)}.$$

**Proposition 5.1.** *Let $A$ be an algebra and $L$ a subset of $A$. If the syntactic congruence $\sigma_L$ has finite index then it is $F$-determined for some finite subset $F$ of $M(A)$.*

*Proof*: Consider the syntactic homomorphism $\eta : A \to A/\sigma_L$, which sends each $a \in A$ to its syntactic class $a/\sigma_L$. By assumption, the algebra $A/\sigma_L$ is finite. Hence, $M(A/\sigma_L)$ is a finite monoid. For each $f \in M(A/\sigma_L)$, we may choose a term $t \in T_\Omega(\{x_1, \ldots, x_{k+1}\})$ which is linear in $x_1$ and elements $a_2, \ldots, a_{k+1} \in A$ such that $f(x/\sigma_L) = t_A(x, a_2, \ldots, a_{k+1})/\sigma_L$ for every $x \in A$. We let $\hat{f}(x) = t_A(x, a_2, \ldots, a_{k+1})$ for each $x \in A$, which defines an element $\hat{f}$ of $M(A)$ such that $\eta \circ \hat{f} = f \circ \eta$. Let $F = \{\hat{f} : f \in M(A/\sigma_L)\}$. We claim that $\sigma_L$ is $F$-determined.

Since $F \subseteq M(A)$, we have $\sigma_L \subseteq \bigcap_{g \in F} (g \times g)^{-1}(\alpha_L)$. Conversely, suppose $a, a' \in A$ are such that, for every $f \in M(A/\sigma_L)$, we have $\hat{f}(a) \in L$ if and only if $\hat{f}(a') \in L$. Since $\eta$ saturates $L$, $x \in L$ is equivalent to $\eta(x) \in \eta(L)$. Thus, we get $f(\eta(a)) \in \eta(L)$ if and only if $f(\eta(a')) \in \eta(L)$ for every $f \in M(A/\sigma_L)$. By definition of the syntactic congruence, we deduce that $(\eta(a), \eta(a')) \in \sigma_{\eta(L)}$, that is, $(a, a') \in (\eta \times \eta)^{-1}(\sigma_{\eta(L)})$. By Proposition 3.4, it follows that $(a, a') \in \sigma_L$, which establishes the claim. ∎



where $\eta : A \to A/\sigma_L$ is the natural quotient mapping. Note that $A/\sigma_L$ is a discrete space under the quotient topology, whence it is locally compact. By Proposition 5.4(1), $\varphi$ is a continuous mapping. As both spaces $A/\sigma_L$ and $\{a, b\}$ are discrete, the space $\mathcal{C}(A/\sigma_L, \{a, b\})$ is in fact the product space $\{a, b\}^{A/\sigma_L}$. Since $\varphi$ is onto and continuous, we deduce that $K$ is compact. The proof is achieved by observing that $\sigma_L$ is $K$-determined. ∎

Note that the second part of the above proof does not use the hypothesis that the algebra $A$ is locally compact. In the locally compact case, the proof could also be given by invoking the Arzelà-Ascoli theorem.

Theorems 5.9 and 4.1 yield the following result.

**Corollary 5.10.** *Let $A$ be a Stone topological algebra, and suppose that for every clopen subset $L$ of $A$, $\sigma_L$ is compactly determined. Then $A$ is profinite.*

Given a profinite algebra $A$ and a clopen subset $L$ of $A$, one may wonder what are the compact subsets of $\mathcal{C}(A)$ determining $\sigma_L$. The following shows that, at the very least, there is always one that is minimal.

**Proposition 5.11.** *Let $A$ be locally compact algebra, $L$ be a clopen subset of $A$. Then, every compact subset of $\mathcal{C}(A)$ determining $\sigma_L$ contains a minimal compact subset determining $\sigma_L$.*

*Proof*: We apply Zorn's lemma. Let $F$ be a compact subset of $\mathcal{C}(A)$ determining $\sigma_L$. Fix a descending chain $\{F_i\}_{i \in I}$ of closed subsets of $F$ that determines $\sigma_L$ and let:

$$F' = \bigcap_{i \in I} F_i.$$

We want to show that $F'$ determines $\sigma_L$. Fixing an arbitrary $i \in I$, the inclusion $F' \subseteq F_i$ gives:

$$\bigcap_{f \in F'} (f \times f)^{-1}(\alpha_L) \supseteq \bigcap_{f \in F_i} (f \times f)^{-1}(\alpha_L) = \sigma_L. \tag{5.1}$$

It remains to show the reverse inclusion. Let us suppose that $x \in A^2 \setminus \sigma_L$. Then, we have

$$\forall i \in I \ \exists f_i \in F_i, \ (f_i \times f_i)(x) \notin \alpha_L.$$

This defines a net $\{f_i\}_{i \in I}$ in $F$, which by compactness has a converging subnet $\{f_{i_j}\}_{j \in J}$; let $f' \in F$ denote its limit. For each $i \in I$, there is $k \in J$ such that $i_k \geqslant i$, and so the net $\{f_{i_j}\}_{j \in J}$ is eventually in $F_i$. Since $F_i$ is closed, it follows that $f' \in F_i$, and this holds for each $i \in I$. Thus, $f'$ belongs to $F'$. Furthermore,

*Proof*: Clearly, we have:

$$\bigcap_{f\in\overline{F}}(f\times f)^{-1}(\alpha_L)\subseteq\bigcap_{f\in F}(f\times f)^{-1}(\alpha_L)=\sigma_L.$$

To prove the reverse inclusion, let us fix $x\in\sigma_L$ and take an arbitrary $f\in\overline{F}$. Let $\{f_i\}_{i\in I}$ be a net in $F$ converging to $f$. By assumption, $(f_i\times f_i)(x)$ belongs to $\alpha_L$ for every $i\in I$. Since $\alpha_L$ is closed and the evaluation and diagonal mappings are continuous, respectively by Proposition 5.4(2) and Lemma 5.6, we deduce that

$$(f\times f)(x)=\lim_{i\in I}(f_i\times f_i)(x)\in\alpha_L.$$

This shows that $x\in(f\times f)^{-1}(\alpha_L)$, as required. ∎

Combining Proposition 5.7 with Theorem 4.4, we get the following result.

**Corollary 5.8.** *Let $A$ be a profinite algebra and $L$ be a clopen subset of $A$. Then $\sigma_L$ is compactly determined.*

The following result shows the significance of the notion of compactly determined syntactic congruence.

**Theorem 5.9.** *Let $A$ be a locally compact algebra $A$ and $L$ a subset of $A$. Then $L$ is clopen and $\sigma_L$ is compactly determined if and only if $\sigma_L$ is clopen.*

*Proof*: Suppose first that $L$ is clopen and $\sigma_L$ is $K$-determined, where $K$ is a compact subset of $\mathcal{C}(A)$. Since $\sigma_L$ is $K$-determined, we may write $\sigma_L=\Psi_{\Delta K}(\alpha_L)$ in the notation of Proposition 5.5 and Lemma 5.6. By Lemma 5.6, $\Delta$ is continuous with respect to the compact-open topologies, so $\Delta K$ is a compact subset of $\mathcal{C}(A\times A)$. By Proposition 5.5 applied to $Y=X\times X$, it follows that $\Psi_{\Delta K}$ maps open relations on $X\times X$ to open relations on $X$. But since $L$ is clopen, so is $\alpha_L$, whence $\sigma_L$ is clopen.

For the converse, assume that $\sigma_L$ is clopen. If $\sigma_L$ is the universal relation, then $L$ is either $\emptyset$ or $A$ and $\sigma_L$ is $K$-determined for every subset $K$ of $\mathcal{C}(A)$. Otherwise, we may choose elements $a,b\in A$ that are not $\sigma_L$-equivalent. Let $K$ be the set of all mappings $A\to\{a,b\}$ that are constant on each $\sigma_L$-class. Since $\sigma_L$ is clopen, $K$ is contained in $\mathcal{C}(A)$. Consider the mapping

$$\begin{aligned}\varphi:\mathcal{C}(A/\sigma_L,\{a,b\})&\to K\\ f&\mapsto f\circ\eta,\end{aligned}$$



Note that Proposition 5.1 applies in particular to clopen syntactic congruences $\sigma_L$ on a compact algebra $A$, which are necessarily determined by a clopen subset $L$ of $A$. One may ask whether the same finite subset $F$ of $M(A)$ may be used to determine all syntactic congruences of clopen subsets $L$ of $A$. This is trivially the case for finite algebras but here is an infinite example for which such a finite set $F$ also exists.

**Example 5.2.** Let $A$ be a Stone space and consider a constant binary operation on $A$. This turns $A$ into a profinite semigroup. Note that, for every $L\subseteq A$, we have $\sigma_L=\alpha_L$. Thus, all syntactic congruences are $F$-determined for every subset $F$ of $M(A)$. The same conclusion would be reached if we took the empty signature instead. ∎

The next result shows that such an example cannot be simultaneously finitely generated and infinite.

**Proposition 5.3.** *If $A$ is an infinite finitely generated profinite algebra then, for every finite subset $F$ of $M(A)$, there exists a clopen subset $L$ of $A$ such that $\sigma_L$ is not $F$-determined.*

*Proof*: Suppose that $F$ is a finite subset of $M(A)$ that determines the syntactic congruence of every clopen subset $L$ of $A$, that is, its syntactic congruence $\sigma_L$ is given by the formula $\sigma_L=\bigcap_{f\in F}\alpha_{f^{-1}(L)}$. Note that each equivalence relation $\alpha_{f^{-1}(L)}$ has at most two classes. Hence, $\sigma_L$ has at most $2^{|F|}$ classes, that is, the syntactic algebra $A/\sigma_L$ has at most $2^{|F|}$ elements. Now, given a finite set $S$ and a finite generating set $X$ of $A$, there are only finitely many functions $X\to S$. Thus, on subsets of $S$ there are only finitely many algebraic structures which are homomorphic images of $A$. There are only finitely many congruences that are kernels of these homomorphisms from $A$ to $S$. In particular, there are only finitely many syntactic congruences $\sigma_L$ of clopen subsets $L$ of $A$. Since the corresponding syntactic homomorphisms $A\to A/\sigma_L$ suffice to separate the points of $A$, we conclude that $A$ is finite, in contradiction with the hypothesis. ∎

**5.2. Compact determination.** Suppose that $A$ is a topological algebra and $L\subseteq A$. We say that the syntactic congruence $\sigma_L$ is *compactly determined* if $\sigma_L$ is $C$-determined for some compact subset $C$ of $\mathcal{C}(A)$. Similarly, $\sigma_L$ is *finitely determined* if $\sigma_L$ is $F$-determined for some finite subset $F$ of $\mathcal{C}(A)$. This not to be confused with $\sigma_L$ being determined by a finite set of terms, which is, a priori, a weaker property in case $F\subseteq M(A)$. Theorem 5.16 at the end

of Subsection 5.3 shows that all these properties are equivalent in case $A$ is compact and $L \subseteq A$ is clopen.

In this subsection, among other results, we establish that, if all syntactic congruences of clopen subsets of a Stone topological algebra $A$ are compactly determined, then $A$ is profinite. Several of our results are stated for locally compact algebras because they hold in that more general setting.

Let us briefly introduce a convenient notation for partial evaluation. Given $f : Y \times Z \to X$, we define $f^\sharp : Z \to X^Y$ by:

$$f^\sharp(z)(y) = f(y, z).$$

The following proposition regroups a few useful properties of the compact-open topology. For proofs, we refer the reader to [13, Chapter X, §3.4, Theorem 3 and Proposition 9].

**Proposition 5.4.** *Let $X$, $Y$ and $Z$ be topological spaces, with $Y$ locally compact.*

(1) *Composition is a continuous mapping $\mathcal{C}(Y, X) \times \mathcal{C}(Z, Y) \to \mathcal{C}(Z, X)$.*
(2) *Evaluation is a continuous mapping $Y \times \mathcal{C}(Y, X) \to X$.*
(3) *If $f \in \mathcal{C}(Y \times Z, X)$, then $f^\sharp \in \mathcal{C}(Z, \mathcal{C}(Y, X))$.*

In particular, from Proposition 5.4(1) it follows that, for every topological space $X$, $\mathcal{C}(X)$ is a topological monoid. In case $A$ is a topological algebra, $M(A)$ is a submonoid of $\mathcal{C}(A)$, whence so is its closure $\overline{M(A)}$.

The following statement is an elementary observation in topology whose proof is presented for the sake of completeness.

**Proposition 5.5.** *Let $X$ and $Y$ be two topological spaces, with $X$ locally compact, and let $K$ be a compact subspace of $\mathcal{C}(X, Y)$.*

(1) *The following mapping is closed:*

$$\begin{aligned} \Phi_K : 2^Y &\to 2^X \\ V &\mapsto \bigcup_{f \in K} f^{-1}(V). \end{aligned}$$

(2) *The following mapping is open:*

$$\begin{aligned} \Psi_K : 2^Y &\to 2^X \\ U &\mapsto \bigcap_{f \in K} f^{-1}(U). \end{aligned}$$



*Proof*: Clearly, (1) and (2) imply each other by taking the complement: $\Phi_K(2^Y \setminus U) = 2^X \setminus \Psi_K(U)$. To prove (1), consider a closed subset $V$ of $Y$. Let $\{x_i\}_{i \in I}$ be a net in $\Phi_K(V)$ converging to $x \in X$. By definition of $\Phi_K$, there exists a net $\{f_i\}_{i \in I}$ in $K$ such that, for all $i \in I$, $f_i(x_i) \in V$. Since $K$ is compact, we may assume by taking a subnet that the first component of $\{(f_i, x_i)\}_{i \in I}$ converges, to say $f \in K$. Since $V$ is closed and evaluation is continuous by Proposition 5.4(2), we obtain

$$f(x) = \lim_{i \in I} f_i(x_i) \in V.$$

It follows that $x \in \Phi_K(V)$, thus showing that $\Phi_K(V)$ is closed. ∎

A further result in topology that we require for later considerations is the following.

**Lemma 5.6.** *Let $X$ be a Hausdorff topological space. The diagonal mapping $\Delta : \mathcal{C}(X) \to \mathcal{C}(X \times X)$ defined by $\Delta(f) = f \times f$ is continuous with respect to the compact-open topologies.*

*Proof*: By [16, Lemma 3.4.6], the sets of the form $[K, U_1 \times U_2]$, where $K$ ranges over all compact subsets of $X \times X$ and $U_1$ and $U_2$ range over all open subsets of $X$, constitute a subbase for the compact-open topology of $\mathcal{C}(X \times X)$. Therefore, it suffices to show that for each compact subset $K \subseteq X \times X$ and open subsets $U_1, U_2 \subseteq X$, the set $\Delta^{-1}[K, U_1 \times U_2]$ is open in the compact-open topology of $\mathcal{C}(X)$.

Let $p_1, p_2$ be the natural projections of $X \times X$ on its first and second components, respectively. Note that, for $i = 1, 2$, we have $p_i \circ \Delta(f) = f \circ p_i$. It follows that:

$$\begin{aligned} \Delta(f)(K) \subseteq U_1 \times U_2 &\iff p_i(\Delta(f)(K)) \subseteq U_i && \text{for } i = 1, 2 \\ &\iff f(p_i(K)) \subseteq U_i && \text{for } i = 1, 2. \end{aligned}$$

This shows that $\Delta^{-1}[K, U_1 \times U_2] = [p_1(K), U_1] \cap [p_2(K), U_2]$. But for $i = 1, 2$, the projection $p_i$ is continuous, so $p_i(K)$ is a compact subset of $X$ and $[p_i(K), U_i]$ is open in the compact-open topology of $\mathcal{C}(X)$. Hence, so is $\Delta^{-1}[K, U_1 \times U_2]$. ∎

The next proposition shows that, in order to be compactly determined, the syntactic congruence of a clopen subset merely needs to be determined by a relatively compact subset of $\mathcal{C}(A)$.

**Proposition 5.7.** *Let $A$ be a locally compact algebra and $L$ be a clopen subset of $A$. If $F \subseteq \mathcal{C}(A)$ is such that $\sigma_L$ is $F$-determined, then $\sigma_L$ is also $\overline{F}$-determined.*