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THE THIRD COHOMOLOGY GROUP OF A MONOID AND ADMISSIBLE ABSTRACT KERNELS

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ABSTRACT: We define the product of admissible abstract kernels of the form $\Phi\colon M \to \frac{End(G)}{Inn(G)}$, where M is a monoid, G is a group, and Φ is a monoid homomorphism. Identifying C-equivalent abstract kernels, where C is the center of G, we obtain that the set $\mathcal{M}(M,C)$ of C-equivalence classes of admissible abstract kernels inducing the same action of M on C is a commutative monoid. Considering the submonoid $\mathcal{L}(M,C)$ of abstract kernels that are induced by special Schreier extensions, we prove that the factor monoid $\mathcal{A}(M,C) = \frac{\mathcal{M}(M,C)}{\mathcal{L}(M,C)}$ is an abelian group. Moreover, we show that this abelian group is isomorphic to the third cohomology group $H^3(M,C)$.

KEYWORDS: monoid, Schreier extension, abstract kernel, Eilenberg-Mac Lane cohomology of monoids.
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1. Introduction

It is well known that every group extension

$$0 \longrightarrow G \rightarrowtail B \twoheadrightarrow \Pi \longrightarrow 1$$

induces, via the conjugation action of B on the normal subgroup G, a group homomorphism $\Phi\colon \Pi \to \frac{Aut(G)}{Inn(G)}$, which is called the *abstract kernel* of the extension. A classical problem in group theory [20, 21] consists in determining what are the abstract kernels $\Phi\colon \Pi \to \frac{Aut(G)}{Inn(G)}$ that are induced by a group extension. A cohomological answer to this question was given by Eilenberg

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and Mac Lane in [7]: they associated to any abstract kernel Φ an element $Obs(\Phi)$, called *obstruction* of the abstract kernel, of the third cohomology group $H^3(\Pi, Z(G))$, where $Z(G)$ is the center of G and the left Π-module structure on $Z(G)$ is induced by Φ. They showed that an abstract kernel Φ is induced by an extension if and only if $Obs(\Phi)$ is the zero element of $H^3(\Pi, Z(G))$. Moreover, if there is an extension inducing Φ, then the set of isomorphism classes of the extensions inducing it is in bijection with the second cohomology group $H^2(\Pi, Z(G))$. The same fact holds in many other contexts, as shown by several authors. Examples of such contexts are associative algebras [9] and Lie algebras [10] over a field, rings [11], *categories of interest* [16], categorical groups [8, 4], semi-abelian action accessible categories [3, 1, 6, 5].

The situation is more complicated for abstract kernels of the form $\Phi\colon M \to \frac{End(G)}{Inn(G)}$, where M is a monoid, G is a group, and Φ is a monoid homomorphism. Every Schreier extension $0 \longrightarrow G \rightarrowtail B \twoheadrightarrow M \longrightarrow 1$ of a monoid M by a group G induces a monoid homomorphism $\Phi\colon M \to \frac{End(G)}{Inn(G)}$ [12]. Here, Similarly to the classical case, arises the problem of describing the abstract kernels $\Phi\colon M \to \frac{End(G)}{Inn(G)}$ that are induced by a Schreier extension. Since Φ need not induce an action of M on $Z(G)$, a cohomological solution of this problem, similar to the one described above, can be obtained only for particular subclasses of abstract kernels [22, 23, 12].

Actually, in [7] Eilenberg and Mac Lane proved something more. They showed that the third cohomology group $H^3(\Pi, Z(G))$ is isomorphic to the group whose elements are equivalence classes (w.r.t. a suitable equivalence relation) of abstract kernels inducing the same Π-action on $Z(G)$, modulo those abstract kernels that are induced by a group extension. This gives a complete interpretation of the third cohomology group in terms of abstract kernels.

The aim of the present paper is to get an interpretation of the third Eilenberg–Mac Lane cohomology group of a monoid M in terms of abstract kernels of the form $\Phi\colon M \to \frac{End(G)}{Inn(G)}$. In [22, 23], the monoid homomorphisms Φ were required to satisfy the following condition: for all $x \in M$ and all $\varphi(x) \in \Phi(x)$, the centralizer of $\varphi(x)(G)$ in G coincides with $Z(G)$. This gives an action of M on $Z(G)$ and allows the author of [22, 23] to involve cohomology groups of M with coefficients in $Z(G)$ in the study of Schreier type extensions of

M by G. The abstract kernels restricted in this way, which we call *admissible abstract kernels*, are used in this paper to get the desired interpretation. We define a product of admissible abstract kernels $\Phi_1\colon M \to \frac{End(G_1)}{Inn(G_1)}$ and $\Phi_2\colon M \to \frac{End(G_2)}{Inn(G_2)}$ with $Z(G_1) = Z(G_2) = C$, inducing the same action of M on C. Identifying abstract kernels that are C-equivalent (see Section 3 for the definition of this equivalence relation), we obtain a commutative monoid $\mathcal{M}(M, C)$. The subset $\mathcal{L}(M, C)$ of *extendable* abstract kernels, namely of those abstract kernels that are induced by a special Schreier extension, is a submonoid of $\mathcal{M}(M, C)$ and we show that the factor monoid $\mathcal{A}(M, C) = \frac{\mathcal{M}(M,C)}{\mathcal{L}(M,C)}$ is an abelian group. Moreover, we prove that the abelian group $\mathcal{A}(M, C)$ is isomorphic to the third cohomology group $H^3(M, C)$ of M with coefficients in the M-module C.

2. Notation and terminology

We begin by fixing some notations we will use throughout the paper. Given a group G, we will denote by $Z(G)$ its center. More generally, if H is a subgroup of G, we will denote by $C_G(H)$ the centralizer of H in G. The monoid $End(G)$ is the monoid of endomorphisms of G, while $Inn(G)$ is the subgroup of inner automorphisms, namely automorphisms of G of the form μ_g, where $g \in G$ and $\mu_g(g') = g + g' - g$ (we will use the additive notation for G, although G will be not necessarily abelian). The identity automorphism of G will be indicated by id_G.

Let M be a monoid (with the operation written multiplicatively). A subgroup H of M (i.e. a subgroup H of the group $U(M)$ of invertible elements of M) is *right normal* if, for all $m \in M$, $mH \subseteq Hm$, where

$$mH = \{mh \mid h \in H\}, \qquad Hm = \{hm \mid h \in H\}.$$

If H is a right normal subgroup of a monoid M, the relation on M defined by

$$m_1 \sim m_2 \quad \Leftrightarrow \quad m_1 = hm_2 \quad \text{for some } h \in H$$

is a congruence on M, called the *right coset relation*. The equivalence class of an element m is $cl(m) = Hm$. Hence the operation

$$Hm_1 \cdot Hm_2 = Hm_1m_2$$

is well defined. We denote by $\frac{M}{H}$ the quotient monoid. For every group G, $Inn(G)$ is right normal in $End(G)$ (indeed, $\varphi\mu_g = \mu_{\varphi(g)}\varphi$, $g \in G$, $\varphi \in End(G)$), so we always have the factor monoid $\frac{End(G)}{Inn(G)}$. Given $\alpha \in End(G)$, we denote the corresponding element in the quotient by $cl(\alpha)$.

Definition 2.1. *Given a monoid M (written multiplicatively) and a group G (written additively), an abstract kernel is a monoid homomorphism $\Phi\colon M \to \frac{End(G)}{Inn(G)}$, also written as (M, G, Φ).*

We will be interested, in particular, in a specific kind of abstract kernels, the *admissible* ones:

Definition 2.2. *An abstract kernel $\Phi\colon M \to \frac{End(G)}{Inn(G)}$ is* admissible *if, for all $x \in M$ and all $\varphi(x) \in \Phi(x)$, one has that $C_G(\varphi(x)(G)) = Z(G)$.*

The notion of admissible abstract kernel first appeared in [22]. In the original definition, another condition was required, namely that, for all $x \in M$, there exists $\varphi(x) \in \Phi(x)$ such that $\varphi(x)(C) \subseteq C$, where $C = Z(G)$. But this condition actually follows from the previous one. Furthermore, it follows that $\varphi(x)(C) \subseteq C$ for all $\varphi(x) \in \Phi(x)$. Indeed, if $\varphi(x) \in \Phi(x)$ and $c \in C$, then for all $g \in G$

$$\varphi(x)(g) + \varphi(x)(c) = \varphi(x)(g + c) = \varphi(x)(c + g) = \varphi(x)(c) + \varphi(x)(g),$$

and so $\varphi(x)(c) \in C_G(\varphi(x)(G)) = Z(G) = C$.

Admissible abstract kernels can be characterized also in the following, simpler way:

Proposition 2.3. *An abstract kernel $\Phi\colon M \to \frac{End(G)}{Inn(G)}$ is* admissible *if and only if for all $x \in M$ there exists $\varphi(x) \in \Phi(x)$ such that $C_G(\varphi(x)(G)) = Z(G)$.*

Proof: This is a corollary of the following lemma. ■

Lemma 2.4. *If $\alpha \in End(G)$ is such that $C_G(\alpha(G)) = Z(G)$, then for all inner automorphisms μ_g one has that $C_G(\mu_g\alpha(G)) = Z(G)$.*

Proof: Let $r \in C_G(\mu_g\alpha(G))$. Then, for all $g' \in G$, we have that

$$r + \mu_g\alpha(g') = \mu_g\alpha(g') + r,$$

or, in other terms,

$$r + g + \alpha(g') - g = g + \alpha(g') - g + r.$$

From this equality we get

$$-g + r + g + \alpha(g') = \alpha(g') - g + r + g,$$

and hence

$$-g + r + g \in C_G(\alpha(G)).$$

Since, by assumption, $C_G(\alpha(G)) = Z(G)$, we get that $-g+r+g = c \in Z(G)$. Then, from $r + g = g + c = c + g$ we obtain $r = c \in Z(G)$ by canceling g on the right. ∎

Any action of a monoid M on an abelian group, i.e. a monoid homomorphism $\varphi\colon M \to End(C)$, where C is an abelian group, is clearly an admissible abstract kernel. It is also clear that any abstract kernel $\Phi\colon M \to \frac{End(G)}{Inn(G)}$ which factors through $\frac{Epi(G)}{Inn(G)}$, where $Epi(G)$ is the monoid of epimorphisms of G on itself, is admissible. Less trivial examples are provided, for instance, by using the fact that, for a non-trivial subgroup of a free group F, the centralizer $C_F(H)$ is different from the trivial group if and only if H is a cyclic subgroup of F. As for concrete examples, let us consider the following:

Example 2.5. *Let $F = F(x, y, z)$ be the free group on three elements. Define $\alpha \in End(F)$ by putting $\alpha(x) = x$, $\alpha(y) = \alpha(z) = y$, and consider the monoid homomorphism $\Phi\colon \mathbb{N} \to \frac{End(F)}{Inn(F)}$ defined by $\Phi(n) = cl(\alpha^n)$, where $\mathbb{N}$ is the monoid of natural numbers with the usual sum. Since the subgroups $\alpha^n(F)$ are not cyclic, $C_F(\alpha^n(F)) = \{1\} = Z(F)$. Hence Φ is an admissible abstract kernel.*

Example 2.6. *Let $\mathcal{F}(a, b)$ be the free monoid on two generators a and b, and let F and α be as in the previous example. Define $\beta \in End(F)$ by putting $\beta(x) = x$, $\beta(y) = \beta(z) = z$, and consider the monoid homomorphism $\Phi\colon \mathcal{F}(a, b) \to \frac{End(F)}{Inn(F)}$ defined by $\Phi(a) = cl(\alpha)$ and $\Phi(b) = cl(\beta)$. It is straightforward to check that $\alpha^n = \alpha$, $\beta^n = \beta$ (for $n \geq 1$), and that $\alpha\beta = \alpha$, $\beta\alpha = \beta$. Hence, for any $w \in \mathcal{F}(a, b) \setminus \{1\}$, we have $\Phi(w) = cl(\alpha)$ or $\Phi(w) = cl(\beta)$. Since the subgroups $\alpha(F)$ and $\beta(F)$ are not cyclic, $C_F(\alpha(F)) = C_F(\beta(F)) = \{1\} = Z(F)$. Hence Φ is an admissible abstract kernel.*

Remark 2.7. *Note that, if $C_G(\alpha(G)) = C_G(\beta(G)) = Z(G)$ for $\alpha, \beta \in End(G)$, it is not true in general that $C_G(\alpha\beta(G)) = Z(G)$ or $C_G(\beta\alpha(G)) =$*

$Z(G)$. As a counterexample, consider F and α as in Example 2.5, and β defined by $\beta(x) = \beta(y) = y$, $\beta(z) = z$. Then $C_F(\alpha(F)) = C_F(\beta(F)) = \{1\} = Z(F)$, while $C_F(\alpha\beta(F))$ and $C_F(\beta\alpha(F))$ coincide with the cyclic subgroup of F generated by y.

Proposition 2.8. *Let $\alpha \in End(G)$. $C_G(\alpha(G)) = Z(G)$ if and only if the following condition is satisfied for any $g \in G$: if $\mu_g\alpha = \alpha$, then $\mu_g = id_G$.*

Proof: Suppose that $C_G(\alpha(G)) = Z(G)$ and that $\mu_g\alpha = \alpha$. Then $\mu_g\alpha(g') = \alpha(g')$ for all $g' \in G$; that is

$$g + \alpha(g') - g = \alpha(g') \text{ for all } g' \in G.$$

This means that $g \in C_G(\alpha(G)) = Z(G)$, and so $\mu_g = id_G$.

Conversely, suppose that, for all $g \in G$, if $\mu_g\alpha = \alpha$, then $\mu_g = id_G$. If $r \in C_G(\alpha(G))$, then for all $g' \in G$:

$$r + \alpha(g') - r = \alpha(g').$$

This means that $\mu_r\alpha = \alpha$; by assumption, we get $\mu_r = id_G$, and hence $r \in Z(G)$. ■

Corollary 2.9. *Given $g_1, g_2 \in G$ and $\alpha \in End(G)$ such that $C_G(\alpha(G)) = Z(G)$, if $\mu_{g_1}\alpha = \mu_{g_2}\alpha$, then $\mu_{g_1} = \mu_{g_2}$.*

We complete this section with the following simple but crucial consequence of Definition 2.2.

Proposition 2.10. *Let $\Phi\colon M \to \frac{End(G)}{Inn(G)}$ be an admissible abstract kernel. Then M acts on $Z(G)$ as follows:*

$$x \cdot c = \varphi(x)(c) \qquad \text{for } x \in M,\ c \in Z(G) \text{ and } \varphi(x) \in \Phi(x).$$

Proof: We have already seen that $\varphi(x)(c) \in Z(G)$ for all $x \in M$ and $c \in Z(G)$ (see the paragraph after Definition 2.2). Now we show that the definition of the action given above does not depend on the choice of $\varphi(x) \in \Phi(x)$. If $\psi(x) \in \Phi(x)$ is another representative, then $\psi(x) = \mu_h\varphi(x)$ for some $h \in G$. Then

$$\psi(x)(c) = \mu_h\varphi(x)(c) = h + \varphi(x)(c) - h = \varphi(x)(c),$$

where the last equality holds since $\varphi(x)(c) \in Z(G)$. The fact that in this way an action of the monoid M on the abelian group $Z(G)$ is defined is a straightforward verification. ■

3. The product of admissible abstract kernels

Let $\Phi_1\colon M \to \frac{End(G_1)}{Inn(G_1)}$ and $\Phi_2\colon M \to \frac{End(G_2)}{Inn(G_2)}$ be admissible abstract kernels such that $Z(G_1) = Z(G_2) = C$, inducing a fixed action $\Phi_0\colon M \to End(C)$ of M on C. We want to define a product of Φ_1 and Φ_2, i.e. an admissible abstract kernel $\Phi\colon M \to \frac{End(G)}{Inn(G)}$ such that $Z(G) = C$ and inducing the same action Φ_0 of M on C. In order to do that, consider, as in [7], the following subgroup of $G_1 \times G_2$:

$$S = \{\, (c, -c) \mid c \in C \,\}.$$

It is immediate to check that S is a normal subgroup of $G_1 \times G_2$. We then define $G = \frac{G_1 \times G_2}{S}$. There is a monomorphism $j\colon C \to G$ defined by $j(c) = cl(c, 0) = cl(0, c)$. Moreover, for all $cl(u_1, u_2) \in Z(G)$ and all $g_1 \in G_1$ we have

$$cl(u_1, u_2) + cl(g_1, 0) = cl(g_1, 0) + cl(u_1, u_2),$$

hence

$$cl(u_1 + g_1, u_2) = cl(g_1 + u_1, u_2).$$

This means that

$$(u_1 + g_1, u_2) - (g_1 + u_1, u_2) \in S,$$

i.e. there exists $c \in C$ such that

$$(u_1 + g_1, u_2) - (g_1 + u_1, u_2) = (c, -c).$$

From this we obtain

$$u_1 + g_1 - (g_1 + u_1) = c \qquad \text{and} \qquad u_2 - u_2 = -c,$$

and so $c = 0$ and $u_1 \in Z(G_1) = C$. Similarly one can prove that $u_2 \in Z(G_2) = C$. Hence

$$j(u_1 + u_2) = cl(u_1 + u_2, 0) = cl(u_1, u_2),$$

and this shows that $j(C) = Z(G)$, so the center of G can be identified with C.

Now we can define $\Phi\colon M \to \frac{End(G)}{Inn(G)}$. For $x \in M$, consider any representatives $\varphi_1(x) \in \Phi_1(x)$, $\varphi_2(x) \in \Phi_2(x)$. We obtain an endomorphism $\varphi_1(x) \times \varphi_2(x)\colon G_1 \times G_2 \to G_1 \times G_2$. For all $c \in C$, $(\varphi_1(x) \times \varphi_2(x))(c, -c) = (\varphi_1(x)(c), -\varphi_2(x)(c)) = (x \cdot c, -x \cdot c) \in S$ since Φ_1 and Φ_2 are admissible (see Proposition 2.10 and its proof). Hence we have $(\varphi_1(x) \times \varphi_2(x))(S) \subseteq S$, giving an endomorphism $\varphi(x)\colon G \to G$ defined by

$$\varphi(x)(cl(g_1, g_2)) = cl(\varphi_1(x)(g_1), \varphi_2(x)(g_2)).$$

If we choose different representatives $\psi_1(x) \in \Phi_1(x)$, $\psi_2(x) \in \Phi_2(x)$, we get another endomorphism $\psi(x)\colon G \to G$ given by

$$\psi(x)(cl(g_1, g_2)) = cl(\psi_1(x)(g_1), \psi_2(x)(g_2)).$$

$\varphi_i(x)$ and $\psi_i(x)$ differ by inner automorphisms, i.e. there are $h_i \in G_i$ ($i = 1, 2$) such that $\varphi_i(x) = \mu_{h_i}\psi_i(x)$. Now we show that $\varphi(x) = \mu_{cl(h_1,h_2)}\psi(x)$:

$$\mu_{cl(h_1,h_2)}\psi(x)(cl(g_1, g_2)) = cl(h_1, h_2) + cl(\psi_1(x)(g_1), \psi_2(x)(g_2)) - cl(h_1, h_2) =$$
$$= cl(h_1+\psi_1(x)(g_1)-h_1, h_2+\psi_2(x)(g_2)-h_2) = cl(\mu_{h_1}\psi_1(x)(g_1), \mu_{h_2}\psi_2(x)(g_2)) =$$
$$= cl(\varphi_1(x)(g_1), \varphi_2(x)(g_2)) = \varphi(x)(cl(g_1, g_2)).$$

Thus we obtain a well defined map $\Phi\colon M \to \frac{End(G)}{Inn(G)}$, given by $\Phi(x) = cl(\varphi(x))$. We have that Φ is a monoid homomorphism. Indeed, for $x, y \in M$, consider representatives $\varphi_i(x) \in \Phi_i(x)$, $\varphi_i(y) \in \Phi_i(y)$; we have

$$\varphi_i(x)\varphi_i(y) = \mu_{h_i}\varphi_i(xy) \qquad \text{for some } h_i \in G_i.$$

Then

$$\varphi(x)\varphi(y)(cl(g_1, g_2)) = \varphi(x)(cl(\varphi_1(y)(g_1), \varphi_2(y)(g_2))) =$$
$$= cl(\varphi_1(x)\varphi_1(y)(g_1), \varphi_2(x)\varphi_2(y)(g_2)) = cl(\mu_{h_1}\varphi_1(xy)(g_1), \mu_{h_2}\varphi_2(xy)(g_2)) =$$
$$= cl(h_1 + \varphi_1(xy)(g_1) - h_1, h_2 + \varphi_2(xy)(g_2) - h_2) =$$
$$= cl(h_1, h_2)+cl(\varphi_1(xy)(g_1), \varphi_2(xy)(g_2))-cl(h_1, h_2) = \mu_{cl(h_1,h_2)}\varphi(xy)(cl(g_1, g_2)).$$

Hence $\varphi(x)\varphi(y) = \mu_{cl(h_1,h_2)}\varphi(xy)$, and so

$$\Phi(xy) = cl(\varphi(xy)) = cl(\varphi(x)\varphi(y)) = cl(\varphi(x))cl(\varphi(y)) \in \Phi(x)\Phi(y),$$

and clearly $\Phi(1) = cl(\varphi(1)) = cl(id_G) = id_{\frac{End(G)}{Inn(G)}}$.

It remains to show that Φ is admissible. Let $cl(r_1, r_2) \in C_G(\varphi(x)(G))$; for every $g_1 \in G_1$ we have

$$cl(r_1, r_2) + \varphi(x)(cl(g_1, 0)) = \varphi(x)(cl(g_1, 0)) + cl(r_1, r_2),$$

hence

$$cl(r_1, r_2) + cl(\varphi_1(x)(g_1), 0) = cl(\varphi_1(x)(g_1), 0) + cl(r_1, r_2),$$

which means that

$$cl(r_1 + \varphi_1(x)(g_1), r_2) = cl(\varphi_1(x)(g_1) + r_1, r_2),$$

or, in other terms,

$$(r_1 + \varphi_1(x)(g_1), r_2) - (\varphi_1(x)(g_1) + r_1, r_2) = (c, -c)$$

for some $c \in C$. Then $c = 0$ and hence

$$r_1 + \varphi_1(x)(g_1) = \varphi_1(x)(g_1) + r_1,$$

from which we get that $r_1 \in C_{G_1}(\varphi_1(x)(G_1)) = C$. Similarly one proves that $r_2 \in C$. Hence

$$cl(r_1, r_2) = cl(r_1 + r_2, 0) = j(r_1 + r_2) \in j(C) = C$$

and Φ is admissible. Finally, the action of M on C induced by Φ is the same as the one induced by Φ_1 and Φ_2, i.e $\Phi_0\colon M \to End(C)$. Indeed:

$$x \cdot j(c) = \varphi(x)(cl(c, 0)) = cl(\varphi_1(x)(c), \varphi_2(x)(0)) = cl(x \cdot c, 0) = x \cdot c$$

for all $x \in M$ and $c \in C$.

Then, on the class of admissible abstract kernels inducing the action Φ_0 of M on C, we have a well defined binary operation $\otimes$, given by

$$(M, G_1, \Phi_1) \otimes (M, G_2, \Phi_2) = (M, G, \Phi).$$

We want this operation to give a monoid structure. In order to have this, we need to identify admissible abstract kernels by means of the following equivalence relation:

Definition 3.1. *Two admissible abstract kernels* $\Phi_1\colon M \to \frac{End(G_1)}{Inn(G_1)}$ *and* $\Phi_2\colon M \to \frac{End(G_2)}{Inn(G_2)}$ *inducing the same M-action on* $C = Z(G_1) = Z(G_2)$ *are C-*equivalent *if there exists a group isomorphism* $\xi\colon G_1 \to G_2$ *satisfying the two following conditions:*

(i) *for all* $c \in C$, $\xi(c) = c$;

(ii) *for all* $x \in M$ *and all* $\varphi_1(x) \in \Phi_1(x)$, $\xi\varphi_1(x)\xi^{-1} \in \Phi_2(x)$.

Condition (ii) can be expressed by the commutativity of the following triangle:

$$\begin{tikzcd} M \arrow[r, "\Phi_1"] \arrow[dr, "\Phi_2"'] & \frac{End(G_1)}{Inn(G_1)} \arrow[d, "\overline{\xi}"] \\ & \frac{End(G_2)}{Inn(G_2)}, \end{tikzcd}$$

where $\overline{\xi}(cl(\alpha)) = cl(\xi\alpha\xi^{-1})$.

We will write $(M, G_1, \Phi_1) \overset{C}{\cong} (M, G_2, \Phi_2)$ to denote that Φ_1 and Φ_2 are C-equivalent. It is clear that $\overset{C}{\cong}$ is an equivalence relation.

The proofs of the following facts are analogous to the corresponding ones in [7] for the case of the classical abstract kernels, that is, for the case of abstract kernels of the form $\Phi\colon \Pi \to \frac{Aut(G)}{Inn(G)}$, where Π and G are groups. We give them in details for the sake of completeness.

Proposition 3.2. *The definition of the binary operation $\otimes$ is compatible with the C-equivalence.*

Proof: Suppose that $(M, G_1, \Phi_1)\otimes(M, G_2, \Phi_2) = (M, G, \Phi)$ and $(M, G_1', \Phi_1')\otimes (M, G_2', \Phi_2') = (M, G', \Phi')$ and that $\Phi_i \overset{C}{\cong} \Phi_i'$. Then there are isomorphisms $\xi_i\colon G_i \to G_i'$ satisfying the conditions (i) and (ii) above. They induce an isomorphism $\xi_1 \times \xi_2\colon G_1 \times G_2 \to G_1' \times G_2'$, and since $(\xi_1 \times \xi_2)(c, -c) = (c, -c)$, we get an isomorphism

$$\xi\colon G = \frac{G_1 \times G_2}{S} \to G' = \frac{G_1' \times G_2'}{S} \quad \text{given by } \xi(cl(g_1, g_2)) = cl(\xi_1(g_1), \xi_2(g_2)),$$

and clearly $\xi(c) = \xi(cl(c, 0)) = cl(\xi_1(c), 0) = cl(c, 0) = c$. It remains to show that the triangle

$$\begin{tikzcd} M \arrow[r, "\Phi"] \arrow[dr, "\Phi'"'] & \frac{End(G)}{Inn(G)} \arrow[d, "\overline{\xi}"] \\ & \frac{End(G')}{Inn(G')} \end{tikzcd}$$

commutes, where $\overline{\xi}$ is defined as in Definition 3.1. For $x \in M$ and $\varphi(x) \in \Phi(x)$, we have

$$\varphi(x)(cl(g_1, g_2)) = cl(\varphi_1(x)(g_1), \varphi_2(x)(g_2)) \quad \text{with } \varphi_i(x) \in \Phi_i(x),$$

and, by assumption, $\varphi_i'(x) = \xi_i \varphi_i(x) \xi_i^{-1} \in \Phi_i'(x)$. Hence, defining $\varphi'(x) \in \Phi'(x)$ by

$$\varphi'(x)(cl(g_1', g_2')) = cl(\varphi_1'(x)(g_1'), \varphi_2'(x)(g_2')),$$

we have that

$$\xi\varphi(x)\xi^{-1}(cl(g_1', g_2')) = cl(\xi_1\varphi_1(x)\xi_1^{-1}(g_1'), \xi_2\varphi_2(x)\xi_2^{-1}(g_2')) =$$
$$= cl(\varphi_1'(x)(g_1'), \varphi_2'(x)(g_2')) = \varphi'(x)(cl(g_1', g_2')),$$

and so $\xi\varphi(x)\xi^{-1} = \varphi'(x) \in \Phi'(x)$. ■

Proposition 3.3. *The neutral element of $\otimes$ is $\Phi_0\colon M \to End(C)$, the fixed M-action on C.*

Proof: Given an admissible abstract kernel $\Phi\colon M \to \frac{End(G)}{Inn(G)}$ inducing the action Φ_0 of M on C, we want to show that $(M, G, \Phi) \otimes (M, C, \Phi_0)$ is C-equivalent to (M, G, Φ). Let us consider the map $\xi\colon G \to \frac{G \times C}{S}$ defined by $\xi(g) = cl(g, 0)$. It is clearly a group homomorphism, and moreover $\xi(c) = cl(c, 0) = c$ (identifying $j(C)$ with C). Its inverse ξ^{-1} given by $\xi^{-1}(cl(g, c)) = g + c$ is well defined. Indeed, $cl(g, c) = cl(g', c')$ if and only if

$$(g, c) - (g', c') = (c_1, -c_1) \quad \text{for some } c_1 \in C,$$

i.e. if and only if

$$g - g' = c_1, \quad c - c' = -c_1 \quad \Leftrightarrow \quad g - g' = c' - c \quad \Leftrightarrow \quad g + c = g' + c'.$$

The fact that $\xi^{-1}\xi$ is the identity is obvious. Concerning the other composition:

$$\xi\xi^{-1}(cl(g, c)) = \xi(g + c) = cl(g + c, 0) = cl(g, c).$$

It remains to show that the following triangle commutes:

$$\begin{tikzcd}
M \arrow[r, "\Phi"] \arrow[dr, "\Psi"'] & \frac{End(G)}{Inn(G)} \arrow[d, "\overline{\xi}"] \\
& \frac{End(\frac{G\times C}{S})}{Inn(\frac{G\times C}{S})},
\end{tikzcd}$$

where $\overline{\xi}$ is defined as in Definition 3.1 and Ψ is given, for $x \in M$ and $\varphi(x) \in \Phi(x)$, by

$$\psi(x)(cl(g, c)) = cl(\varphi(x)(g),\ x \cdot c).$$

We have to show that, if $\varphi(x) \in \Phi(x)$, then $\xi\varphi(x)\xi^{-1} \in \Psi(x)$. We have that

$$\psi(x)(cl(g, c)) = cl(\varphi(x)(g),\ x \cdot c) = cl(\varphi(x)(g) + x \cdot c,\ 0) =$$

$$= \xi(\varphi(x)(g)+x\cdot c) = \xi(\varphi(x)(g)+\varphi(x)(c)) = \xi\varphi(x)(g+c) = \xi\varphi(x)\xi^{-1}(cl(g, c)),$$

and so $\xi\varphi(x)\xi^{-1} = \psi(x) \in \Psi(x)$. ■

So we proved that the set $\mathcal{M}(M, C)$ of C-equivalence classes $[M, G, \Phi]$ of admissible abstract kernels, inducing the fixed M-action $\Phi_0\colon M \to End(C)$ on the group C, is a unitary magma w.r.t. the product defined above. Our aim now is to show that it is actually a commutative monoid. In order to prove associativity, we start with some preliminary lemmas.

Lemma 3.4. *Given two admissible abstract kernels* $\Phi_1\colon M \to \frac{End(G_1)}{Inn(G_1)}$ *and* $\Phi_2\colon M \to \frac{End(G_2)}{Inn(G_2)}$, *inducing the same* M*-action on* $C = Z(G_1) = Z(G_2)$, *and their product* $\Phi\colon M \to \frac{End(G)}{Inn(G)}$, *in* G *we have that*

$cl(g_1, g_2) = c \in C$ *if and only if* $g_i = c_i$ *for some* $c_1, c_2 \in C$ *with* $c_1 + c_2 = c$.

Proof: If $cl(g_1, g_2) = cl(c, 0) = c$, then there exists $c' \in C$ such that

$$(g_1, g_2) - (c, 0) = (c', -c');$$

then

$$g_1 - c = c', \quad g_2 = -c',$$

so, putting $c_1 = c + c'$ and $c_2 = -c'$ we get the thesis. Conversely,

$$cl(g_1, g_2) = cl(c_1, c_2) = cl(c_1 + c_2, 0) = cl(c, 0) = c.$$

■

Lemma 3.5. *Given three admissible abstract kernels* $\Phi_i\colon M \to \frac{End(G_i)}{Inn(G_i)}$, $i = 1, 2, 3$, *inducing the same* M*-action on* $C = Z(G_i)$, *consider the product* $\Phi\colon M \to \frac{End(G)}{Inn(G)}$ *of* Φ_1 *and* Φ_2 *and the product* $\Phi^\sharp\colon M \to \frac{End(G^\sharp)}{Inn(G^\sharp)}$ *of* Φ *and* Φ_3, *so that*

$$(M, G^\sharp, \Phi^\sharp) = ((M, G_1, \Phi_1) \otimes (M, G_2, \Phi_2)) \otimes (M, G_3, \Phi_3).$$

Then, in $G^\sharp = \frac{((G_1 \times G_2)/S) \times G_3}{S}$, *we have that* $cl(cl(g_1, g_2), g_3) = cl(cl(g_1', g_2'), g_3')$ *if and only if*

$$g_1 - g_1' = c_1, \;\; g_2 - g_2' = c_2, \;\; g_3 - g_3' = -(c_1 + c_2), \;\; \textit{with} \;\; c_1, c_2 \in C.$$

Proof: If $cl(cl(g_1, g_2), g_3) = cl(cl(g_1', g_2'), g_3')$, then there exists $c \in C$ such that

$$(cl(g_1, g_2), g_3) - (cl(g_1', g_2'), g_3') = (c, -c),$$

hence

$$cl(g_1 - g_1', g_2 - g_2') = c, \quad g_3 - g_3' = -c.$$

Thanks to the previous lemma, we know that there exist $c_1, c_2 \in C$ such that

$$g_1 - g_1' = c_1, \quad g_2 - g_2' = c_2, \quad c_1 + c_2 = c,$$

and so

$$g_1 - g_1' = c_1, \quad g_2 - g_2' = c_2, \quad g_3 - g_3' = -c = -(c_1 + c_2).$$

Conversely,

$$(cl(g_1, g_2), g_3) - (cl(g'_1, g'_2), g'_3) = (cl(g_1, g_2) - cl(g'_1, g'_2),\ g_3 - g'_3) =$$
$$= (cl(g_1 - g'_1, g_2 - g'_2),\ g_3 - g'_3) = (cl(c_1, c_2), -(c_1 + c_2)) =$$
$$= (cl(c_1 + c_2, 0), -(c_1 + c_2)) = (c_1 + c_2, -(c_1 + c_2)),$$

and so $cl(cl(g_1, g_2), g_3) = cl(cl(g'_1, g'_2), g'_3)$. ■

In the same way one can prove the following

Lemma 3.6. *Given three admissible abstract kernels* $\Phi_i\colon M \to \frac{End(G_i)}{Inn(G_i)}$, $i = 1, 2, 3$, *inducing the same* M*-action on* $C = Z(G_i)$, *consider the product* $\Psi\colon M \to \frac{End(H)}{Inn(H)}$ *of* Φ_2 *and* Φ_3 *and the product* $\Psi^\sharp\colon M \to \frac{End(H^\sharp)}{Inn(H^\sharp)}$ *of* Φ_1 *and* Ψ, *so that*

$$(M, H^\sharp, \Psi^\sharp) = (M, G_1, \Phi_1) \otimes ((M, G_2, \Phi_2) \otimes (M, G_3, \Phi_3)).$$

Then, in $H^\sharp = \frac{G_1 \times ((G_2 \times G_3)/S)}{S}$, *we have that* $cl(g_1, cl(g_2, g_3)) = cl(g'_1, cl(g'_2, g'_3))$ *if and only if*

$$g_2 - g'_2 = c_2,\ \ g_3 - g'_3 = c_3,\ \ g_1 - g'_1 = -(c_2 + c_3),\ \ \textit{with}\ \ c_2, c_3 \in C.$$

Proposition 3.7. *The unitary magma* $\mathcal{M}(M, C)$ *is a monoid.*

Proof: Using the notation of the previous lemmas, we have to show that $\Phi^\sharp\colon M \to \frac{End(G^\sharp)}{Inn(G^\sharp)}$ and $\Psi^\sharp\colon M \to \frac{End(H^\sharp)}{Inn(H^\sharp)}$ are C-equivalent. To do that, first we have to build a group isomorphism

$$\xi\colon \frac{((G_1 \times G_2)/S) \times G_3}{S} \to \frac{G_1 \times ((G_2 \times G_3)/S)}{S}.$$

ξ is defined by $\xi(cl(cl(g_1, g_2), g_3)) = cl(g_1, cl(g_2, g_3))$. It is well defined, indeed, if

$$cl(cl(g_1, g_2), g_3) = cl(cl(g'_1, g'_2), g'_3),$$

then, thanks to Lemma 3.5, there exist $c_1, c_2 \in C$ such that

$$g_1 - g'_1 = c_1, \quad g_2 - g'_2 = c_2, \quad g_3 - g'_3 = -(c_1 + c_2).$$

Putting $c'_2 = c_2$ and $c'_3 = -(c_1 + c_2)$, we get that

$$g_2 - g'_2 = c'_2,\ \ g_3 - g'_3 = c'_3,\ \ g_1 - g'_1 = c_1 = -(c_2 - (c_1 + c_2)) = -(c'_2 + c'_3),$$

and then, by Lemma 3.6, we conclude that

$$cl(g_1, cl(g_2, g_3)) = cl(g'_1, cl(g'_2, g'_3)).$$

The fact that ξ is a group homomorphism is obvious. Its inverse ξ^{-1} is defined by

$$\xi^{-1}(cl(g_1, cl(g_2, g_3))) = (cl(cl(g_1, g_2), g_3)).$$

The proof that ξ^{-1} is a well defined map is similar to the one for ξ, and it is obvious that these two maps are inverse to each other. Moreover, for all $c \in C$:

$$\xi(c) = \xi(cl(c, 0)) = cl(c, cl(0, 0)) = cl(c, 0) = c.$$

It remains to show that the following triangle commutes:

$$\begin{tikzcd} M \arrow[r, "\Phi^\sharp"] \arrow[dr, "\Psi^\sharp"'] & \frac{End(G^\sharp)}{Inn(G^\sharp)} \arrow[d, "\overline{\xi}"] \\ & \frac{End(H^\sharp)}{Inn(H^\sharp)}, \end{tikzcd}$$

where $\overline{\xi}(cl(\alpha)) = cl(\xi\alpha\xi^{-1})$. Consider the representatives $\varphi^\sharp(x) \in \Phi^\sharp(x)$, $\psi^\sharp(x) \in \Psi^\sharp(x)$, where

$$\varphi^\sharp(x)(cl(cl(g_1, g_2), g_3)) = (cl(cl(\varphi_1(x)(g_1), \varphi_2(x)(g_2)), \varphi_3(x)(g_3))),$$
$$\psi^\sharp(x)(cl(g_1, cl(g_2, g_3))) = cl(\varphi_1(x)(g_1), cl(\varphi_2(x)(g_2), \varphi_3(x)(g_3))),$$

with $\varphi_i(x) \in \Phi_i(x)$. Then

$$\begin{aligned} \xi\varphi^\sharp(x)\xi^{-1}(cl(g_1, cl(g_2, g_3))) &= \xi\varphi^\sharp(x)(cl(cl(g_1, g_2), g_3)) = \\ &= \xi((cl(cl(\varphi_1(x)(g_1), \varphi_2(x)(g_2)), \varphi_3(x)(g_3)))) = \\ &= cl(\varphi_1(x)(g_1), cl(\varphi_2(x)(g_2), \varphi_3(x)(g_3))) = \psi^\sharp(x)(cl(g_1, cl(g_2, g_3))), \end{aligned}$$

and hence $\xi\varphi^\sharp(x)\xi^{-1} = \psi^\sharp(x) \in \Psi^\sharp(x)$. ∎

Proposition 3.8. *The monoid $\mathcal{M}(M, C)$ is commutative.*

Proof: Given two admissible abstract kernels $\Phi_i \colon M \to \frac{End(G_i)}{Inn(G_i)}$, $i = 1, 2$, inducing the same M-action on $C = Z(G_i)$, consider the products $(M, G_1, \Phi_1) \otimes (M, G_2, \Phi_2) = (M, G, \Phi)$ and $(M, G_2, \Phi_2) \otimes (M, G_1, \Phi_1) = (M, G', \Psi)$, where $G = \frac{G_1 \times G_2}{S}$ and $G' = \frac{G_2 \times G_1}{S}$. It is clear that the twisting isomorphism $G_1 \times G_2 \to G_2 \times G_1$ gives an isomorphism $\xi \colon G \to G'$, defined by $\xi(cl(g_1, g_2)) = cl(g_2, g_1)$, such that $\xi(c) = c$ for all $c \in C$. To conclude the proof, consider the representatives $\varphi(x) \in \Phi(x)$, $\psi(x) \in \Psi(x)$, where

$$\varphi(x)(cl(g_1, g_2)) = cl(\varphi_1(x)(g_1), \varphi_2(x)(g_2)),$$
$$\psi(x)(cl(g_2, g_1)) = cl(\varphi_2(x)(g_2), \varphi_1(x)(g_1)),$$

with $\varphi_i(x) \in \Phi_i(x)$. Then

$$\xi\varphi(x)\xi^{-1}(cl(g_2,g_1)) = \xi\varphi(x)(cl(g_1,g_2)) = \xi(cl(\varphi_1(x)(g_1),\varphi_2(x)(g_2))) = \\ cl(\varphi_2(x)(g_2),\varphi_1(x)(g_1)) = \psi(x)(cl(g_2,g_1)),$$

hence $\xi\varphi(x)\xi^{-1} = \psi(x) \in \Psi(x)$. ■

Our aim, now, is to introduce a suitable submonoid $\mathcal{L}(M,C)$ of $\mathcal{M}(M,C)$ such that the quotient monoid becomes an abelian group. In order to do that, in the next section we will consider the notion of *extendable* admissible abstract kernel.

4. Extendable admissible abstract kernels

We recall from [2, 13, 14] the following

Definition 4.1. *Let*

$$E:\ 0 \longrightarrow G \overset{\kappa}{\rightarrowtail} B \overset{\sigma}{\twoheadrightarrow} M \longrightarrow 1 \tag{1}$$

be a sequence of monoids and monoid homomorphisms such that σ is a surjection, κ is an injection and $\kappa(G) = \{b \in B \mid \sigma(b) = 1\}$ (i.e. κ is the kernel of σ). E is a special Schreier extension *of M by G (some authors would say "G by M") if, for every $b_1, b_2 \in B$ such that $\sigma(b_1) = \sigma(b_2)$, there exists a unique $g \in G$ such that*

$$b_2 = g + b_1,$$

where we treat κ just as an inclusion (again, we use the multiplicative notation for M and the additive one for the other monoids involved).

The word "special" is motivated by the fact that these extensions are special cases of the Schreier extensions in the sense of [19] (see also [17, 18]). It is easily seen that, in a special Schreier extension (1), the monoid G is necessarily a group.

Let us now show how to associate an abstract kernel to a special Schreier extension (1). First note that σ is the cokernel of κ. Indeed, suppose that $f\colon B \to M'$ is a monoid homomorphism such that $f(g) = 1_{M'}$ for all $g \in G$. Define $f'\colon M \to M'$ by putting $f'(x) = f(b)$, $b \in \sigma^{-1}(x)$. If $\sigma(b_1) = x = \sigma(b_2)$, then $b_2 = g + b_1$, whence $f(b_2) = f(b_1)$. Hence f' is well defined. Clearly, f' is a monoid homomorphism and $f'\sigma = f$. The uniqueness of such a homomorphism f' is also clear. Furthermore, for every $b \in B$ and every

$g \in G$, there is a unique $g' \in G$ such that $b + g = g' + b$. This defines an endomorphism $\theta(b)\colon G \to G$ sending g to g' ($b + g_1 + g_2 = \theta(b)(g_1) + b + g_2 = \theta(b)(g_1) + \theta(b)(g_2) + b$, whence $\theta(b)(g_1 + g_2) = \theta(b)(g_1) + \theta(b)(g_2)$). Moreover, we get a monoid homomorphism $\theta\colon B \to End(G)$, which sends b to $\theta(b)$ ($\theta(0) = 1_G$ and $b_1 + b_2 + g = b_1 + \theta(b_2)(g) + b_2 = \theta(b_1)(\theta(b_2)(g)) + b_1 + b_2$, whence $\theta(b_1 + b_2)(g) = (\theta(b_1)\theta(b_2))(g)$). For $g \in G$, it is immediate to see that $\theta(g) = \mu_g \in Inn(G)$. Hence, since σ is the cokernel of κ, we get the abstract kernel Φ via the universal property of the cokernel, as in the following diagram:

$$\begin{tikzcd} G \arrow[r, tail, "\kappa"] & B \arrow[r, two heads, "\sigma"] \arrow[d, "\theta"'] & M \arrow[d, "\Phi"] \\ & End(G) \arrow[r, "p"'] & \frac{End(G)}{Inn(G)}. \end{tikzcd} \tag{2}$$

More explicitly, $\Phi(x) = p\theta(b) = cl(\theta(b))$ for any b such that $\sigma(b) = x$.

Given a special Schreier extension (1), for every $b \in B$ one always has that $b + G \subseteq G + b$ (and so G is right normal in B), but the other inclusion is false, in general. The set

$$G_b = \{\ \overline{g} \in G \mid \overline{g} + b = b + g \text{ for some } g \in G\ \}$$

measures the difference between the two cosets (in other words, the sets G_b measure how far G is from being a normal subgroup of B).

Lemma 4.2. *G_b is a subgroup of G.*

Proof: If $\overline{g_1}, \overline{g_2} \in G_b$, then

$$\overline{g_1} + b = b + g_1, \quad \overline{g_2} + b = b + g_2 \quad \text{for some } g_1, g_2 \in G.$$

Then

$$\overline{g_1} + \overline{g_2} + b = \overline{g_1} + b + g_2 = b + g_1 + g_2,$$

and so $\overline{g_1} + \overline{g_2} \in G_b$. Furthermore, if $\overline{g} \in G_b$, then $\overline{g} + b = b + g$ for some $g \in G$. Hence $-\overline{g} + b = b + (-g)$, and $-\overline{g} \in G_b$. ■

Definition 4.3. *A special Schreier extension* (1) *is* admissible *if, for all $b \in B$, $C_G(G_b) = Z(G)$.*

Lemma 4.4. *In the notation of diagram* (2)*, for all $b \in B$ one has $\theta(b)(G) = G_b$.*

Proof: If $g \in \theta(b)(G)$, then $g = \theta(b)(g_1)$ for some $g_1 \in G$. Hence

$$b + g_1 = \theta(b)(g_1) + b = g + b$$

and $g \in G_b$. Conversely, if $g \in G_b$, then there exists $g_1 \in G$ such that $g + b = b + g_1$. Thus, we have

$$b + g_1 = g + b \quad \text{and} \quad b + g_1 = \theta(b)(g_1) + b,$$

whence, by the uniqueness in the Schreier condition, we get that $g = \theta(b)(g_1)$, and so $g \in \theta(b)(G)$. ■

Proposition 4.5. *If an abstract kernel* $\Phi \colon M \to \frac{End(G)}{Inn(G)}$ *is induced by a special Schreier extension* E*, then* Φ *is admissible if and only if* E *is admissible.*

Proof: Suppose E is admissible. Then, for all $b \in B$, $C_G(G_b) = Z(G)$. By the previous lemma, this means that $C_G(\theta(b)(G)) = Z(G)$. Let $x = \sigma(b)$. Then $\theta(b) \in \Phi(x)$, and hence Φ is admissible (see Proposition 2.3). Conversely, suppose Φ is admissible. If $b \in B$, then $\Phi(\sigma(b)) = cl(\theta(b))$. By admissibility of Φ, we know that $C_G(\theta(b)(G)) = Z(G)$. Since $\theta(b)(G) = G_b$, thanks to the previous lemma, we get that E is admissible. ■

Definition 4.6. *We say that an admissible abstract kernel* $\Phi \colon M \to \frac{End(G)}{Inn(G)}$ *is* extendable *if it is induced by a special Schreier extension (which is necessarily admissible because of the previous proposition).*

Suppose that admissible abstract kernels (M, G, Φ) and (M, G', Φ') inducing the same M-action on $C = Z(G) = Z(G')$ are C-equivalent. Then, (M, G, Φ) is extendable if and only if so is (M, G', Φ'). Indeed, if (M, G, Φ) is induced by a special Schreier extension (1), then (M, G', Φ') is induced by a special Schreier extension $E' : \ 0 \longrightarrow G' \overset{\kappa\xi^{-1}}{\rightarrowtail} B \overset{\sigma}{\twoheadrightarrow} M \longrightarrow 1$, where $\xi \colon G \to G'$ is an isomorphism realizing the C-equivalence $(M, G, \Phi) \overset{C}{\cong} (M, G', \Phi')$ (see Definition 3.1). The set of C-equivalence classes of extendable admissible abstract kernels inducing the same M-action on C will be denoted by $\mathcal{L}(M, C)$.

Proposition 4.7. *If* (M, G_1, Φ_1) *and* (M, G_2, Φ_2) *are extendable admissible abstract kernels inducing the same action on* C*, then their product*

$$(M, G, \Phi) = (M, G_1, \Phi_1) \otimes (M, G_2, \Phi_2)$$

is extendable as well.

Proof: If Φ_1 and Φ_2 are extendable, then they are induced by admissible special Schreier extensions E_1 and E_2, as in the following diagrams:

$$\begin{tikzcd} E_1: G_1 \arrow[r, tail, "\kappa_1"] & B_1 \arrow[r, two heads, "\sigma_1"] \arrow[d, "\theta_1"'] & M \arrow[d, "\Phi_1"] \\ & End(G_1) \arrow[r, "p_1"'] & \frac{End(G_1)}{Inn(G_1)}, \end{tikzcd} \qquad \begin{tikzcd} E_2: G_2 \arrow[r, tail, "\kappa_2"] & B_2 \arrow[r, two heads, "\sigma_2"] \arrow[d, "\theta_2"'] & M \arrow[d, "\Phi_2"] \\ & End(G_2) \arrow[r, "p_2"'] & \frac{End(G_2)}{Inn(G_2)}. \end{tikzcd}$$

Consider the pullback

$$\begin{tikzcd} R \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B_2 \arrow[d, "\sigma_2"] \\ B_1 \arrow[r, "\sigma_1"'] & M, \end{tikzcd}$$

i.e. the monoid $R = \{(b_1, b_2) \in B_1 \times B_2 \mid \sigma_1(b_1) = \sigma_2(b_2)\}$. Clearly $S = \{(c, -c) \mid c \in C\}$ is a submonoid of R. Moreover, S is right normal in R, i.e. $(b_1, b_2) + S \subseteq S + (b_1, b_2)$ for all $(b_1, b_2) \in R$. Indeed, if $(b_1, b_2) \in R$ and $(c, -c) \in S$, using the admissibility of Φ_1 and Φ_2 we get that

$$\theta_1(b_1)(c) = \theta_2(b_2)(c) = x \cdot c,$$

where $x = \sigma_1(b_1) = \sigma_2(b_2)$. Then we have

$$(b_1, b_2) + (c, -c) = (b_1 + c, b_2 - c) = (\theta_1(b_1)(c) + b_1, \theta_2(b_2)(-c) + b_2) =$$
$$= (x \cdot c + b_1, -x \cdot c + b_2) = (x \cdot c, -x \cdot c) + (b_1, b_2),$$

and $(x \cdot c, -x \cdot c) \in S$. Let us then put $B = \frac{R}{S}$ and consider the following sequence:

$$E: \qquad 0 \longrightarrow G \overset{\kappa}{\rightarrowtail} B \overset{\sigma}{\twoheadrightarrow} M \longrightarrow 1,$$

where

$$G = \frac{G_1 \times G_2}{S}, \quad \kappa(cl(g_1, g_2)) = cl(g_1, g_2), \quad \text{and } \sigma(cl(b_1, b_2)) = \sigma_1(b_1) = \sigma_2(b_2).$$

We want to show that E is a special Schreier extension which induces the product Φ of Φ_1 and Φ_2. It is immediate to see that κ is a well defined injective homomorphism. σ is well defined, too. Indeed, if $cl(b_1, b_2) = cl(b'_1, b'_2)$, then there exists $c \in C$ such that

$$(b'_1, b'_2) = (c, -c) + (b_1, b_2).$$

Then $b'_1 = c + b_1$, $b'_2 = -c + b_2$, and so $\sigma_i(b'_i) = \sigma_i(b_i)$, $i = 1, 2$. Clearly σ is a monoid homomorphism. It is surjective, since for all $x \in M$ there exist

$b_i \in B_i$, $i = 1, 2$, with $\sigma_i(b_i) = x$; then $\sigma(cl(b_1, b_2)) = x$. Moreover $\sigma\kappa = 0$, indeed $\sigma\kappa(cl(g_1, g_2)) = \sigma_1(g_1) = 1$. So, $\kappa(G) \subseteq Ker(\sigma)$. To show the other inclusion, suppose that $\sigma(cl(b_1, b_2)) = 1$. Then $\sigma_1(b_1) = \sigma_2(b_2) = 1$. Since κ_i is the kernel of σ_i, we know that $b_i = \kappa_i(g_i)$ for some $g_i \in G_i$. Hence $\kappa(cl(g_1, g_2)) = cl(b_1, b_2)$.

Let us now check the Schreier condition. Suppose that $\sigma(cl(b'_1, b'_2)) = \sigma(cl(b_1, b_2))$. Then

$$\sigma_1(b'_1) = \sigma_1(b_1) = \sigma_2(b_2) = \sigma_2(b'_2).$$

E_1 and E_2 are special Schreier extensions, so there are unique $g_i \in G_i$ such that $b'_i = g_i + b_i$. Hence

$$cl(b'_1, b'_2) = cl(g_1, g_2) + cl(b_1, b_2).$$

To prove the uniqueness of the element $cl(g_1, g_2)$ satisfying the last equality, it suffices to show that, if $cl(g_1, g_2) + cl(b_1, b_2) = cl(b_1, b_2)$, then $cl(g_1, g_2) = 0$. So, suppose that $cl(g_1, g_2) + cl(b_1, b_2) = cl(b_1, b_2)$. Then

$$(g_1 + b_1, g_2 + b_2) = (c, -c) + (b_1, b_2) \quad \text{for some } c \in C.$$

Then $g_1 + b_1 = c + b_1$, $g_2 + b_2 = -c + b_2$. Being E_1 and E_2 special Schreier extensions, this gives that $(g_1, g_2) = (c, -c)$, and so $cl(g_1, g_2) = 0$.

It remains to show that E induces the admissible abstract kernel Φ. Let us call Ψ the abstract kernel induced by E, as in the following diagram:

$$\begin{tikzcd}
G \arrow[r, tail, "\kappa"] & B \arrow[r, two heads, "\sigma"] \arrow[d, "\theta"'] & M \arrow[d, "\Psi"] \\
& End(G) \arrow[r, "p"'] & \frac{End(G)}{Inn(G)}.
\end{tikzcd}$$

Then, for $x \in M$, $\Psi(x) = cl(\theta(cl(b_1, b_2)))$, where $\sigma(cl(b_1, b_2)) = \sigma_1(b_1) = \sigma_2(b_2) = x$. By construction of θ_1 and θ_2, we have that

$$b_1 + g_1 = \theta_1(b_1)(g_1) + b_1, \qquad b_2 + g_2 = \theta_2(b_2)(g_2) + b_2.$$

Hence, on one hand

$$cl(b_1, b_2) + cl(g_1, g_2) = cl(\theta_1(b_1)(g_1), \theta_2(b_2)(g_2)) + cl(b_1, b_2);$$

on the other hand, since E is a special Schreier extension, we have that

$$cl(b_1, b_2) + cl(g_1, g_2) = \theta(cl(b_1, b_2))(cl(g_1, g_2)) + cl(b_1, b_2),$$

by construction of θ. Thanks to the uniqueness in the Schreier condition, we obtain that

$$\theta(cl(b_1, b_2))(cl(g_1, g_2)) = cl(\theta_1(b_1)(g_1), \theta_2(b_2)(g_2)).$$

Moreover, we know that Φ is defined by $\Phi(x) = cl(\varphi(x))$, where

$$\varphi(x)(cl(g_1, g_2)) = cl(\theta_1(b_1)(g_1), \theta_2(b_2)(g_2)).$$

Hence $\theta(cl(b_1, b_2)) = \varphi(x)$, and consequently $\Psi(x) = \Phi(x)$ for all $x \in M$. ■

Since the "zero" abstract kernel $\Phi_0 \colon M \to End(C)$ is clearly extendable (it is induced by the special Schreier extension given by the semidirect product of M and C), we get the following

Corollary 4.8. *The set $\mathcal{L}(M, C)$ of C-equivalence classes of extendable admissible abstract kernels inducing the same action of M on C is a submonoid of the monoid $\mathcal{M}(M, C)$.*

Using the fact that the monoid $\mathcal{M}(M, C)$ is commutative, in the next section we will observe that we can consider a suitable factor monoid $\mathcal{A}(M, C) = \frac{\mathcal{M}(M,C)}{\mathcal{L}(M,C)}$, and we will prove that this factor monoid is actually an abelian group, following essentially the same idea of [7] for the case of abstract kernels of the form $\Phi \colon \Pi \to \frac{Aut(G)}{Inn(G)}$ with Π and G are groups.

5. The group structure of admissible abstract kernels

We begin this section by recalling a general fact. If A is a commutative monoid, and $B \subseteq A$ is a submonoid, the relation $\sim$ on A defined by

$$a_1 \sim a_2 \quad \iff \quad \exists\ b_1, b_2 \in B \text{ such that } a_1 + b_1 = a_2 + b_2$$

is a congruence on A. We denote the factor monoid $\frac{A}{\sim}$ by $\frac{A}{B}$. It is easy to check that $\frac{A}{B}$ is a group as soon as the following condition is satisfied: for all $a \in A$ there exists $a' \in A$ such that $a + a' \in B$. We will use this fact, together with the results of the previous sections, to show that the factor monoid $\mathcal{A}(M, C) = \frac{\mathcal{M}(M,C)}{\mathcal{L}(M,C)}$ is an abelian group.

Given an admissible abstract kernel $\Phi \colon M \to \frac{End(G)}{Inn(G)}$, let us denote by G^* the opposite group of G: as a set, $G^* = G$, but we will denote the elements of G^* with g^*, for $g \in G$. Then the group operation in G^* is defined by $g^* + h^* = (h + g)^*$, and so the inverse of an element g^* is $(-g)^*$. We will simply write c for the elements c^* of $C^* = Z(G^*) = Z(G) = C$. Given an endomorphism $\alpha\colon G \to G$, we get an endomorphism $\alpha^*\colon G^* \to G^*$ simply by putting $\alpha^*(g^*) = (\alpha(g))^*$. In this way, it is obvious that $(\beta\alpha)^* = \beta^*\alpha^*$ and that $(\mu_g)^* = \mu_{-g^*}$ for $g \in G$.

Hence we can define an abstract kernel $\Phi^*\colon M \to \frac{End(G^*)}{Inn(G^*)}$ by putting, for $x \in M$, $\Phi^*(x) = cl(\varphi^*(x))$, where $\varphi^*(x) = (\varphi(x))^*$, with $\varphi(x) \in \Phi(x)$.

Lemma 5.1. *The abstract kernel Φ^* is admissible.*

Proof: Let $r^* \in C_{G^*}(\varphi^*(x)(G^*))$. Then, for all $g \in G$,

$$r^* + \varphi^*(x)(g^*) = \varphi^*(x)(g^*) + r^*.$$

This means that

$$r^* + (\varphi(x)(g))^* = (\varphi(x)(g))^* + r^*,$$

hence

$$(\varphi(x)(g) + r)^* = (r + \varphi(x)(g))^*,$$

i.e.

$$\varphi(x)(g) + r = r + \varphi(x)(g).$$

But the abstract kernel Φ is admissible, so $r = r^* \in C$, giving that Φ^* is admissible, too. ∎

We also observe that the action of M on C induced by Φ^* is the same that Φ induces, since

$$\varphi^*(x)(c) = (\varphi(x)(c))^* = (x \cdot c)^* = x \cdot c$$

for all $x \in M$ and all $c \in C$.

Our goal now is to show that, for any admissible abstract kernel (M, G, Φ), the product $(M, G, \Phi) \otimes (M, G^*, \Phi^*)$ is extendable. In order to do that, we first build an admissible special Schreier extension, and then we will show that the abstract kernel induced by it is C-equivalent to $(M, G, \Phi) \otimes (M, G^*, \Phi^*)$.

Let B be the set

$$B = \{(g, \alpha, x) \mid g \in G, x \in M, \alpha \in \Phi(x)\}.$$

We define on B the following binary operation:

$$(g, \alpha, x) + (h, \beta, y) = (g + \alpha(h), \alpha\beta, xy).$$

It is easy to see that $(B,+)$ is a monoid, with neutral element $(0, id_G, 1)$. Consider then the following sequence:

$$E: \; 0 \longrightarrow K \overset{\kappa}{\rightarrowtail} B \overset{\sigma}{\twoheadrightarrow} M \longrightarrow 1,$$

where $\sigma(g,\alpha,x) = x$, K is the kernel of σ and κ is the canonical inclusion. Explicitly

$$K = \{(g,\alpha,1) \mid g \in G, \alpha \in \Phi(1)\} = \{(g,\mu_h,1) \mid g,h \in G\}.$$

Moreover, K is a group. Indeed, for all $g,h \in G$ we have that

$$(g,\mu_h,1) + (-h-g+h,\ \mu_{-h},1) = (g+\mu_h(-h-g+h),\ \mu_h\mu_{-h},1) =$$
$$= (g+h-h-g+h-h,\ \mu_{h-h},1) = (0,\mu_0,1) = (0,id_G,1) = 1_K.$$

Lemma 5.2. $Z(K) = \{(c, id_G, 1) \mid c \in C\}$ *and so can be identified with* C.

Proof: If $(s,\mu_h,1) \in Z(K)$, then for all $r \in G$ we have

$$(s,\mu_h,1) + (0,\mu_r,1) = (0,\mu_r,1) + (s,\mu_h,1);$$

hence

$$(s,\mu_{h+r},1) = (\mu_r(s),\mu_{r+h},1),$$

and from this we get that $s+r = r+s$ for all $r \in G$, i.e. $s \in C$. Furthermore, for all $g \in G$ we have that

$$(s,\mu_h,1) + (g,id_G,1) = (g,id_G,1) + (s,\mu_h,1),$$

and so

$$(s+\mu_h(g),\mu_h,1) = (g+s,\mu_h,1).$$

From this, using that $C = Z(G)$ is a group, we get that, for all $g \in G$

$$s+\mu_h(g) = g+s \quad \Longrightarrow \quad s+\mu_h(g) = s+g \quad \Longrightarrow \quad \mu_h(g) = g,$$

so that $\mu_h = id_G$ and $(s,\mu_h,1) \in \{\ (c,id_G,1) \mid c \in C\ \}$. The converse inclusion is obvious. ∎

Lemma 5.3. *E is a special Schreier extension.*

Proof: Given $x \in M$ and $(g,\alpha,x),(h,\beta,x) \in \sigma^{-1}(x)$, we have that there exists $s \in G$ such that $\alpha = \mu_s\beta$ (because α and β both belong to $\Phi(x)$). Hence

$$(g+s-h-s,\mu_s,1) + (h,\beta,x) = (g+s-h-s+\mu_s(h),\mu_s\beta,x) =$$
$$= (g+s-h-s+s+h-s,\alpha,x) = (g,\alpha,x).$$

As for uniqueness, if

$$(g, \mu_s, 1) + (h, \alpha, x) = (h, \alpha, x),$$

then

$$(g + \mu_s(h), \mu_s\alpha, x) = (h, \alpha, x),$$

and so

$$g + \mu_s(h) = h \quad \text{and } \mu_s\alpha = \alpha.$$

From the first equality we get $g+s+h-s = h$, while from the second, using the admissibility of Φ, we obtain that $\mu_s = id_G$ (see Proposition 2.8). Hence $s \in C$, and consequently $g + h = h$. But h is cancellable, so we get $g = 0$, whence $(g, \mu_s, 1) = (0, id_G, 1) = 1_B$. ■

Consider now the abstract kernel Ψ induced by the special Schreier extension E:

$$\begin{tikzcd} K \arrow[r, tail, "\kappa"] & B \arrow[r, two heads, "\sigma"] \arrow[d, "\theta"'] & M \arrow[d, "\Psi"] \\ & End(K) \arrow[r, "p"'] & \frac{End(K)}{Inn(K)}. \end{tikzcd}$$

Given $x \in M$ and choosing $(0, \alpha, x) \in \sigma^{-1}(x)$, we know that Ψ is defined by $\Psi(x) = cl(\theta(0, \alpha, x))$. Given $(g, \mu_h, 1) \in K$, on one hand, since E is a special Schreier extension, we have

$$(0, \alpha, x) + (g, \mu_h, 1) = \theta(0, \alpha, x)(g, \mu_h, 1) + (0, \alpha, x)$$

by construction of θ, and on the other hand

$$(0, \alpha, x) + (g, \mu_h, 1) = (\alpha(g), \alpha\mu_h, x) = (\alpha(g), \mu_{\alpha(h)}\alpha, x) =$$
$$= (\alpha(g), \mu_{\alpha(h)}, 1) + (0, \alpha, x).$$

Thanks to the uniqueness in the Schreier condition, we get that $\theta(0, \alpha, x)(g, \mu_h, 1) = (\alpha(g), \mu_{\alpha(h)}, 1)$.

Lemma 5.4. *The abstract kernel Ψ is admissible.*

Proof: We have to show that, for $(0, \alpha, x) \in B$, $C_K(\theta(0, \alpha, x)(K)) = C$. If $(r, \mu_h, 1) \in C_K(\theta(0, \alpha, x)(K))$ and $g \in G$, then

$$(r, \mu_h, 1) + \theta(0, \alpha, x)(0, \mu_g, 1) = \theta(0, \alpha, x)(0, \mu_g, 1) + (r, \mu_h, 1).$$

Using the previous expression for θ, this is the same as

$$(r, \mu_h, 1) + (0, \mu_{\alpha(g)}, 1) = (0, \mu_{\alpha(g)}, 1) + (r, \mu_h, 1),$$

i.e.

$$(r, \mu_h \mu_{\alpha(g)}, 1) = (\mu_{\alpha(g)}(r), \mu_{\alpha(g)} \mu_h, 1).$$

Hence, for all $g \in G$:

$$r = \mu_{\alpha(g)}(r) \quad \Longrightarrow \quad r = \alpha(g) + r - \alpha(g),$$

which means that $r \in C_G(\alpha(G)) = C$ (because Φ was admissible). Moreover, for all $g \in G$

$$(r, \mu_h, 1) + \theta(0, \alpha, x)(g, \mu_0, 1) = \theta(0, \alpha, x)(g, \mu_0, 1) + (r, \mu_h, 1),$$

which is the same as

$$(r, \mu_h, 1) + (\alpha(g), \mu_0, 1) = (\alpha(g), \mu_0, 1) + (r, \mu_h, 1),$$

i.e.

$$(r + \mu_h(\alpha(g)), \mu_h, 1) = (\alpha(g) + r, \mu_h, 1).$$

From this we get

$$r + \mu_h(\alpha(g)) = \alpha(g) + r = r + \alpha(g),$$

so that

$$h + \alpha(g) - h = \alpha(g).$$

Hence $h \in C_G(\alpha(G)) = C$ since Φ is admissible. Thus $\mu_h = id_G$ and $(r, \mu_h, 1) = (r, id_G, 1) \in C$. ∎

So we conclude that the admissible abstract kernel (M, K, Ψ) is extendable. Furthermore, the action of M on C induced by Ψ is the same as the one induced by (M, G, Φ), indeed:

$$\theta(0, \alpha, x)(c) = \theta(0, \alpha, x)(c, id_G, 1) = \theta(0, \alpha, x)(c, \mu_0, 1) =$$
$$= (\alpha(x)(c), \mu_{\alpha(x)(0)}, 1) = (x \cdot c, id_G, 1) = x \cdot c.$$

Proposition 5.5. *The product $(M, G, \Phi) \otimes (M, G^*, \Phi^*)$ is C-equivalent to (M, K, Ψ).*

Proof: Let us denote by $(M, \frac{G \times G^*}{S}, \Phi')$ the product $(M, G, \Phi) \otimes (M, G^*, \Phi^*)$. In order to show that it is C-equivalent to (M, K, Ψ), we consider the map $\xi \colon \frac{G \times G^*}{S} \to K$ defined by

$$\xi(cl(g, h^*)) = (g + h, \mu_{-h}, 1).$$

This definition is well given. Indeed, if $cl(g_1, h_1^*) = cl(g_2, h_2^*)$, then

$$(g_1, h_1^*) - (g_2, h_2^*) = (c, -c) \qquad \text{for some } c \in C.$$

Hence

$$g_1 - g_2 = c, \quad (-h_2 + h_1)^* = h_1^* - h_2^* = -c = -c^* = (-c)^*,$$

and from this we get

$$g_1 = g_2 + c, \quad h_1 = h_2 - c,$$

whence

$$g_1 + h_1 = g_2 + c + h_2 - c = g_2 + h_2.$$

Moreover, from the equality $-h_1 = c - h_2$ we get that $\mu_{-h_1} = \mu_{-h_2}$, and so

$$(g_1 + h_1, \mu_{-h_1}, 1) = (g_2 + h_2, \mu_{-h_2}, 1).$$

The map ξ is a group homomorphism:

$$\xi(cl(g_1, h_1^*) + cl(g_2, h_2^*)) = \xi(cl(g_1 + g_2, (h_2 + h_1)^*)) =$$

$$= (g_1 + g_2 + h_2 + h_1,\ \mu_{-(h_2+h_1)}, 1) = (g_1 + h_1 + \mu_{-h_1}(g_2 + h_2),\ \mu_{-h_1}\mu_{-h_2}, 1) =$$

$$= (g_1 + h_1, \mu_{-h_1}, 1) + (g_2 + h_2, \mu_{-h_2}, 1) = \xi(cl(g_1, h_1^*)) + \xi(cl(g_2, h_2^*)).$$

The inverse of ξ is the map $\xi^{-1}\colon K \to \frac{G\times G^*}{S}$ defined by

$$\xi^{-1}(g, \mu_h, 1) = cl(g + h, -h^*).$$

It is well defined. Indeed, if $(g, \mu_{h_1}, 1) = (g, \mu_{h_2}, 1)$, then $\mu_{h_1} = \mu_{h_2}$, and so $h_1 - h_2 = c \in C$. We need to check that

$$cl(g + h_1, -h_1^*) = cl(g + h_2, -h_2^*),$$

i.e. that

$$(g + h_1, -h_1^*) - (g + h_2, -h_2^*) \in S.$$

We have that

$$(g + h_1, -h_1^*) - (g + h_2, -h_2^*) = (g + h_1 - (g + h_2), -h_1^* + h_2^*) =$$

$$= (g + h_1 - h_2 - g, (-h_1)^* + h_2^*) = (g + c - g, (h_2 - h_1)^*) =$$

$$= (c, (-c)^*) = (c, -c),$$

and so $cl(g + h_1, -h_1^*) = cl(g + h_2, -h_2^*)$. The maps ξ and ξ^{-1} are inverse to each other:

$$\xi\xi^{-1}(g, \mu_h, 1) = \xi(cl(g + h, (-h)^*)) = (g + h - h,\ \mu_{-(-h)}, 1) = (g, \mu_h, 1),$$

and

$$\xi^{-1}\xi(cl(g, h^*)) = \xi^{-1}(g + h, \mu_{-h}, 1) = cl(g + h - h, -(-h)^*) = cl(g, h^*).$$

Moreover, for all $c \in C$:

$$\xi(c) = \xi(cl(c, 0^*)) = (c + 0, \mu_0, 1) = (c, id_G, 1) = c.$$

To conclude the proof, it remains to show that the following triangle commutes:

$$\begin{tikzcd} M \arrow[r, "\Phi'"] \arrow[dr, "\Psi"'] & \frac{End(\frac{G\times G^*}{S})}{Inn(\frac{G\times G^*}{S})} \arrow[d, "\overline{\xi}"] \\ & \frac{End(K)}{Inn(K)}, \end{tikzcd}$$

where $\overline{\xi}(cl(\alpha)) = cl(\xi\alpha\xi^{-1})$. If $\varphi(x) \in \Phi'(x)$, then

$$\varphi(x)(cl(g, h^*)) = cl(\alpha(x)(g), \alpha^*(x)(h^*)),$$

where $\alpha(x) \in \Phi(x)$ and $\alpha^*(x) \in \Phi^*(x)$ is given by $\alpha^*(x) = (\alpha(x))^*$. Then

$$\xi\varphi(x)\xi^{-1}(g, \mu_h, 1) = \xi\varphi(x)(cl(g + h, -h^*)) =$$

$$= \xi(cl(\alpha(x)(g + h), \alpha^*(x)(-h^*))) = \xi(cl(\alpha(x)(g + h), (\alpha(x)(-h))^*)) =$$

$$= (\alpha(x)(g + h) + \alpha(x)(-h),\ \mu_{-\alpha(x)(-h)}, 1) =$$

$$= (\alpha(x)(g) + \alpha(x)(h) + \alpha(x)(-h),\ \mu_{\alpha(x)(h)}, 1) =$$

$$= (\alpha(x)(g),\ \mu_{\alpha(x)(h)}, 1) = \theta(0, \alpha(x), 1)(g, \mu_h, 1),$$

hence $\xi\varphi(x)\xi^{-1} = \theta(0, \alpha(x), 1) \in \Psi(x)$. ■

We know that the definition of extendability of an admissible abstract kernel is compatible with the C-equivalence (see the paragraph after Definition 4.6). Hence, since the admissible abstract kernel (M, K, Ψ) is extendable, the previous proposition gives

Corollary 5.6. *For any admissible abstract kernel* (M, G, Φ)*, the product* $(M, G, \Phi) \otimes (M, G^*, \Phi^*)$ *is extendable.*

This corollary, Proposition 3.8 and Corollary 4.8, according to the first paragraph of this section, imply the following

Theorem 5.7. *The factor monoid* $\mathcal{A}(M, C) = \frac{\mathcal{M}(M,C)}{\mathcal{L}(M,C)}$ *is an abelian group.*

6. The isomorphism of $\mathcal{A}(M, C)$ with the third cohomology group

The aim of this section is to prove that the abelian group $\mathcal{A}(M, C) = \frac{\mathcal{M}(M,C)}{\mathcal{L}(M,C)}$ described above is isomorphic to the third cohomology group $H^3(M, C)$. In order to do that, the first step is to associate with every admissible abstract kernel $\Phi\colon M \to \frac{End(G)}{Inn(G)}$ an element $Obs(\Phi)$ of $H^3(M, Z(G))$. Due to the admissibility condition, the construction of $Obs(\Phi)$ is, as shown in [23], analogous to the one described in [7] for the case of the classical abstract kernels. A very detailed construction of $Obs(\Phi)$ is given in [12, Section 5], in a slightly different context. Here we just give a brief sketch of the construction, stressing the difference with the one in [12].

Given an admissible abstract kernel $\Phi\colon M \to \frac{End(G)}{Inn(G)}$, we choose a representative $\varphi(x) \in \Phi(x)$ for any $x \in M$, with $\varphi(1) = id_G$. We have that

$$\varphi(x)\varphi(y) = \mu_{f(x,y)}\varphi(xy)$$

for some $f(x, y) \in G$, with $f(x, 1) = f(1, y) = 0$. Now, given $x, y, z \in M$, we have, on one hand

$$\varphi(x)\varphi(y)\varphi(z) = \varphi(x)\mu_{f(y,z)}\varphi(yz) = \mu_{\varphi(x)(f(y,z))}\varphi(x)\varphi(yz) =$$
$$= \mu_{\varphi(x)(f(y,z))}\mu_{f(x,yz)}\varphi(xyz) = \mu_{\varphi(x)(f(y,z))+f(x,yz)}\varphi(xyz),$$

and, on the other hand

$$\varphi(x)\varphi(y)\varphi(z) = \mu_{f(x,y)}\varphi(xy)\varphi(z) = \mu_{f(x,y)}\mu_{f(xy,z)}\varphi(xyz) = \mu_{f(x,y)+f(xy,z)}\varphi(xyz).$$

Comparing the two expressions, and using Corollary 2.9, we get the equality

$$\mu_{\varphi(x)(f(y,z))+f(x,yz)} = \mu_{f(x,y)+f(xy,z)},$$

namely

$$\mu_{\varphi(x)(f(y,z))+f(x,yz)-(f(x,y)+f(xy,z))} = id_G,$$

which tells us that

$$\varphi(x)(f(y, z)) + f(x, yz) - (f(x, y) + f(xy, z)) \in Z(G).$$

This means that there exists a unique element $k(x, y, z) \in Z(G)$ such that

$$\varphi(x)(f(y, z)) + f(x, yz) = k(x, y, z) + f(x, y) + f(xy, z).$$

Clearly, $k(x, y, 1) = k(x, 1, z) = k(1, y, z) = 0$.

It is shown in [23] that the map $k\colon M \times M \times M \to C$ obtained from an admissible abstract kernel Φ as above is a 3-cocycle of the cohomology of M with coefficients in the M-module $Z(G)$, and that the cohomology class of k does not depend on the choices made in the construction. Note that the same conclusion can be drawn from [12, Section 5] using Corollary 2.9 instead of the surjectivity of the homomorphisms $\varphi(x)$, for $x \in M$.

Let us now show that C-equivalent admissible abstract kernels determine cohomologous 3-cocycles. Given two abstract kernels $\Phi\colon M \to \frac{End(G)}{Inn(G)}$ and $\Phi'\colon M \to \frac{End(G')}{Inn(G')}$, suppose that $(M, G, \Phi) \overset{C}{\cong} (M, G', \Phi')$. That is, there exists a group isomorphism $\xi\colon G \to G'$ such that $\xi(c) = c$ for all $c \in C$ and $\xi\varphi(x)\xi^{-1} \in \Phi'(x)$ for all $x \in M$ and all $\varphi(x) \in \Phi(x)$ (see Definition 3.1). Suppose that k is the 3-cocycle associated with (M, G, Φ) as above. If we choose $\varphi'(x) \in \Phi'(x)$ and $f'(x, y) \in G'$ by putting

$$\varphi'(x) = \xi\varphi(x)\xi^{-1} \quad \text{and} \quad f'(x, y) = \xi(f(x, y)),$$

then the 3-cocycle we get from (M, G', Φ') by means of this choice is precisely k. Indeed, for all $x, y, z \in M$, we have that

$$\varphi'(x)\varphi'(y) = \xi\varphi(x)\xi^{-1}\xi\varphi(y)\xi^{-1} = \xi\varphi(x)\varphi(y)\xi^{-1} =$$
$$= \xi\mu_{f(x,y)}\varphi(xy)\xi^{-1} = \mu_{\xi(f(x,y))}\xi\varphi(xy)\xi^{-1} = \mu_{f'(x,y)}\varphi'(xy),$$

and

$$\varphi'(x)(f'(y, z)) + f'(x, yz) = \xi\varphi(x)\xi^{-1}\xi(f(y, z)) + \xi(f(x, yz)) =$$
$$= \xi\varphi(x)(f(y, z)) + \xi(f(x, yz)) = \xi(\varphi(x)(f(y, z)) + f(x, yz)) =$$
$$= \xi(k(x, y, z) + f(x, y) + f(xy, z)) = \xi(k(x, y, z)) + \xi(f(x, y)) + \xi(f(xy, z)) =$$
$$= k(x, y, z) + f'(x, y) + f'(xy, z).$$

Hence we get a well-defined map

$$\zeta\colon \mathcal{M}(M, C) \to H^3(M, C), \quad \zeta([M, G, \Phi]) = Obs(\Phi) = cl(k).$$

Proposition 6.1. *The map $\zeta\colon \mathcal{M}(M, C) \to H^3(M, C)$ is a monoid homomorphism.*

Proof: Let $\zeta([M, G_1, \Phi_1]) = cl(k_1)$ and $\zeta([M, G_2, \Phi_2]) = cl(k_2)$. According to the beginning of this section, there are $\varphi_i(x) \in \Phi_i(x)$ and $f_i(x, y) \in G_i$, $i = 1, 2$, for $x, y \in M$, with $\varphi_i(1) = 1_G$ and $f_i(x, 1) = f_i(1, y) = 0$, such that

$$\varphi_i(x)\varphi_i(y) = \mu_{f_i(x,y)}\varphi_i(xy)$$

and

$$\varphi_i(x)(f_i(y,z)) + f_i(x,yz) = k_i(x,y,z) + f_i(x,y) + f_i(xy,z), \quad i = 1,2,$$

for all $x, y, z \in M$. Let now $\Phi\colon M \to \frac{End(G)}{Inn(G)}$ be the product of Φ_1 and Φ_2. Consider the representatives $\varphi(x) \in \Phi(x)$ defined by

$$\varphi(x)(cl(g_1, g_2)) = cl(\varphi_1(x)(g_1), \varphi_2(x)(g_2))$$

(see Section 3) and the map $f\colon M \times M \to G$ defined by

$$f(x,y) = cl(f_1(x,y), f_2(x,y)).$$

Clearly, $\varphi(1) = 1_G$ and $f(x,1) = f(1,y) = 0$. Furthermore, for all $x, y, z \in M$, we have

$$\begin{gathered}\varphi(x)\varphi(y)(cl(g_1,g_2)) = cl(\varphi_1(x)\varphi_1(y)(g_1),\ \varphi_2(x)\varphi_2(y)(g_2)) = \\ = cl(\mu_{f_1(x,y)}\varphi_1(xy)(g_1),\ \mu_{f_2(x,y)}\varphi_2(xy)(g_2)) = \\ = cl(f_1(x,y), f_2(x,y)) + cl(\varphi_1(xy)(g_1),\ \varphi_2(xy)(g_2)) - cl(f_1(x,y), f_2(x,y)) = \\ = f(x,y) + \varphi(xy)(cl(g_1,g_2)) - f(x,y) = \mu_{f(x,y)}\varphi(xy)(cl(g_1,g_2)),\end{gathered}$$

and

$$\begin{gathered}\varphi(x)(f(y,z)) + f(x,yz) = \varphi(x)(cl(f_1(y,z), f_2(y,z))) + cl(f_1(x,yz), f_2(x,yz)) = \\ = cl(\varphi_1(x)(f_1(y,z)) + f_1(x,yz),\ \varphi_2(x)(f_2(y,z)) + f_2(x,yz)) = \\ = cl(k_1(x,y,z) + f_1(x,y) + f_1(xy,z),\ k_2(x,y,z) + f_2(x,y) + f_2(xy,z)) = \\ = cl(k_1(x,y,z), k_2(x,y,z)) + cl(f_1(x,y), f_2(x,y)) + cl(f_1(xy,z), f_2(xy,z)) = \\ = cl(k_1(x,y,z), 0) + cl(0, k_2(x,y,z)) + f(x,y) + f(xy,z) = \\ = k_1(x,y,z) + k_2(x,y,z) + f(x,y) + f(xy,z)\end{gathered}$$

(recall that $cl(c,0) = cl(0,c) = c$ for all $c \in C$). So, for all $x, y, z \in M$, we get

$$\varphi(x)\varphi(y) = \mu_{f(x,y)}\varphi(xy)$$

and

$$\varphi(x)(f(y,z)) + f(x,yz) = k_1(x,y,z) + k_2(x,y,z) + f(x,y) + f(xy,z).$$

Thus we have

$$\zeta([M,G,\Phi]) = cl(k_1 + k_2) = cl(k_1) + cl(k_2) = \zeta([M,G_1,\Phi_1]) + \zeta([M,G_2,\Phi_2]).$$

■

Proposition 6.2. *The monoid homomorphism $\zeta\colon \mathcal{M}(M,C) \to H^3(M,C)$ is surjective.*

Proof: Let $cl(k) \in H^3(M, C)$. We have to show that there exists an admissible abstract kernel $\Phi\colon M \to \frac{End(G)}{Inn(G)}$ with $Z(G) = C$, inducing the given action on C, and such that $\zeta([M, G, \Phi]) = cl(k)$ (cf. [7, Lemma 9.1]). First consider the case in which the monoid M has at least 3 elements. Let F be the free group on the set of symbols

$$\{[x, y] \mid x, y \in M, x, y \neq 1\}$$

and let G be the direct product $C \times F$. Define a map $f\colon M \times M \to G$ by

$$f(x, y) = [x, y] \quad \text{if } x, y \neq 1 \quad \text{and} \quad f(x, 1) = f(1, y) = 0,$$

where we identify $0 \times F$ with F. Next, we identify $Z(G)$ with C and define an endomorphism $\varphi(x) \in End(G)$ by putting, on the generators of G:

$$\varphi(x)(c) = x \cdot c,$$

where the action of M on C is the given one, and

$$\varphi(x)([y, z]) = k(x, y, z) + f(x, y) + f(xy, z) - f(x, yz). \tag{3}$$

Then, for all $x, y, z, t \in M$, we have

$$\begin{aligned} &\varphi(x)\varphi(y)([z, t]) = \varphi(x)(k(y, z, t) + f(y, z) + f(yz, t) - f(y, zt)) = \\ &= x \cdot k(y, z, t) + k(x, y, z) + f(x, y) + f(xy, z) - f(x, yz) + k(x, yz, t) + f(x, yz) \\ &\quad + f(xyz, t) - f(x, yzt) - (k(x, y, zt) + f(x, y) + f(xy, zt) - f(x, yzt)) = \\ &= f(x, y) + x \cdot k(y, z, t) + k(x, y, z) + k(x, yz, t) - k(x, y, zt) + f(xy, z) - f(x, yz) \\ &\quad + f(x, yz) + f(xyz, t) - f(x, yzt) + f(x, yzt) - f(xy, zt) - f(x, y). \end{aligned}$$

Since k is a 3-cocycle, this last expression is equal to

$$\begin{aligned} &f(x, y) + k(xy, z, t) + f(xy, z) + f(xyz, t) - f(xy, zt) - f(x, y) = \\ &= f(x, y) + \varphi(xy)([z, t]) - f(x, y) = \mu_{f(x,y)}\varphi(xy)([z, t]). \end{aligned}$$

Hence

$$\varphi(x)\varphi(y) = \mu_{f(x,y)}\varphi(xy)$$

for all $x, y \in M$. So we obtain an abstract kernel

$$\Phi\colon M \to \frac{End(G)}{Inn(G)}, \quad \Phi(x) = cl(\varphi(x)).$$

Let us show that Φ is admissible. Suppose that x and y are two distinct non-trivial elements of M. If $xy = 1$ then $x^2 \neq 1$ (since otherwise $x = y$) and $\varphi(x)([x, x])$ does not commute with $\varphi(x)([x, y])$. If $xy \neq 1$ then

$\varphi(x)([x,x])$ and $\varphi(x)([y,x])$ do not commute. Hence $\varphi(x)(C\times F)$ is a non-abelian subgroup of $C\times F$ for all $x\in M$. Next, denoting $C_{C\times F}(\varphi(x)(C\times F))$ by H and using elementary properties of centralizers, we have

$$\varphi(x)(C\times F)\subseteq C_{C\times F}(H)=\bigcap_{(c,u)\in H} C_{C\times F}(c,u)=$$

$$=\bigcap_{(c,u)\in H}(C_C(c)\times C_F(u))=\bigcap_{(c,u)\in H}(C\times C_F(u))=C\times\bigcap_{(c,u)\in H} C_F(u).$$

Now, if we let $H\neq C$, then $\bigcap_{(c,u)\in H} C_F(u)$ is a cyclic subgroup of F (since the centralizer of any non-trivial element of a free group is a cyclic subgroup of that group) and hence $\varphi(x)(C\times F)$ is an abelian subgroup of $C\times F$, a contradiction which shows that $C_{C\times F}(\varphi(x)(C\times F))=C$ for all $x\in M$. So Φ is an admissible abstract kernel, inducing the given action on C. The fact that $\zeta([M,G,\Phi])=cl(k)$ is an immediate consequence of (3).

It remains to consider the cases in which M has less than 3 elements. If M has only one element, then clearly $H^3(M,C)=0$, and so the result is obvious. If M is the two element group, then the abstract kernels involved, as well as the cohomology group $H^3(M,C)$, lie inside groups. Hence one can apply to this case the proof of [7, Lemma 9.1]. If $M=M_2=\{1,x\}$ is the two element monoid that is not a group, then x is an absorbing element. It is known that if a monoid possesses an absorbing element, then all its cohomology groups of order greater than zero are trivial (see e.g. [15]), but for the sake of the reader's convenience, let us check here that $H^3(M_2,C)=0$. Suppose that $k\colon M_2\times M_2\times M_2\to C$ is a 3-cocycle. Then $x\cdot k(x,x,x)=0$. Define a 2-cochain $g\colon M_2\times M_2\to C$ by $g(x,x)=-k(x,x,x)$. Then for the coboundary $\delta_g\colon M_2\times M_2\times M_2\to C$ of g, one has

$$\begin{aligned}\delta_g(x,x,x)&=x\cdot g(x,x)-g(x,x)+g(x,x)-g(x,x)=x\cdot g(x,x)-g(x,x)=\\&=-x\cdot k(x,x,x)+k(x,x,x)=k(x,x,x).\end{aligned}$$

Thus, the cohomology group $H^3(M_2,C)$ vanishes. This clearly implies the result. ■

Proposition 6.3. *For an admissible abstract kernel $\Phi\colon M\to\frac{End(G)}{Inn(G)}$, we have that $\zeta([M,G,\Phi])=0$ if and only if Φ is extendable.*

Proof: Suppose that $\Phi\colon M \to \frac{End(G)}{Inn(G)}$ is extendable. That is, there exists a diagram

$$\begin{tikzcd} E:\ G \arrow[r, tail, "\kappa"] & B \arrow[r, two heads, "\sigma"] \arrow[d, "\theta"'] & M \arrow[d, "\Phi"] \\ & End(G) \arrow[r, "p"'] & \frac{End(G)}{Inn(G)}, \end{tikzcd}$$

where E is a special Schreier extension, the monoid homomorphism θ is defined thanks to the uniqueness in the Schreier condition,

$$b + g = \theta(b)(g) + b \qquad \text{for every } b \in B \quad \text{and every } g \in G,$$

and $\Phi(x) = p\theta(b) = cl(\theta(b))$ for any b such that $\sigma(b) = x$ (see the beginning of Section 4). Let us choose, for every $x \in M$, an element $u_x \in \sigma^{-1}(x)$ with $u_1 = 0$, and denote $\theta(u_x)$ by $\varphi(x)$. Clearly, $\varphi(x) \in \Phi(x)$ and $\varphi(1) = 1_G$. Since E is a special Schreier extension, for all $x, y \in M$, there exists a unique element $f(x, y) \in G$ such that $u_x + u_y = f(x, y) + u_{xy}$. This defines a map $f\colon M \times M \to G$ such that $f(x, 1) = f(1, y) = 0$, and implies

$$\varphi(x)\varphi(y) = \mu_{f(x,y)}\varphi(xy) \qquad \text{for all } x, y \in M.$$

Indeed,

$$\varphi(x)\varphi(y) = \theta(u_x)\theta(u_y) = \theta(u_x + u_y) =$$
$$= \theta(f(x, y) + u_{xy}) = \theta(f(x, y))\theta(u_{xy}) = \mu_{f(x,y)}\varphi(xy)$$

(clearly, $\theta(g) = \mu_g$ for every $g \in G$). Then, thanks to Corollary 2.9, we get, as in the beginning of this section, that

$$\varphi(x)(f(y, z)) + f(x, yz) = k(x, y, z) + f(x, y) + f(xy, z) \qquad \text{for all } x, y, z \in M, \tag{4}$$

where $k\colon M \times M \times M \to C$ is a 3-cocycle. Hence, by definition of ζ, we have $\zeta([M, G, \Phi]) = cl(k)$. Next, on one hand

$$u_x + u_y + u_z = f(x, y) + u_{xy} + u_z = f(x, y) + f(xy, z) + u_{xyz},$$

and, on the other hand

$$u_x + u_y + u_z = u_x + f(y, z) + u_{yz} = \varphi(x)(f(y, z)) + u_x + u_{yz} =$$
$$= \varphi(x)(f(y, z)) + f(x, yz) + u_{xyz},$$

whence

$$\varphi(x)(f(y, z)) + f(x, yz) = f(x, y) + f(xy, z) \qquad \text{for all } x, y, z \in M.$$

Comparing the last equality with (4), we obtain that $k = 0$. Thus $\zeta([M, G, \Phi]) = 0$.

Conversely, suppose that $\zeta([M, G, \Phi]) = 0$. Then there are $\varphi(x) \in \Phi(x)$ and $f(x, y) \in G$ for $x, y \in M$, with $\varphi(1) = 1_G$ and $f(x, 1) = f(1, y) = 0$, such that

$$\varphi(x)\varphi(y) = \mu_{f(x,y)}\varphi(xy)$$

and, in addition, $f(x, y)$ can be chosen so that

$$\varphi(x)(f(y, z)) + f(x, yz) = f(x, y) + f(xy, z) \qquad \text{for all } x, y, z \in M$$

(cf. [12, Proposition 5.6]). Then the set $[G, \varphi, f, M]$ of all pairs $(g, x) \in G \times M$ with the operation defined by

$$(g_1, x) + (g_2, y) = (g_1 + \varphi(x)(g_2) + f(x, y),\ xy)$$

is a monoid, and the sequence

$$G \overset{i}{\rightarrowtail} [G, \varphi, f, M] \overset{p}{\twoheadrightarrow} M, \quad i(g) = (g, 1), \quad p(g, x) = x,$$

is a special Schreier extension of M by G inducing the given admissible abstract kernel $\Phi\colon M \to \frac{End(G)}{Inn(G)}$ $((0, x) + (g, 1) = (\varphi(x)(g), x) = (\varphi(x)(g), 1) + (0, x))$. ■

Now, as an immediate consequence of Propositions 6.2 and 6.3, we have the following

Theorem 6.4. *The map*

$$\zeta'\colon \mathcal{A}(M, C) = \frac{\mathcal{M}(M, C)}{\mathcal{L}(M, C)} \to H^3(M, C), \quad \zeta'(cl([M, G, \Phi])) = \zeta([M, G, \Phi]),$$

is a group isomorphism.

If $M = \Pi$ is a group then $\Phi\colon \Pi \to \frac{End(G)}{Inn(G)}$ factors through $\frac{Aut(G)}{Inn(G)}$ and Theorem 6.4 turns into the classical interpretation of the third cohomology group of Π in terms of the abstract kernels of the form $\Phi\colon \Pi \to \frac{Aut(G)}{Inn(G)}$ [7, Theorem 10.1].

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