Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 21–40

DIFFERENTIAL IDENTITIES AND VARIETIES OF ALMOST POLYNOMIAL GROWTH

FABRIZIO MARTINO AND CARLA RIZZO

Abstract: Let $\mathcal{V}$ be an L-variety of associative L-algebras, i.e., algebras where a Lie algebra L acts on them by derivations, and let $c_n^L(\mathcal{V})$, $n \geq 1$, be its L-codimension sequence. If $\mathcal{V}$ is generated by a finite dimensional L-algebra, then such a sequence is polynomially bounded if and only if $\mathcal{V}$ does not contain UT_2, the 2×2 upper triangular matrix algebra with trivial L-action, and UT_2^ε where L acts on UT_2 as the 1-dimensional Lie algebra spanned by the inner derivation ε induced by e_{11}. In this paper we completely classify all the L-subvarieties of $\mathrm{var}^L(UT_2)$ and $\mathrm{var}^L(UT_2^\varepsilon)$ by giving a complete list of finite dimensional L-algebras generating them.

Keywords: polynomial identity, differential identity, variety of algebras, codimension growth.
Math. Subject Classification (2020): Primary 16R10, 16R50; Secondary 16W25, 16P90.

1. Introduction

Let F be a field of characteristic zero, let $F\langle X\rangle$ be the free associative algebra on a countable set X of variables over F and let A be an associative F-algebra. A polynomial of $F\langle X\rangle$ vanishing under every evaluation in A is called a polynomial identity of A and we denote by $\mathrm{Id}(A)$ the T-ideal of polynomial identities satisfied by A. One of the most challenging problem in the theory of algebras with polynomial identities (PI theory) is to find some numerical invariants allowing us to classify such T-ideals of $F\langle X\rangle$. Since there is a one-to-one correspondence between T-ideals and varieties of algebras, often it is convenient to translate a given issue about T-ideals into the language of varieties of algebras.

If P_n is the space of multilinear polynomials in the variables $x_1, \dots, x_n$, then we set

$$c_n(A) = \dim_F \frac{P_n}{P_n \cap \mathrm{Id}(A)},$$

Received October 25, 2021.
This work was partially supported by the Centre for Mathematics of the University of Coimbra - UIDB/00324/2020, funded by the Portuguese Government through FCT/MCTES.

for all $n \geq 1$, and we call it the codimension sequence of A. If $\mathcal{V}$ is a variety of algebras and $\mathrm{Id}(\mathcal{V})$ is its corresponding T-ideal, then we can similarly define $c_n(\mathcal{V})$. Moreover, if $\mathcal{V} = \mathrm{var}(A)$ is the variety generated by the algebra A, then we refer to the codimension sequence of $\mathcal{V}$ as the one of A. Such a numerical sequence was introduced by Regev in [28] and it measures the rate of growth of the multilinear polynomials lying in the corresponding T-ideal. In the same paper, Regev also showed that if A is an associative algebra satisfying a non-trivial polynomial identity, then $c_n(A)$ is exponentially bounded. Later on, Kemer in [18] and [19] proved several properties about the codimension sequence. On one hand, he showed that $c_n(A)$ is polynomially bounded or grows exponentially, on the other he gave a characterization of the varieties of polynomial growth of the codimension proving that $c_n(A)$ is polynomially bounded if and only if $G, UT_2 \notin \mathrm{var}(A)$, where G is the infinite dimensional Grassmann algebra and UT_2 is the algebra of 2×2 upper triangular matrices. Hence $\mathrm{var}(G)$ and $\mathrm{var}(UT_2)$ are the only varieties of almost polynomial growth, i.e., they grow exponentially but any proper subvariety grows polynomially.

Varieties of poylnomial growth were extensively studied in the past years in various settings. We refer the interested reader to [5], [6], [22] for some results about ordinary algebras; to [8], [23], [24], [32] for superalgebras and more generally group graded algebras; to [3], [7], [10], [20], [21], [25] for algebras with involution, graded involution, superinvolution and pseudoinvolution; to [27] for special Jordan algebras.

In this paper we deal with associative algebras with a Lie algebra action by derivations. If L is such a Lie algebra, then its action can be naturally extended to the action of the universal enveloping algebra $U(L)$ of L and in this case we say that the algebra A is an algebra with derivations or an L-algebra. In this context it is natural to define the differential identities of A, i.e. the polynomials in the variables $x^h = h(x)$, $h \in U(L)$, vanishing on A. In analogy with the ordinary case, one defines the sequence of L-codimensions and studies their asymptotic behaviour. In [13] it was proved that, in case of finite dimensional L-algebras, the sequence of L-codimensions is exponentially bounded or grows polynomially. Moreover, in [9] the authors studied the algebra UT_2^{ε} of 2×2 upper triangular matrices with the action of the 1-dimensional Lie algebra spanned by the inner derivation ε induced by e_{11}. In that paper, they show that such algebra generates an L-variety of almost polynomial growth. Finally, in [31] it was proved that the L-codimension sequence of an L-variety $\mathcal{V}$ generated by a finite dimensional L-algebra is polynomially bounded if and only if $UT_2, UT_2^\varepsilon \notin \mathcal{V}$, where UT_2 stands for the algebra of 2×2 upper triangular matrices and L acts trivially on it.

The main purpose of this paper is to classify, up to PI-equivalence, all the L-subvarieties of $\mathrm{var}^L(UT_2)$ and $\mathrm{var}^L(UT_2^\varepsilon)$ in terms of generators of the corresponding T_L-ideals and to provide a complete list of L-algebras generating such L-subvarieties. Concerning $\mathrm{var}^L(UT_2^\varepsilon)$, the main result is given by Theorem 28 below. We also highlight that if L acts trivially on UT_2, then such a classification coincides to the one of the ordinary case given in [22]. We chose to include it here for sake of completeness.

2. On differential identities

Throughout this paper F will denote a field of characteristic zero and L a Lie algebra over F.

Recall that a derivation of an associative algebra A is a linear map $\delta : A \to A$ such that $(ab)^\delta = a^\delta b + ab^\delta$, for all $a, b \in A$. In particular, an inner derivation induced by $a \in A$ is the derivation ad_a acting on the left on A by $b^{\mathrm{ad}_a} = [a, b] = ab - ba$, for all $b \in A$. Clearly, the set $\mathrm{Der}(A)$ of all derivations of A is a Lie algebra.

Let $U(L)$ be the universal enveloping algebra of L. By the Poincaré–Birkhoff–Witt Theorem, if L has a ordered basis $\{\delta_i \mid i \in I\}$, then $U(L)$ has a basis $\{\delta_{i_1} \cdots \delta_{i_p} \mid i_1 < \cdots < i_p,\ i_k \in I,\ p \geq 0\}$. Thus if A is an associative F-algebra with an L-action by derivations, then this action can be naturally extended to an $U(L)$-action. In this case we call A algebra with derivations or L-algebra.

Let $X = \{x_1, x_2, \dots\}$ be a countable set and $\mathcal{B} = \{d_j \mid j \geq 0\}$ be a basis of $U(L)$. We denote by $F\langle X|L\rangle$ the free associative algebra over F with free formal generators $x_i^{d_j}$, $i > 0$ and $j \geq 0$, where we identify $x_i = x_i^1$, $1 = d_0 \in U(L)$. Notice that $U(L)$ acts on $F\langle X|L\rangle$ by setting

$$(x_{i_1}^{d_{j_1}} x_{i_2}^{d_{j_2}} \dots x_{i_n}^{d_{j_n}})^\delta = x_{i_1}^{\delta d_{j_1}} x_{i_2}^{d_{j_2}} \dots x_{i_n}^{d_{j_n}} + x_{i_1}^{d_{j_1}} x_{i_2}^{\delta d_{j_2}} \dots x_{i_n}^{d_{j_n}} + \cdots + x_{i_1}^{d_{j_1}} x_{i_2}^{d_{j_2}} \dots x_{i_n}^{\delta d_{j_n}},$$

where $\delta \in L$ and $x_{i_1}^{d_{j_1}} x_{i_2}^{d_{j_2}} \dots x_{i_n}^{d_{j_n}} \in F\langle X|L\rangle$. Thus we call $F\langle X|L\rangle$ the free associative algebra with derivations on X over F and we refer to its elements as differential polynomials or L-polynomials.

Let A be an L-algebra over F. Recall that an L-polynomial $f(x_1,\dots,x_n)\in F\langle X|L\rangle$ is a differential polynomial identity of A (or simply an L-identity), and we write $f\equiv 0$, if $f(a_1,\dots,a_n)=0$ for all $a_i\in A$, $1\le i\le n$. We denote by $\mathrm{Id}^L(A)=\{f\in F\langle X|L\rangle \mid f\equiv 0 \text{ on } A\}$ the T_L-ideal of L-identities of A, i.e., $\mathrm{Id}^L(A)$ is an ideal of $F\langle X|L\rangle$ invariant under all endomorphisms φ of $F\langle X|L\rangle$ such that $\varphi(f^h)=\varphi(f)^h$, for all $f\in F\langle X|L\rangle$ and $h\in U(L)$ (see for example [14, 17, 29, 30]).

Let H be a Lie subalgebra of L. If A is an L-algebra, then by restricting the action, A can be regarded as a H-algebra. In this case we can identify the T_L-ideal $\mathrm{Id}^L(A)$ and the T_H-ideal $\mathrm{Id}^H(A)$, i.e., in $\mathrm{Id}^L(A)$ we omit the differential identities $x^\delta\equiv 0$, for all $\delta\in L\backslash H$. Furthermore, any algebra A can be regarded as L-algebra by letting L act on A trivially, i.e., L acts on A as the trivial Lie algebra. Hence the theory of differential identities generalizes the ordinary theory of polynomial identities.

As in the ordinary case, in characteristic zero, every L-identity is equivalent to a system of multilinear ones. We denote by

$$P_n^L=\mathrm{span}\{x_{\sigma(1)}^{d_{i_1}}\dots x_{\sigma(n)}^{d_{i_n}}\mid \sigma\in S_n, d_{i_k}\in\mathcal{B}\}$$

the vector space of multilinear differential polynomials in the variables $x_1,\dots$, x_n, $n\ge 1$. Since $\mathrm{Id}^L(A)$ is generated, as T_L-ideal, by the multilinear L-polynomials it contains, the study of $\mathrm{Id}^L(A)$ is equivalent to the study of $P_n^L\cap \mathrm{Id}^L(A)$ for all $n\ge 1$. In case $U(L)$ acts on A as a suitable finite dimensional subalgbera of the endomorphism ring of A, then P_n^L is finite dimensional and we denote by

$$c_n^L(A)=\dim_F\frac{P_n^L}{P_n^L\cap \mathrm{Id}^L(A)},\quad n\ge 1,$$

the nth differential codimension of A or the nth L-codimension of A. From now on, we will assume that the action of $U(L)$ is always of this type.

Given a variety $\mathcal{V}$ of L-algebras the growth of $\mathcal{V}$ is defined as the growth of the sequence of differential codimensions of any L-algebra A generating $\mathcal{V}$, i.e., $\mathcal{V}=\mathrm{var}^L(A)$. In this case we set $c_n^L(\mathcal{V})=c_n^L(A)$, $n\ge 1$. Then we say that $\mathcal{V}$ has polynomial growth if there exist C,t such that $c_n^L(\mathcal{V})\le Cn^t$ and that $\mathcal{V}$ has almost polynomial growth if $c_n^L(\mathcal{V})$ is not polynomially bounded but every proper subvariety has polynomial growth.

In [31] the authors proved that there exists only two L-varieties generated by finite dimensional L-algebras of almost polynomial growth. Next we are going to describe such varieties.

Let us denote by UT_2 the L-algebra of 2×2 upper triangular matrices over F where L acts trivially on it. Since $x^\delta \equiv 0$, for all $\delta \in L$, is a differential identity of UT_2, we are dealing with ordinary identities. Thus by [19], it follows that the algebra UT_2 generates an L-variety of almost polynomial growth. Moreover, we have the following result (see [26]).

Theorem 1.

1. $Id^L(UT_2) = \langle [x_1, x_2][x_3, x_4]\rangle_{T_L}$;
2. $c_n^L(UT_2) = 2^{n-1}(n-2) + 2$.

Let now denote by UT_2^ε the L-algebra of 2×2 upper triangular matrices over F where L acts on it as the 1-dimensional Lie algebra spanned by the inner derivation ε induced by e_{11}, i.e.

$$(ae_{11} + be_{22} + ce_{12})^\varepsilon = ce_{12},$$

for all $a, b, c \in F$, where e_{ij}'s are the usual matrix units. In [9], the authors proved that UT_2^ε has almost polynomial growth and also they proved the following.

Theorem 2. [9, Theorem 5]

1. $Id^L(UT_2^\varepsilon) = \langle x_1^{\varepsilon^2} - x_1^\varepsilon,\ x_1^\varepsilon x_2^\varepsilon,\ [x_1, x_2]^\varepsilon - [x_1, x_2]\rangle_{T_L}$;
2. $c_n^L(UT_2^\varepsilon) = 2^{n-1}n - 1$.

The above algebras characterize the L-varieties of polynomial growth.

Theorem 3. [31, Theorem 18] *Let A be a finite dimensional L-algebra. Then the sequence $c_n^L(A)$, $n \geq 1$, is polynomially bounded if and only if $UT_2, UT_2^\varepsilon \notin var^L(A)$.*

Recall that given two L-algebras A and B, A is T_L-equivalent to B and we write $A \sim_{T_L} B$ in case $\mathrm{Id}^L(A) = \mathrm{Id}^L(B)$. Thus as a consequence of the above theorem, we have that the algebras UT_2 and UT_2^ε are the only two finite dimensional L-algebras, up to T_L-equivalence, generating L-varieties of almost polynomial growth.

As in the ordinary case, a useful tool when studying L-identities of algebras with 1 is provided by the so-called proper polynomials.

Recall that a left normed commutator of length $n \geq 2$ in the variables x_i's is defined inductively by setting

$$[x_1^{h_1}, \ldots, x_{n-1}^{h_{n-1}}, x_n^{h_n}] = -[x_1^{h_1}, \ldots, x_{n-1}^{h_{n-1}}]^{\mathrm{ad}_{x_n^{h_n}}},$$

where $h_1, \dots h_n \in U(L)$. An L-polynomial $f(x_1, \dots, x_n) \in F\langle X|L\rangle$ is a proper L-polynomial if it is a linear combination of elements of the type

$$x_{i_1}^{h_1} \dots x_{i_k}^{h_k} w_1 \dots w_m$$

where $h_i \in U(L)$, $h_i \neq 1_{U(L)}$, for all $1 \leq i \leq k$, and $w_1, \dots, w_m$ are (eventually empty) left normed Lie commutators in x_i's.

In characteristic zero, if A is an unitary L-algebra, then $\mathrm{Id}^L(A)$ is generated, as T_L-ideal, by the multilinear proper L-polynomials (see [1, Lemma 2.1]). Thus if Γ_n^L denotes the subspace of P_n^L of multilinear proper L-polynomials in $n \geq 1$ variables, and $\Gamma_0^L = \mathrm{span}\{1\}$, then we define the sequence of proper L-codimensions of A as

$$\gamma_n^L(A) = \dim_F \frac{\Gamma_n^L}{\Gamma_n^L \cap \mathrm{Id}^L(A)}, \quad n \geq 0.$$

For unitary L-algebra A, the relation between the L-codimensions and the proper L-codimensions is given by the following

$$c_n^L(A) = \sum_{i=0}^{n} \binom{n}{i} \gamma_i^L(A), \quad n \geq 1.$$

This relation can be proved following closely the proof in the ordinary case in Theorem 4.3.1 of [2].

Next we present a result on proper L-polynomials which will be useful later. Recall that given two sets of L-polynomials $S, S' \subseteq F\langle X|L\rangle$, we say that S' is a consequence of S if $S' \subseteq \langle S\rangle_{T_L}$.

Proposition 4. *Let $i \geq 1$. If k is even, then Γ_{k+i}^L is a consequence of Γ_k^L. Otherwise, Γ_{k+i}^L is a consequence of Γ_k^L plus the polynomial $[x_1, x_2] \cdots [x_k, x_{k+1}]$.*

Proof: We start by proving the statement in case k is even.

Let $w \in \Gamma_{k+i}^L$ be a generator and $i \geq 1$. Suppose first that w is a product of commutators. If w is a product of commutators of length 2, then

$$w = [x_1^{h_1}, x_2^{h_2}] \cdots [x_{k-1}^{h_{k-1}}, x_k^{h_k}] \cdots [x_{k+i-1}^{h_{k+i-1}}, x_{k+i}^{h_{k+i}}],$$

where $h_1, \dots, h_{k+i} \in U(L)$. Thus w is a consequence of

$$[x_1^{h_1}, x_2^{h_2}] \cdots [x_{k-1}^{h_{k-1}}, x_k^{h_k}] \in \Gamma_k^L$$

and we are done. On the other hand, if w contains a commutator u of length $m > 2$, then u can be viewed as a consequence of a commutator of length $< m$. Thus by the above, we get also that w is a consequence of Γ_k^L. Hence we may assume that $w = x_{i_1}^{h_1} \cdots x_{i_t}^{h_t}[\cdots]\cdots[\cdots]$ with $t > 0$ and $h_1, \ldots, h_t \in U(L)$. If $t \leq i$, then by the previous case, w is a consequence of Γ_k^L. Otherwise, if $t > i$, then w is a consequence of the polynomial $x_{i_r}^{h_r} \cdots x_{i_t}^{h_t}[\cdots]\cdots[\cdots]$, where $r = i+1$, and we are done also in this case.

Now suppose that k is odd.

If we prove that Γ_{k+1}^L is a consequence of Γ_k^L and the polynomial $[x_1, x_2]\cdots[x_k, x_{k+1}]$, by the first part of the proof we reach the desired conclusion. Thus let $w \in \Gamma_{k+1}^L$ be a generator. If either w contains a commutator of length greater than 2 or $w = x_{i_1}^{h_1} \cdots x_{i_t}^{h_t}[\cdots]\cdots[\cdots]$ with $t > 0$, $h_1, \ldots, h_t \in U(L)$, we have, as above, that w is a consequence of Γ_k^L. Thus we may assume that w is product of commutators of length 2, i.e.,

$$w = [x_1^{h_1}, x_2^{h_2}] \cdots [x_k^{h_k}, x_{k+1}^{h_{k+1}}],$$

where $h_1, \ldots, h_{k+1} \in U(L)$. If $h_i \in \operatorname{span}_F\{1_{U(L)}\}$ for all $1 \leq i \leq k+1$, then $w = \beta[x_1, x_2]\cdots[x_k, x_{k+1}]$, for some $\beta \in F$, and we are done. Hence we may assume that $h_i \notin \operatorname{span}_F\{1_{U(L)}\}$, for some i. We write

$$\begin{aligned} w =& [x_1^{h_1}, x_2^{h_2}] \cdots [x_{i-3}^{h_{i-3}}, x_{i-2}^{h_{i-2}}] x_i^{h_i} x_{i+1}^{h_{i+1}} [x_{i+1}^{h_{i+1}}, x_{i+2}^{h_{i+2}}][x_k^{h_k}, x_{k+1}^{h_{k+1}}] \\ &- [x_1^{h_1}, x_2^{h_2}] \cdots [x_{i-3}^{h_{i-3}}, x_{i-2}^{h_{i-2}}] x_{i+1}^{h_{i+1}} x_i^{h_i} [x_{i+1}^{h_{i+1}}, x_{i+2}^{h_{i+2}}][x_k^{h_k}, x_{k+1}^{h_{k+1}}]. \end{aligned}$$

Since $yz = [y, z] + zy$ and a commutator of length $m > 2$ is a consequence of a commutator of length $< m$, then

$$\begin{aligned} w \equiv& (x_i^{h_i} x_{i+1}^{h_{i+1}} [x_1^{h_1}, x_2^{h_2}] \cdots [x_{i-3}^{h_{i-3}}, x_{i-2}^{h_{i-2}}][x_{i+1}^{h_{i+1}}, x_{i+2}^{h_{i+2}}][x_k^{h_k}, x_{k+1}^{h_{k+1}}] \\ &- x_{i+1}^{h_{i+1}} x_i^{h_i} [x_1^{h_1}, x_2^{h_2}] \cdots [x_{i-3}^{h_{i-3}}, x_{i-2}^{h_{i-2}}][x_{i+1}^{h_{i+1}}, x_{i+2}^{h_{i+2}}][x_k^{h_k}, x_{k+1}^{h_{k+1}}]) \pmod{\langle \Gamma_k \rangle_{T_L}}. \end{aligned}$$

If $h_{i+1} \notin \operatorname{span}_F\{1_{U(L)}\}$, then w is a consequence of

$$y^h [x_1^{h_1}, x_2^{h_2}] \cdots [x_{i-3}^{h_{i-3}}, x_{i-2}^{h_{i-2}}][x_{i+1}^{h_{i+1}}, x_{i+2}^{h_{i+2}}][x_k^{h_k}, x_{k+1}^{h_{k+1}}] \in \Gamma_k^L$$

and we are done. Hence suppose that $h_{i+1} = 1_{U(L)}$, then

$$w \equiv x_i^{h_i} x_{i+1} [x_1^{h_1}, x_2^{h_2}] \cdots [x_{i-3}^{h_{i-3}}, x_{i-2}^{h_{i-2}}][x_{i+1}^{h_{i+1}}, x_{i+2}^{h_{i+2}}][x_k^{h_k}, x_{k+1}^{h_{k+1}}] \pmod{\langle \Gamma_k \rangle_{T_L}}.$$

Without loss of generality, we may assume that $h_i = \delta_1 \cdots \delta_s$, where $\delta_1, \ldots, \delta_s$ are in L. Let $I = \{i_1, \ldots, i_r\}$ and $J = \{j_1, \ldots, j_t\}$ be two disjoint sets such that $I \cup J = \{1, \ldots, s\}$, $i_1 < \cdots < i_r$ and $j_1 < \cdots < j_t$, respectively. We set $c_I = \delta_{i_1} \cdots \delta_{i_r}$ and $c_J = \delta_{j_1} \cdots \delta_{j_t}$, then by definition of derivation, we have the following

$$x_i^{h_i}x_{i+1} = (x_ix_{i+1})^{h_i} - x_ix_{i+1}^{h_i} - \sum_{I,J} x_i^{c_I}x_{i+1}^{c_J}.$$

Since $c_I, c_J \in U(L)$ for all I, J, it follows that w is a consequence of

$$y^h[x_1^{h_1}, x_2^{h_2}]\cdots[x_{i-3}^{h_{i-3}}, x_{i-2}^{h_{i-2}}][x_{i+1}^{h_{i+1}}, x_{i+2}^{h_{i+2}}][x_k^{h_k}, x_{k+1}^{h_{k+1}}] \in \Gamma_k^L$$

and this completes the proof. ∎

As a consequence we have the following.

Corollary 5. *Let A be an L-algebra with 1. If $\gamma_k^L(A) = 0$ for some $k \geq 1$, then $\gamma_m^L(A) = 0$ for all $m \geq k$.*

Remark that these results are general facts not related to the L-identities of UT_2.

One of the main tool in the study of T_L-ideals is the representation theory of the symmetric group S_n. In fact, the natural left S_n-action $\sigma(x_i^h) = x_{\sigma(i)}^h$ turns P_n^L into a S_n-module and therefore the space $P_n^L(A) = P_n^L/(P_n^L \cap \mathrm{Id}^L(A))$ becomes a left S_n-module. Similarly $\Gamma_n^L(A) = \Gamma_n^L/(\Gamma_n^L \cap \mathrm{Id}^L(A))$ is an S_n-module under the induced action. We denote by $\chi_n^L(A)$ and $\psi_n^L(A)$ the S_n-characters of $P_n^L(A)$ and $\Gamma_n^L(A)$, respectively, and we refer to them as the nth L-cocharacter and the nth proper L-cocharacter of A. Since $\operatorname{char} F = 0$, by complete reducibility we can write

$$\chi_n^L(A) = \sum_{\lambda\vdash n} m_\lambda\chi_\lambda, \quad \psi_n^L(A) = \sum_{\lambda\vdash n} m'_\lambda\chi_\lambda,$$

where λ is a partition of n, χ_λ is the irreducible S_n-character associated to λ, and $m_\lambda, m'_\lambda \geq 0$ are the corresponding multiplicities. It is clear that by knowing the decomposition of the (proper) cocharacter of A, one can get informations about the corresponding (proper) codimensions.

3. Constructing L-subvarieties in $\mathrm{var}^L(UT_2^\varepsilon)$

The main goal of this section is to construct some suitable finite dimensional L-algebras belonging to the L-variety generated by UT_2^ε whose L-codimension sequence grows polynomially.

For all $k \geq 2$ let

$$A_k^\varepsilon = \mathrm{span}_F\Big\{e_{11}, E, E^2, \dots, E^{k-2}, e_{12}, e_{13}, \dots, e_{1k}\Big\}$$

be the subalgebra of $UT_k(F)$ where $E = \sum_{i=2}^{k-1} e_{i,i+1}$ and L acts on A_k^ε as the 1-dimensional Lie algebra spanned by the derivation $\varepsilon = \mathrm{ad}_{e_{11}}$. Remark that $e_{11}^\varepsilon = (E^j)^\varepsilon = 0$, for all $1 \leq j \leq k-2$, and $e_{1i}^\varepsilon = e_{1i}$, for all $2 \leq i \leq k$.

We also denote by $(A_k^\varepsilon)^*$ the subalgebra of $UT_k(F)$ obtained by flipping A_k^ε along its secondary diagonal. Hence

$$(A_k^\varepsilon)^* = \mathrm{span}_F\Big\{e_{kk}, E, E^2, \ldots, E^{k-2}, e_{1k}, e_{2k}, \ldots, e_{k-1,k}\Big\}.$$

In this case, L acts on $(A_k^\varepsilon)^*$ as the 1-dimensional Lie algebra spanned by the derivation $\varepsilon = \mathrm{ad}_{e_{kk}}$. Notice that one can determine the L-polynomial identities of such an L-algebra via the ones of A_k^ε. In fact, if $f \in F\langle X|L\rangle$ and f^* is the L-polynomial obtained by reversing the order of the variables in each monomial of f, then one can easily check that $f \in \mathrm{Id}^L(A_k^\varepsilon)$ if and only if $f^* \in \mathrm{Id}^L((A_k^\varepsilon)^*)$. Notice that such kind of algebras was first studied in the ordinary case in [5]

In what follows, we explicitly describe the L-identities of A_k^ε and $(A_k^\varepsilon)^*$ for any $k \geq 2$.

Lemma 6 ([30], Theorem 3)**.** *Let $k = 2$, then:*

1. *$Id^L(A_2^\varepsilon) = \langle x_1^{\varepsilon^2} - x_1^\varepsilon,\ x_1^\varepsilon x_2,\ x_1 x_2^\varepsilon - x_2 x_1^\varepsilon - [x_1, x_2]\rangle_{T_L}$;*
2. *$Id^L((A_2^\varepsilon)^*) = \langle x_1^{\varepsilon^2} - x_1^\varepsilon,\ x_1 x_2^\varepsilon,\ x_1^\varepsilon x_2 - x_2^\varepsilon x_1 - [x_1, x_2]\rangle_{T_L}$;*
3. *$c_n^L(A_2^\varepsilon) = c_n^L((A_2^\varepsilon)^*) = n + 1$.*

Lemma 7. *Let $k \geq 3$, then:*

1. *$Id^L(A_k^\varepsilon) = \langle x_1^{\varepsilon^2} - x_1^\varepsilon,\ x_1^\varepsilon x_2^\varepsilon,\ [x_1, x_2]^\varepsilon - [x_1, x_2],\ x_1^\varepsilon x_2 \cdots x_k\rangle_{T_L}$;*
2. *$c_n^L(A_k^\varepsilon) = 2+n+\sum_{l=0}^{k-2}\binom{n}{l}(n-l+1)+\sum_{l=1}^{k-2}\sum_{j=2}^{n-l+1}\binom{n-j}{l-1}(j-1) \approx qn^{k-1}$, for some $q > 0$.*

Hence $Id^L((A_k^\varepsilon)^*) = \langle x_1^{\varepsilon^2} - x_1^\varepsilon,\ x_1^\varepsilon x_2^\varepsilon,\ [x_1, x_2]^\varepsilon - [x_1, x_2],\ x_1 \cdots x_{k-1} x_k^\varepsilon\rangle_{T_L}$ and $c_n^L((A_k^\varepsilon)^*) = c_n^L(A_k^\varepsilon) \approx qn^{k-1}$.

Proof: Write $I = \langle x_1^{\varepsilon^2} - x_1^\varepsilon,\ x_1^\varepsilon x_2^\varepsilon,\ [x_1, x_2]^\varepsilon - [x_1, x_2],\ x_1^\varepsilon x_2 \cdots x_k\rangle_{T_L}$. It is clear that $I \subseteq \mathrm{Id}^L(A_k^\varepsilon)$. In order to prove the opposite inclusion, first we find a set of generators of P_n^L modulo $P_n^L \cap I$, for all $n \geq 1$.

Let $f \in P_n^L$ be a multilinear L-polynomial of degree n. Because of the L-identities $x_1^{\varepsilon^2} - x_1^\varepsilon \equiv 0$ and $x_1^\varepsilon x_2^\varepsilon \equiv 0$, in each monomial of f can occur at most one differential variable x_j^ε. Moreover, $[x_1, x_2]x_3^\varepsilon \equiv 0$ and $x_3^\varepsilon[x_1, x_2] \equiv 0$

are a consequences of $x_1^\varepsilon x_2^\varepsilon \equiv 0$ and $[x_1,x_2]^\varepsilon - [x_1,x_2] \equiv 0$. Furthermore, from $[x_1,x_2]^\varepsilon - [x_1,x_2] \equiv 0$ and $x_1^\varepsilon x_2 \cdots x_k \equiv 0$, it follows also $[x_1,x_2]x_3 \cdots x_{k+1} \equiv 0$. Finally, since $[x_1,x_2][x_3,x_4] \equiv 0$ is a consequence of $x_1^\varepsilon x_2^\varepsilon \equiv 0$ and $[x_1,x_2]^\varepsilon - [x_1,x_2] \equiv 0$, every left normed commutator $[x_{j_1},\dots,x_{j_t}]$ can be written as a linear combination of $[x_{i_1},\dots,x_{i_t}]$ where $i_1 > i_2 < \dots < i_t$ (see for instance [2, Theorem 5.2.1]).

By taking into account the previous remarks plus the Poincaré-Birkhoff-Witt theorem, modulo I, f is a linear combination of L-polynomials of the type

$$\begin{aligned} &x_1 \cdots x_n, \quad x_{i_1} \cdots x_{i_t}[x_i,x_j]x_{j_1} \cdots x_{j_l}, \\ &x_2 \cdots x_n x_1^\varepsilon, \quad x_{p_1} \cdots x_{p_r} x_m^\varepsilon x_{q_1} \cdots x_{q_s}, \end{aligned} \tag{1}$$

where $t+l = n-2$, $r+s = n-1$, $l < k-1$, $s < k-1$, $i > j < i_1 < \dots < i_t$, $j_1 < \dots < j_l$, $m < p_1 < \dots < p_r$ and $q_1 < \dots < q_s$. It follows that the space P_n^L is generated modulo $P_n^L \cap I$ by the above polynomials.

We next show that they are linearly independent modulo $\mathrm{Id}^L(A_k^\varepsilon)$. To that end, let $f \in \mathrm{Id}^L(A_k^\varepsilon)$ be a linear combination of such polynomials and write

$$\begin{aligned} f = \alpha x_1 \cdots x_n + \beta x_2 \cdots x_n x_1^\varepsilon + \sum_{l<k-1} \sum_{I,J} \alpha_{I,J} x_{i_1} \cdots x_{i_t}[x_i,x_j]x_{j_1} \cdots x_{j_l} + \\ + \sum_{s<k-1} \sum_{P,Q} \beta_{P,Q} x_{p_1} \cdots x_{p_r} x_m^\varepsilon x_{q_1} \cdots x_{q_s}, \end{aligned}$$

where $I = \{i, j, i_1, \dots, i_t\}$, $J = \{j_1, \dots, j_l\}$, $P = \{m, p_1, \dots, p_r\}$ and $Q = \{q_1, \dots, q_s\}$ are disjoint sets of indices subjected to the above conditions.

First suppose that $\alpha \neq 0$, then by making the evaluation $x_1 = \cdots = x_n = e_{11}$ one gets $\alpha = 0$, a contradiction.

Suppose that there exists $\alpha_{I,J} \neq 0$ for some $l < k-1$, I and J. Then by making the evaluation $x_i = e_{12}$, $x_j = x_{i_1} = \cdots = x_{i_t} = e_{11}$ and $x_{j_1} = \cdots = x_{j_l} = E$, we get $\alpha_{I,J} = 0$, a contradiction.

Now suppose that $\beta \neq 0$, then if one considers the evaluation $x_1 = e_{12}$ and $x_2 = \cdots = x_n = e_{11}$, we get $\beta = 0$, a contradiction.

Finally, if $\beta_{P,Q} \neq 0$ for some $s < k-1$, P and Q, then let $x_m = e_{12}$, $x_{p_1} = \cdots = x_{p_r} = e_{11}$ and $x_{q_1} = \cdots = x_{q_s} = E$, obtaining $\beta_{P,Q} = 0$, a contradiction.

Therefore the elements in (1) are linearly independent modulo $P_n^L \cap \mathrm{Id}^L(A_k^\varepsilon)$ and, since $P_n^L \cap \mathrm{Id}^L(A_k^\varepsilon) \supseteq P_n^L \cap I$, they form a basis of P_n^L modulo $P_n^L \cap \mathrm{Id}^L(A_k^\varepsilon)$ and $\mathrm{Id}^L(A_k^\varepsilon) = I$.

Thus, by counting we get

$$c_n^L(A_k^\varepsilon) = 2 + n + \sum_{l=0}^{k-2}\binom{n}{l}(n-l+1) + \sum_{l=1}^{k-2}\sum_{j=2}^{n-l+1}\binom{n-j}{l-1}(j-1) \approx qn^{k-1},$$

for some $q > 0$ and we are done.

Notice that, from the previous results, it follows also that $\mathrm{Id}^L((A_k^\varepsilon)^*) = \langle x_1^{\varepsilon^2} - x_1^\varepsilon,\ x_1^\varepsilon x_2^\varepsilon,\ [x_1,x_2]^\varepsilon - [x_1,x_2],\ x_1\cdots x_{k-1}x_k^\varepsilon\rangle_{T_L}$ and $c_n^L((A_k^\varepsilon)^*) = c_n^L(A_k^\varepsilon) \approx qn^{k-1}$. ■

We now introduce, for any fixed $k \geq 2$, a unitary L-algebra in $\mathrm{var}^L(UT_2^\varepsilon)$ which codimension sequence grows as n^{k-1}.

To this end, for all $k \geq 2$, let

$$N_k^\varepsilon = \mathrm{span}_F\Big\{I, E, E^2, \dots, E^{k-2}, e_{12}, e_{13}, \dots, e_{1k}\Big\}$$

where I is the identity $k \times k$ matrix and L acts on N_k^ε as the 1-dimensional Lie algebra spanned by the derivation $\varepsilon = \mathrm{ad}_{e_{11}}$. In this case $I^\varepsilon = (E^j)^\varepsilon = 0$, for all $1 \leq j \leq k-1$, and $e_{1i}^\varepsilon = e_{1i}$, for all $2 \leq i \leq k$.

Lemma 8. *Let $k \geq 2$, then:*

1. *$Id^L(N_k^\varepsilon) = \langle x_1^{\varepsilon^2} - x_1^\varepsilon,\ x_1^\varepsilon x_2^\varepsilon,\ [x_1,x_2]^\varepsilon - [x_1,x_2],\ [x_1,\dots,x_k]\rangle_{T_L}$;*
2. *$c_n^L(N_k^\varepsilon) = 1 + \sum_{j=1}^{k-1}\binom{n}{j}j \approx qn^{k-1}$, for some $q > 0$.*

Proof: Let $Q = \langle x_1^{\varepsilon^2} - x_1^\varepsilon,\ x_1^\varepsilon x_2^\varepsilon,\ [x_1,x_2]^\varepsilon - [x_1,x_2],\ [x_1,\dots,x_k]\rangle_{T_L}$. It is easily proved that $Q \subseteq \mathrm{Id}^L(N_k^\varepsilon)$.

Let now f be an L-identity of N_k^ε. We may assume that f is multilinear and since N_k^ε is an unitary algebra, we may take f proper.

After reducing f modulo Q, we get that f is the zero polynomial if $\deg f \geq k$ and it is a linear combination of commutators

$$[x_1^\varepsilon, x_2, \dots, x_n] \quad [x_i, x_1, \dots, \widehat{x}_i, \dots, x_n]$$

if $\deg f < k$, where $2 \leq i \leq n$ and the symbol $\widehat{x}_i$ means that the variable x_i is omitted.

Hence, modulo Q,

$$f = \alpha[x_1^\varepsilon, x_2, \dots, x_n] + \sum_{i=2}^{n}\beta_i[x_i, x_1, \dots, \widehat{x}_i, \dots, x_n],$$

where $n \le k-1$. We claim that such commutators are linearly independent modulo $\mathrm{Id}^L(N_k^\varepsilon)$, i.e. f is the zero polynomial modulo $\mathrm{Id}^L(N_k^\varepsilon)$ and this will imply that $Q = \mathrm{Id}^L(N_k^\varepsilon)$, as required.

Suppose that $\beta_i \neq 0$ for some i, then we consider the evaluation $x_i = e_{12}$, $x_j = E$ for all $j \neq i$ and we get $\beta_i = 0$, a contradiction. Now, if $\alpha \neq 0$, then we make the evaluation $x_1 = \cdots = x_n = E$ and we get $\alpha = 0$, a contradiction. This says that $f \in Q$ and so $Q = \mathrm{Id}^L(N_k^\varepsilon)$ as claimed.

The arguments above also prove that

$$\gamma_j^L(N_k^\varepsilon) = \begin{cases} j, & \text{if } j \le k-1 \\ 0, & \text{if } j \ge k \end{cases} .$$

Hence we also get that

$$c_n^L(N_k^\varepsilon) = 1 + \sum_{j=1}^{k-1} \binom{n}{j} \gamma_j^L(N_k^\varepsilon) = 1 + \sum_{j=1}^{k-1} \binom{n}{j} j \approx q n^{k-1},$$

for some $q > 0$. ∎

We want to highlight that the case $k = 2$ was already studied in [30, Theorem 1]. Moreover, it is clear that if $k = 1$, then $N_1^\varepsilon = F$, $\mathrm{Id}^L(N_1^\varepsilon) = \langle [x_1, x_2],\ x_1^\varepsilon \rangle_{T_L}$ and $c_n^L(N_1^\varepsilon) = 1$ for all $n \ge 1$.

4. On the structure of algebras generating L-subvarieties of $\mathrm{var}^L(UT_2^\varepsilon)$

In this section we shall study the structure of L-algebras belonging to the L-variety generated by UT_2^ε.

Notice that in what follows we may assume, without loss of generality, that L is a 1-dimensional Lie algebra spanned by ε.

We start by proving that any L-algebra inside $\mathrm{var}^L(UT_2^\varepsilon)$ satisfies the same L-identities of a finite dimensional L-algebra.

Theorem 9. *If $A \in var^L(UT_2^\varepsilon)$ is a finitely generated L-algebra over an algebraically closed field F of characteristic zero, then A is T_L-equivalent to a finite dimensional L-algebra over F.*

Proof: If $A \in \mathrm{var}^L(UT_2^\varepsilon)$, then by Theorem 2, $x^{\varepsilon^2} - x^\varepsilon \in \mathrm{Id}^L(A)$. Hence $U(L)$ acts on A as the 2-dimensional semisimple Hopf algebra H with basis $\{1_H, \bar\varepsilon\}$ where $\bar\varepsilon^2 = \bar\varepsilon$. Thus A can be regarded as an algebra with H-action and we may restrict the T_L-ideal $\mathrm{Id}^L(A)$ to the T_H-ideal $\mathrm{Id}^H(A)$. Thus the claim follows from [16, Theorem 1.1]. ■

We refer the reader to [13, 16] for an account on algebras with an Hopf algebra action and the related theory of polynomial identities.

Now we recall the following result characterizing the nth L-cocharacter of UT_2^ε.

Theorem 10 ([9], Theorem 12)**.** *If $\chi_n^L(UT_2^\varepsilon) = \sum_{\lambda\vdash n} m_\lambda \chi_\lambda$ is the nth cocharacter of UT_2^ε, then*

$$m_\lambda = \begin{cases} n+1, & \lambda = (n) \\ 2(q+1), & \lambda = (p+q,p) \\ q+1, & \lambda = (p+q,p,1) \\ 0, & \textit{otherwise} \end{cases},$$

where $p, q \geq 0$.

In order to characterize the L-subvariety of $\mathrm{var}^L(UT_2^\varepsilon)$ we are going to prove the following.

Theorem 11. *If $A \in var^L(UT_2^\varepsilon)$, then A is T_L-equivalent to a finitely generated L-algebra.*

Proof: Let B be the relatively free algebra of $\mathrm{var}^L(A)$ with 3 generators. We claim that $\mathrm{Id}^L(A) = \mathrm{Id}^L(B)$. Clearly $\mathrm{Id}^L(A) \subseteq \mathrm{Id}^L(B)$, thus we shall prove the opposite inclusion.

Let $f \in \mathrm{Id}^L(B)$ be a multilinear polynomial of degree n and let M be the S_n-module generated by f. Without loss of generality, we may assume that M is irreducible. In fact, if $M = M_1 \oplus \cdots \oplus M_k$ is the decomposition into irreducible components, where M_i is generated by f_i as S_n-module, $1 \leq i \leq k$, then $f_i \in \mathrm{Id}^L(A)$ for all i implies that also $f \in \mathrm{Id}^L(A)$.

Let χ_λ be the irreducible character of M, where $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$, and let

$$e_{T_\lambda} = \sum_{\substack{\tau \in R_{T_\lambda} \\ \sigma \in C_{T_\lambda}}} (\operatorname{sgn} \sigma)\tau\sigma$$

be the corresponding essential idempotent (see for instance [12, Chapter 2]). Here recall that R_{T_λ} and C_{T_λ} stand for the row-stabilizer and the column-stabilizer of the Young tableau T_λ, respectively.

If $\lambda_4 \neq 0$ or $\lambda_3 > 1$, then by Theorem 10, it follows that $f \in \mathrm{Id}^L(A)$. Thus we may assume that $\lambda_4 = 0$ and $\lambda_3 \leq 1$.

Let now consider $g = \left(\sum_{\tau \in R_{T_\lambda}} \tau\right) f$ and notice that g is symmetric in at most two disjoint subsets X_1, X_2 of differential variables. If we identify all the variables of X_1 with x_1 and all the variables of X_2 with x_2 in g, we obtain the homogeneous polynomial $p = p(x_1, x_2, x_3)$ which is still an L-identity of B. But from the definition of relatively free algebra, it follows that $p \in \mathrm{Id}^L(A)$. By multilinearizing the polynomial p, we get the polynomial $\lambda_1!\lambda_2!g(x_1, \ldots x_n)$. Hence $g \in \mathrm{Id}^L(A)$ and, since M is irreducible and $g \neq 0$, it follows that also $f \in \mathrm{Id}^L(A)$. This completes the proof. ■

As a consequence of Theorems 9 and 11 we have the following.

Corollary 12. *If $A \in var^L(UT_2^\varepsilon)$ is an L-algebra over an algebraically closed field F of characteristic zero, then $var^L(A) = var^L(B)$ for some finite dimensional L-algebra B over F.*

According to Corollary 12, from now on we will always assume, without loss of generality, that if $\mathcal{V} \subseteq \mathrm{var}^L(UT_2^\varepsilon)$, then $\mathcal{V} = \mathrm{var}^L(A)$ where A is a finite dimensional L-algebra.

Now we are going to describe the structure of such finite dimensional L-algebras belonging to $\mathrm{var}^L(UT_2^\varepsilon)$.

First we recall some definitions. A subalgebra (ideal) B of A is an L-subalgebra (ideal) if it is a subalgebra (ideal) such that $B^L \subseteq B$, where B^L denotes the set of all $h(b)$, for all $b \in B$ and $h \in U(L)$.

Let A be a finite dimensional L-algebras over an algebraically closed field. By the Wedderburn-Malcev Theorem for associative algebras, we can write

$$A = B + J \tag{2}$$

where B is a maximal semisimple unitary subalgebra of A and $J = J(A)$ is its Jacobson radical. Notice that although J is an L-invariant ideal of A (see [15]), it may does not exist an L-invariant Wedderburn-Malcev decomposition, i.e., it may happen that all semisimple subalgebras B of A that satisfy (2) are not L-subalgebras of A. For example, the algebra UT_2^δ of 2×2 upper triangular matrices where L acts as the 1-dimensional Lie algebra spanned by the inner derivation δ induced by e_{12} has no L-invariant Wedderburn-Malcev decomposition (see [31, Example 2]). Things are different inside $\mathrm{var}^L(UT_2^\varepsilon)$, in fact at the end of the section, we will prove that, up to T_L-equivalence, we can always assume that a subvariety of $\mathrm{var}^L(UT_2^\varepsilon)$ is generated by an L-algebra with an L-invariant Wedderburn–Malcev decomposition.

To this end, first recall that J can be decompose into direct sum of B-bimodules

$$J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11}$$

where for $i, k \in \{0, 1\}$, J_{ik} is a left faithful module or a 0-left module according as $i = 1$ or $i = 0$, respectively. Similarly, J_{ik} is a right faithful module or a 0-right module according as $k = 1$ or $k = 0$, respectively. Moreover, for $i, k, l, m \in \{0, 1\}$, $J_{ik}J_{lm} \subseteq \delta_{k,l}J_{im}$ where $\delta_{k,l}$ is the Kronecker delta and $J_{11} = BN$ for some nilpotent subalgebra N of A commuting with B. For a proof of this result see [11, Lemma 2].

Proposition 13. *Let $A = B + J$ be a finite dimensional L-algebra. If $\delta \in \mathrm{Der}(A)$, then $1_B^\delta \in J_{01} + J_{10}$. Moreover, $J_{00}^\delta \subseteq J_{00} + J_{01} + J_{10}$, $J_{01}^\delta \subseteq J_{00} + J_{01} + J_{11}$, $J_{10}^\delta \subseteq J_{00} + J_{10} + J_{11}$ and $J_{11}^\delta \subseteq J_{01} + J_{10} + J_{11}$.*

Proof: Since $\delta \in \mathrm{Der}(A)$, by [15, Theorem 4.3] $\delta = \mathrm{ad}_b + \mathrm{ad}_j + \delta'$, where $b \in B$, $j \in J$ and $\delta' \in \mathrm{Der}(A)$ is such that $B^{\delta'} = 0$. Thus since $[J_{11}, 1_B] = J_{00}1_B = 1_BJ_{00} = 0$, we get $1_B^\delta = 1_B^{\mathrm{ad}_j} \in J_{01} + J_{10}$.

Let $j_{00} \in J_{00}$. Since $1_BJ_{00} = 0$, we get $0 = 1_B^\delta j_{00} + 1_Bj_{00}^\delta$ and so it follows that $1_Bj_{00}^\delta \in J_{10}$. On the other hand, since $J_{00}1_B = 0$, we have $0 = j_{00}^\delta 1_B + j_{00}1_B^\delta$. Then $j_{00}^\delta 1_B \in J_{01}$. Thus it follows that $J_{00}^\delta \subseteq J_{00} + J_{01} + J_{10}$.

Let now $j_{11} \in J_{11}$. Then $j_{11}^\delta = (j_{11}1_B)^\delta = j_{11}^\delta 1_B + j_{11}1_B^\delta \in J_{01} + J_{10} + J_{11}$. Thus we get $J_{11}^\delta \subseteq J_{01} + J_{10} + J_{11}$. Similarly it can be proved for J_{01} and J_{10}. ■

In case of algebras belonging to $\mathrm{var}^L(UT_2^\varepsilon)$, the action of L on J and its components can be assumed to be much more simpler.

Lemma 14. *If $A = B + J \in var^L(UT_2^\varepsilon)$ with $J = J_{00} + J_{10} + J_{01} + J_{11}$, then $j^\varepsilon = j$ for all $j \in J_{01} \cup J_{10}$.*

Proof: If $j \in J_{01}$ (resp. $j \in J_{10}$), then $j = [j, 1_B]$ (resp. $j = [1_B, j]$). Thus the claim follows since $[x_1, x_2]^\varepsilon - [x_1, x_2] \in \mathrm{Id}^L(A)$. ■

Lemma 15. *Let $A = B + J \in var^L(UT_2^\varepsilon)$. Then $J_{00}^\varepsilon J_{01} = J_{10}J_{00}^\varepsilon = J_{11}^\varepsilon J_{10} = J_{01}J_{11}^\varepsilon = J_{01}J_{10} = J_{10}J_{01} = J_{01}[J_{11}, J_{11}] = [J_{11}, J_{11}]J_{10} = [J_{00}, J_{00}]J_{01} = J_{10}[J_{00}, J_{00}] = 0$.*

Proof: Since $[x_1, x_2]^\varepsilon - [x_1, x_2] \equiv 0$ and $x_1^\varepsilon x_2^\varepsilon \equiv 0$ on $\mathrm{var}^L(UT_2^\varepsilon)$, the result immediately follows applying Lemma 14. ■

Theorem 16. *If $A = B + J \in var^L(UT_2^\varepsilon)$ then, up to T_L-equivalence, $B^L \subseteq B$.*

Proof: If $J = 0$ there is nothing to prove, so let $J \neq 0$ and since $A \in \mathrm{var}^L(UT_2^\varepsilon)$ then $B = F \oplus \cdots \oplus F$. By Proposition 13 it readily follows that if either $1_B^\varepsilon = 0$ or $J_{01} = J_{10} = 0$, then $B^\varepsilon \subseteq B$ and we are done also in this case.

So, let suppose $1_B^\varepsilon = j \neq 0$, where $j \in J_{01} + J_{10}$, and let consider $\varepsilon' = \varepsilon - \mathrm{ad}_j \in \mathrm{Der}(A)$. Remark that $\varepsilon' \neq 0$ in fact if $\varepsilon' = 0$, then $\varepsilon = \mathrm{ad}_j$ and $1_B^{\varepsilon^2} = j^\varepsilon = 0$, since $J_{01}^2 = J_{10}^2 = 0$ and, by Lemma 15, $J_{01}J_{10} = J_{10}J_{01} = 0$. This is a contradiction since $x^{\varepsilon^2} - x^\varepsilon \in \mathrm{Id}^L(A)$.

Let now $A_{\varepsilon'}$ be the L-algebra A where L acts on it as the 1-dimensional Lie algebra spanned by ε'. Clearly $1_B^{\varepsilon'} = 0$, $B^{\varepsilon'} \subseteq B$ and a straightforward computation can also prove that $A_{\varepsilon'} \in \mathrm{var}^L(UT_2^\varepsilon)$. We claim that $\mathrm{Id}^L(A) = \mathrm{Id}^L(A_{\varepsilon'})$ and this will complete the proof.

Let $f \in \mathrm{Id}^L(A)$ be a multilinear polynomial of degree n. According to [9, Theorem 5] we can write f as

$$f = \alpha x_1 \dots x_n + \sum_{k=1}^{n} \beta_k x_{i_1} \dots x_{i_{n-1}} x_k^\varepsilon + \sum_{P,t} \gamma_{P,t} x_{p_1} \dots x_{p_m} [x_t^\varepsilon, x_{j_1}, \dots, x_{j_{n-m-1}}] + g,$$

where $g \in \mathrm{Id}^L(UT_2^\varepsilon) \subseteq \mathrm{Id}^L(A)$, $i_1 < \cdots < i_{n-1}$, $p_1 < \cdots < p_m$ and $j_1 < \cdots < j_{n-m-1}$. Notice that if we make the evaluation $x_1 = \cdots = x_n = 1_B$, we get $\alpha = 0$. Thus

$$f = \sum_{k=1}^{n} \beta_k x_{i_1} \dots x_{i_{n-1}} x_k^\varepsilon + \sum_{P,t} \gamma_{P,t} x_{p_1} \dots x_{p_m} [x_t^\varepsilon, x_{j_1}, \dots, x_{j_{n-m-1}}] + g \in \mathrm{Id}^L(A). \tag{3}$$

In order to prove that $f \in \mathrm{Id}^L(A_{\varepsilon'})$, we have to show that

$$f = \sum_{k=1}^{n} \beta_k x_{i_1} \dots x_{i_{n-1}} x_k^{\varepsilon'} + \sum_{P,t} \gamma_{P,t} x_{p_1} \dots x_{p_m} [x_t^{\varepsilon'}, x_{j_1}, \dots, x_{j_{n-m-1}}] + \tilde{g}$$

vanishes under every evaluation of elements of A. Here $\tilde{g}$ stands for the polynomial g in which we substituted every differential variable x^ε with $x^{\varepsilon'}$.

Since $\varepsilon' = \varepsilon - \mathrm{ad}_j$, it is enough to prove that

$$\sum_{k=1}^{n} \beta_k x_{i_1} \dots x_{i_{n-1}} x_k^{\mathrm{ad}_j} + \sum_{P,t} \gamma_{P,t} x_{p_1} \dots x_{p_m} [x_t^{\mathrm{ad}_j}, x_{j_1}, \dots, x_{j_{n-m-1}}] \in \mathrm{Id}^L(A_{\varepsilon'}).$$

But by definition of inner derivation, the claim follows since $[j,x]^\varepsilon - [j,x] \in \mathrm{Id}^L(A)$ and (3) holds. Hence $f \in \mathrm{Id}^L(A_{\varepsilon'})$.

Similarly it can be proved that $\mathrm{Id}^L(A_{\varepsilon'}) \subseteq \mathrm{Id}^L(A)$. Thus $A \sim_{T_L} A_{\varepsilon'}$ and the claim is proved. ■

As a consequence of Proposition 13, Lemma 14 and Theorem 16 we get the following.

Corollary 17. *If $A = B + J \in var^L(UT_2^\varepsilon)$ with $J = J_{00} + J_{10} + J_{01} + J_{11}$, then $J_{ik}^L \subseteq J_{ik}$, for all $i, k \in \{0,1\}$.*

According to the previous results, from now on we will assume that the Wedderburn–Malcev decomposition and the Jacobson radical decomposition into bimodules of every considered L-algebra are L-invariant.

5. On minimal L-subvarieties in var$^L(UT_2^\varepsilon)$

In this section we shall prove that the L-algebras A_k^ε, $(A_k^\varepsilon)^*$ and N_k^ε introduced in section 3 generate minimal L-varieties of polynomial growth. We start with the definition of minimal L-variety.

Definition 1. *An L-variety $\mathcal{V}$ is said to be minimal of polynomial growth if $c_n^L(\mathcal{V}) \approx qn^k$, for some $q > 0$, and for any proper L-subvariety $\mathcal{U} \subsetneq \mathcal{V}$, we have that $c_n^L(\mathcal{U}) \approx q'n^t$ with $t < k$.*

Algebras generating minimal varieties will play an important role in the main result, since we shall prove that any L-algebra inside $\mathrm{var}^L(UT_2^\varepsilon)$ has the same differential identities of a direct sum of such kind of algebra plus a nilpotent algebra, eventually.

Remark 18 ([25], Remark 2)**.** *Let $A = F + J$ be an L-algebra with $J = J_{00} + J_{10} + J_{01} + J_{11}$. If A satisfies the identity $[x_1, \dots, x_t] \equiv 0$ for some $t \geq 2$, then $J_{01} = J_{10} = 0$.*

Proof: The proof immediately follows from the fact that $J_{10} = [J_{10}, \underbrace{F, \dots, F}_{t-1}]$ and $J_{01} = [J_{01}, \underbrace{F, \dots, F}_{t-1}]$. ■

Theorem 19. *For all $k \geq 2$, N_k^ε generates a minimal L-variety of polynomial growth.*

Proof: Let suppose that $A \in \mathrm{var}^L(N_k^\varepsilon)$ and $c_n^L(A) \approx qn^{k-1}$ for some $q > 0$. We shall prove that $A \sim_{T_L} N_k^\varepsilon$ and this will complete the proof.

Since $c_n^L(A)$ is polynomially bounded, by [31, Theorem 8] we may assume that

$$A = B_1 \oplus \cdots \oplus B_m$$

where $B_1, \dots, B_m$ are finite dimensional L-algebras such that $\dim_F \frac{B_i}{J(B_i)} \leq 1$, for all $1 \leq i \leq m$. This implies that either $B_i \cong F + J(B_i)$ or $B_i \cong J(B_i)$ is a nilpotent algebra. Moreover, since

$$c_n^L(A) \leq c_n^L(B_1) + \cdots + c_n^L(B_m),$$

then there exists B_i such that $c_n^L(B_i) \approx bn^{k-1}$, for some $b > 0$. Thus

$$\mathrm{var}^L(N_k^\varepsilon) \supseteq \mathrm{var}^L(A) \supseteq \mathrm{var}^L(F + J(B_i)) \supseteq \mathrm{var}^L(F + J_{11}(B_i))$$

and $c_n^L(F + J(B_i)) \approx bn^{k-1.}$ Here remark that $F + J_{11}(B_i)$ is an L-subalgebra of $F + J(B_i)$ since, according to Theorem 16, in $\mathrm{var}^L(UT_2^\varepsilon)$ we may assume $F^\varepsilon = 0$ and $J_{ij}^\varepsilon \subseteq J_{ij}$ for all $i, j \in \{0, 1\}$.

Moreover, by Remark 18, we get that $J_{10}(B_i) = J_{01}(B_i) = 0$ and so

$$F + J(B_i) = \Big(F + J_{11}(B_i)\Big) \oplus J_{00}(B_i),$$

as L-algebras, and $c_n^L(F + J(B_i)) = c_n^L(F + J_{11}(B_i))$ for n large enough.

It turns out that, in order to prove $A \sim_{T_L} N_k^\varepsilon$, it suffices to show that $F + J_{11}(B_i) \sim_{T_L} N_k^\varepsilon$. Hence we assume, as we may, that A is a unitary L-algebra and we shall look at its proper codimension and cocharacter sequences.

Since $c_n^L(A) \approx qn^{k-1}$, then

$$c_n^L(A) = \sum_{i=0}^{k-1} \binom{n}{i} \gamma_i^L(A)$$

and by Corollary 5, $\gamma_i^L(A) \neq 0$ for all $0 \leq i \leq k-1$ and $\gamma_i^L(A) = 0$ for all $i \geq k$.

Moreover, recall that since $\mathrm{Id}^L(N_k^\varepsilon) \subseteq \mathrm{Id}^L(A)$, then $\frac{\Gamma_i^L}{\Gamma_i^L \cap \mathrm{Id}^L(A)}$ is isomorphic to a quotient module of $\frac{\Gamma_i^L}{\Gamma_i^L \cap \mathrm{Id}^L(N_k^\varepsilon)}$. Thus if $\psi_i^L(A) = \sum_{\lambda \vdash i} m_\lambda \chi_\lambda$ and $\psi_i^L(N_k^\varepsilon) = \sum_{\lambda \vdash i} m'_\lambda \chi_\lambda$ are the i-th proper L-cocharacters of A and N_k^ε, respectively, then $m_\lambda \leq m'_\lambda$ for all $\lambda \vdash i$.

From now on, suppose $k \geq 3$. For all $3 \leq i \leq k-1$, let $f_1 = [x_1, \underbrace{x_2, \dots, x_2}_{i-1}]$ and $f_2 = [x_1^\varepsilon, \underbrace{x_1, \dots, x_1}_{i-1}]$ be the highest weight vectors corresponding to the partitions $\lambda_1 = (i-1, 1)$ and $\lambda_2 = (i)$, respectively. It is clear that f_1 and f_2 are not differential identities of N_k^ε, thus $\chi_{(i-1,1)}$ and $\chi_{(i)}$ participate in the i-th proper L-cocharacter of N_k^ε with a non-zero multiplicities.

Moreover, since $\gamma_i^L(N_k^\varepsilon) = i = \deg \chi_{(i-1,1)} + \deg \chi_{(i)}$, for all $2 \leq i \leq k-1$, we get that

$$\psi_i^L(N_k^\varepsilon) = \chi_{(i-1,1)} + \chi_{(i)}.$$

Now, since $\gamma_{k-1}^L(A) \neq 0$ then either $\psi_{k-1}^L(A) = \chi_{(k-1)}$ or $\psi_{k-1}^L(A) = \chi_{(k-2,1)}$ or $\psi_{k-1}^L(A) = \chi_{(k-1)} + \chi_{(k-2,1)}$.

Fist suppose that $\psi_{k-1}^L(A) = \chi_{(k-1)}$. Then $[x_1, \underbrace{x_2, \dots, x_2}_{k-2}] \equiv 0$ on A and this trivially implies $[x_1^\varepsilon, \underbrace{x_1, \dots, x_1}_{k-2}] \equiv 0$ on A. Thus $\psi_{k-1}^L(A) = 0$ and $\gamma_{k-1}^L(A) = 0$, a contradiction.

Now suppose $\psi_{k-1}^L(A) = \chi_{(k-2,1)}$, then $[x_1^\varepsilon, \underbrace{x_1, \dots, x_1}_{k-2}] \equiv 0$ on A. Let substitute the variable x_1 with $x_1 + x_2$ and let consider the multihomogeneous component with degree $k-2$ in x_1 and 1 in x_2. As a consequence, we get the following identity modulo $\mathrm{Id}^L(UT_2^\varepsilon)$:

$$[x_2^\varepsilon, \underbrace{x_1, \dots, x_1}_{k-2}] + (k-2)[x_1^\varepsilon, x_2, \underbrace{x_1, \dots, x_1}_{k-3}] \equiv 0. \tag{4}$$

Since $[x_1, x_2] - [x_1^\varepsilon, x_2] - [x_1, x_2^\varepsilon] \in \mathrm{Id}^L(UT_2^\varepsilon) \subseteq \mathrm{Id}^L(A)$, we get

$$[x_2, x_1^\varepsilon, \underbrace{x_1, \dots, x_1}_{k-3}] \equiv [x_2, \underbrace{x_1, \dots, x_1}_{k-2}] - [x_2^\varepsilon, \underbrace{x_1, \dots, x_1}_{k-2}].$$

By putting together the latter one with (4) we get the identity

$$(k-2)[x_2, \underbrace{x_1, \dots, x_1}_{k-2}] \equiv (k-1)[x_2^\varepsilon, \underbrace{x_1, \dots, x_1}_{k-2}]. \tag{5}$$

Moreover, by substituting the variable x_2 with x_2^ε in (4) and recalling that $x_2^{\varepsilon^2} \equiv x_2^\varepsilon$, we also obtain $[x_2^\varepsilon, \underbrace{x_1, \dots, x_1}_{k-2}] \equiv 0$. From this one plus identity

(5), we finally get the identity $[x_2, \underbrace{x_1, \ldots, x_1}_{k-2}] \equiv 0$, thus $\psi^L_{k-1}(A) = 0$ and $\gamma^L_{k-1}(A) = 0$, a contradiction. Hence it must be $\psi^L_{k-1}(A) = \chi_{(k-1)} + \chi_{(k-2,1)}$.

Now, for all $2 \le i \le k-2$, as before either $\psi^L_i(A) = \chi_{(i)}$ or $\psi^L_i(A) = \chi_{(i-1,1)}$ or $\psi^L_i(A) = \chi_{(i)} + \chi_{(i-1,1)}$.

If $\psi^L_i(A) = \chi_{(i)}$ then $[x_1, \underbrace{x_2, \ldots, x_2}_{i-1}] \equiv 0$ on A. Thus also $[x_1, \underbrace{x_2, \ldots, x_2}_{k-2}] \equiv 0$ that is absurd for the first part of the proof. Analogously, if $\psi^L_i(A) = \chi_{(i-1,1)}$ then $[x_1^\varepsilon, \underbrace{x_1, \ldots, x_1}_{i-1}] \equiv 0$ on A and so $[x_1^\varepsilon, \underbrace{x_1, \ldots, x_1}_{k-2}] \equiv 0$, a contradiction.

Thus $\psi^L_i(A) = \chi_{(i)} + \chi_{(i-1,1)} = \psi^L_i(N_k^\varepsilon)$, for all $1 \le i \le k-1$ and

$$c^L_n(A) = \sum_{i=0}^{k-1} \binom{n}{i} \gamma^L_i(A) = 1 + \sum_{i=1}^{k-1} \binom{n}{i} i = c^L_n(N_k^\varepsilon).$$

Hence A and N_k^ε have the same codimension sequence and, since $\mathrm{Id}^L(N_k^\varepsilon) \subseteq \mathrm{Id}^L(A)$, we get the equality $\mathrm{Id}^L(N_k^\varepsilon) = \mathrm{Id}^L(A)$, as required.

Notice that if $k = 2$, then $\psi_2(N_2^\varepsilon) = \chi_{(1,1)} + \chi_{(2)}$ and with similar arguments as in the first part of the proof we get $\psi_2(A) = \psi_2(N_2^\varepsilon)$, $c^L_n(A) = c^L_n(N_2^\varepsilon)$ and so $A \sim_{T_L} N_2^\varepsilon$. ■

We now recall a result about the Jacobson radical of an algebra belonging to $\mathrm{var}^L(A_k^\varepsilon)$ that will be very useful hereafter.

Lemma 20. *Let $A = F + J \in var^L(A_k^\varepsilon)$ (resp. $A = F + J \in var^L((A_k^\varepsilon)^*)$). Then $J_{11}^\varepsilon = 0$ and $J_{01} = [J_{11}, J_{11}] = 0$ (resp. $J_{10} = [J_{11}, J_{11}] = 0$).*

Proof: We will prove the statement in case $A \in \mathrm{var}^L(A_k^\varepsilon)$. The other one will follow analogously.

Recall that according to Corollary 17, $J_{11}^\varepsilon \subseteq J_{11}$. Moreover, since $x_1^\varepsilon x_2 \cdots x_k \in \mathrm{Id}^L(A_k^\varepsilon) \subseteq \mathrm{Id}^L(A)$, for all $j \in J_{11}$ we get $j^\varepsilon \underbrace{1_F \cdots 1_F}_{k-1} = 0$. Thus, if we let $j^\varepsilon = \tilde{j} \in J_{11}$ then

$$0 = \tilde{j} \underbrace{1_F \cdots 1_F}_{k-1} = \tilde{j},$$

since 1_F acts as a unit element on J_{11}. Now, due to the identity $[x_1, x_2]^\varepsilon - [x_1, x_2] \equiv 0$, we get also $[J_{11}, J_{11}] = 0$.

Finally, by Lemma 14, for all $a \in J_{01}$, $a^\varepsilon = a$. Thus by using the same argument as before, we get that $J_{01} = J_{01}^\varepsilon = 0$. ■

Lemma 21. *Let $A = F + J \in var^L(A_k^\varepsilon)$ (resp. $A = F + J \in var^L((A_k^\varepsilon)^*)$). If $c_n^L(A) \approx qn^{k-1}$, for some $q > 0$, then $A \sim_{T_L} A_k^\varepsilon$ (resp. $A \sim_{T_L} (A_k^\varepsilon)^*$).*

Proof: We prove the statement for $A = F + J \in \mathrm{var}^L(A_k^\varepsilon)$. The case $A = F + J \in \mathrm{var}^L((A_k^\varepsilon)^*)$ will follow with similar arguments.

By the previous Lemma, $J_{01} = [J_{11}, J_{11}] = 0$, so we may assume $A = F + J_{00} + J_{10} + J_{11}$ and J_{11} commutative. Moreover $J_{11}^\varepsilon = 0$.

First suppose that $J_{10}J_{00}^{k-2} = 0$.

If $J^m = 0$, then for all $n \geq m$ we shall prove that $g = x_k \cdots x_n x_1^\varepsilon x_2 \cdots x_{k-1} \in \mathrm{Id}^L(A)$. Since such a monomial is multilinear, we can evaluate each variable in a basis of A consisting in an union of a basis of J_{00}, J_{10}, J_{11} and 1_F. Since $n \geq m$ and $J^m = 0$, if we evaluate all the variables in J then we get zero, thus at least one variable must be evaluated in 1_F.

Let focus our attention to the variable x_1. It is clear that if x_1 is evaluated in 1_F or on J_{11}, then g vanishes since $F^\varepsilon = J_{11}^\varepsilon = 0$. If we evaluate x_1 in an element $j_{10} \in J_{10}$, then $j_{10}^\varepsilon = j_{10}$ and we are forced to evaluate $x_2, \ldots, x_{k-1}$ on elements of J_{00}. Since $J_{10}J_{00}^{k-2} = 0$, we get zero. Finally, let evaluate x_1 on an element $j_{00} \in J_{00}$. Then $j_{00}^\varepsilon \in J_{00}$ and since there exists t such that x_t is evaluated in 1_F, also in this case we get zero.

Therefore we have proved that $x_k \cdots x_n x_1^\varepsilon x_2 \cdots x_{k-1} \in \mathrm{Id}^L(A)$. From this identity and from $[x_1, x_2]^\varepsilon - [x_1, x_2] \equiv 0$ follows also that

$$x_{k+1} \cdots x_n [x_1, x_2] x_3 \cdots x_k \in \mathrm{Id}^L(A).$$

Since $A \in \mathrm{var}^L(A_k^\varepsilon)$, if $f \in P_n^L$ with $\deg f = n \geq m$, then after reducing f modulo the T_L-ideal generated by the differential identities of A_k^ε and by g, by using also Lemma 7, we have that f is a linear combination of the L-polynomials

$$\begin{gathered} x_1 \cdots x_n, \quad x_{i_1} \cdots x_{i_t} [x_i, x_j] x_{j_1} \cdots x_{j_l}, \\ x_2 \cdots x_n x_1^\varepsilon, \quad x_{p_1} \cdots x_{p_r} x_m^\varepsilon x_{q_1} \cdots x_{q_s}, \end{gathered}$$

where $t + l = n - 2$, $r + s = n - 1$, $l < k - 2$, $s < k - 2$, $i > j < i_1 < \ldots < i_t$, $j_1 < \ldots < j_l$, $m < p_1 < \ldots < p_r$ and $q_1 < \ldots < q_s$. Remark that $l, s < k - 2$ since $g \equiv 0$ on A.

Therefore

$$c_n^L(A) \leq 2 + n + \sum_{l=0}^{k-3} \binom{n}{l}(n-l+1) + \sum_{l=1}^{k-3} \sum_{j=2}^{n-l+1} \binom{n-j}{l-1}(j-1) \approx q'n^{k-2},$$

for some $q' > 0$. This is a contradiction, since we are assuming that $c_n^L(A) \approx qn^{k-1}$.

Thus $J_{10}J_{00}^{k-2} \neq 0$ and there exist $a \in J_{10}$ and $b_1, \dots, b_{k-2} \in J_{00}$ such that $ab_1 \cdots b_{k-2} \neq 0$. Let $f \in \mathrm{Id}^L(A)$ be a multilinear L-polynomial of degree n. By Lemma 7, f modulo $\mathrm{Id}^L(A_k^\varepsilon)$ can be written as

$$f = \alpha x_1 \cdots x_n + \beta x_2 \cdots x_n x_1^\varepsilon + \sum_{l<k-1} \sum_{I,J} \alpha_{I,J} x_{i_1} \cdots x_{i_t}[x_i, x_j] x_{j_1} \cdots x_{j_l} + \sum_{s<k-1} \sum_{P,Q} \beta_{P,Q} x_{p_1} \cdots x_{p_r} x_m^\varepsilon x_{q_1} \cdots x_{q_s} + f',$$

where $f' \in \mathrm{Id}^L(A_k^\varepsilon)$, $I = \{i, j, i_1, \dots, i_t\}$, $J = \{j_1, \dots, j_l\}$, $P = \{m, p_1, \dots, p_r\}$ and $Q = \{q_1, \dots, q_s\}$ with $t + l = n - 2$, $r + s = n - 1$, $l < k - 1$, $s < k - 1$, $i > j < i_1 < \ldots < i_t$, $j_1 < \ldots < j_l$, $m < p_1 < \ldots < p_r$ and $q_1 < \ldots < q_s$.

By choosing $x_1 = \cdots = x_n = 1_F$ we get $\alpha = 0$. Moreover, by induction on l, for fixed I and J, the evaluation $x_i = a$, $x_j = x_{i_1} = \cdots = x_{i_t} = 1_F$ and $x_{j_h} = b_h$, for all $1 \leq h \leq l$, gives $\alpha_{I,J} = 0$. If we let $x_1 = a$ and $x_2 = \cdots = x_n = 1_F$, then we get $\beta = 0$. Finally, by induction on s, for fixed P and Q, the evaluation $x_m = a$, $x_{p_1} = \cdots = x_{p_r} = 1_F$ and $x_{q_h} = b_h$, for all $1 \leq h \leq s$, gives $\beta_{P,Q} = 0$.

Thus $f = f' \in \mathrm{Id}^L(A_k^\varepsilon)$ and $\mathrm{Id}^L(A_k^\varepsilon) = \mathrm{Id}^L(A)$, as claimed. ■

We are now in a position to prove that A_k^ε and $(A_k^\varepsilon)^*$ generate minimal L-varieties.

Theorem 22. *For all $k \geq 2$, A_k^ε and $(A_k^\varepsilon)^*$ generate minimal L-varieties of polynomial growth.*

Proof: Let $A \in \mathrm{var}^L(A_k^\varepsilon)$ such that $c_n^L(A) \approx qn^{k-1}$, for some $q > 0$. By [31, Theorem 8] we assume

$$A = B_1 \oplus \cdots \oplus B_m$$

where $B_1, \dots, B_m$ are finite dimensional L-algebras such that $\dim_F \frac{B_i}{J(B_i)} \leq 1$. This says that either $B_i \cong F + J(B_i)$ or $B_i \cong J(B_i)$ is a nilpotent L-algebra, for all $1 \leq i \leq m$. Since

$$c_n^L(A) \leq c_n^L(B_1) + \cdots + c_n^L(B_m),$$

there exists B_i such that $c_n^L(B_i) \approx bn^{k-1}$, for some $b > 0$. Thus $B_i = F + J(B_i)$ and by the previous Lemma, $B_i \sim_{T_L} A_k^\varepsilon$. Hence

$$\mathrm{var}^L(A_k^\varepsilon) = \mathrm{var}^L(B_i) \subseteq \mathrm{var}^L(A) \subseteq \mathrm{var}^L(A_k^\varepsilon)$$

and so $\mathrm{var}^L(A) = \mathrm{var}^L(A_k^\varepsilon)$.

Similarly one can prove the statement for $(A_k^\varepsilon)^*$. ■

6. Classifying subvarieties of $\mathbf{var}^L(UT_2^\varepsilon)$

In this section we present the main result about the L-variety generated by UT_2^ε, i.e., we will classify up to T_L-equivalence all the L-algebras generating L-subvarieties of $\mathrm{var}^L(UT_2^\varepsilon)$.

To this end, we start with the following lemma concerning algebras with slow codimension growth.

Lemma 23. *Let $A = F + J_{10} + J_{11} \in var^L(UT_2^\varepsilon)$ with $J_{10} \neq 0$ (resp. $A = F + J_{01} + J_{11} \in var^L(UT_2^\varepsilon)$ with $J_{01} \neq 0$). If $J_{11}^\varepsilon = 0$, then $A \sim_{T_L} A_2^\varepsilon$ (resp. $A \sim_{T_L} (A_2^\varepsilon)^*$).*

Proof: Since $F^\varepsilon = J_{11}^\varepsilon = 0$ and $J_{10}^2 = 0$, it is clear that $x_1x_2^\varepsilon - x_2x_1^\varepsilon - [x_1, x_2] \in \mathrm{Id}^L(A)$, thus $\mathrm{Id}^L(A_2^\varepsilon) \subseteq \mathrm{Id}^L(A)$.

In order to prove the opposite inclusion, let $f \in \mathrm{Id}^L(A)$ be a multilinear L-polynomial of degree n. By [30, Theorem 3], f can be written as

$$f = \sum_{j=1}^{n} \alpha_j x_{i_1} \cdots x_{i_{n-1}} x_j + \beta x_2 \cdots x_n x_1^\varepsilon + g,$$

where $g \in \mathrm{Id}^L(A_2^\varepsilon)$ and $i_1 < \cdots < i_{n-1}$.

Suppose that there exists $j \neq 1$ such that $\alpha_j \neq 0$. Then by making the evaluation $x_j = b \in J_{10}$, for some $b \neq 0$, and $x_{i_1} = \cdots = x_{i_{n-1}} = 1_F$, we get $\alpha_j = 0$, a contradiction. Now, if $\alpha_1 \neq 0$, then by making the evaluation $x_1 = \cdots = x_n = 1_F$ we get $\alpha_1 = 0$, a contradiction. Finally, if $\beta \neq 0$, then we let $x_1 = b$ and $x_2 = \cdots = x_n = 1_F$ getting $\beta = 0$, a contradiction.

Hence $f = g \in \mathrm{Id}^L(A_2^\varepsilon)$ and so $A \sim_{T_L} A_2^\varepsilon$.

Similarly, if $A = F + J_{01} + J_{11}$, we get $A \sim_{T_L} (A_2^\varepsilon)^*$. ■

Lemma 24. *Let $A = F + J_{11} \in var^L(UT_2^\varepsilon)$. Then $A \sim_{T_L} N_k^\varepsilon$, for some $k \geq 1$.*

Proof: Since $A \in \mathrm{var}^L(UT_2^\varepsilon)$, then $c_n^L(A) \approx qn^{k-1}$ for some $q > 0$ and $k \geq 1$.

If $J_{11}^\varepsilon = 0$, then $x^\varepsilon \equiv 0$ on A and so $[x_1, x_2] \in \mathrm{Id}^L(A)$. This trivially implies that A is a commutative algebra with trivial derivation, i.e., $A \sim_{T_L} N_1^\varepsilon = F$.

Let now $J_{11}^\varepsilon \neq 0$. Since A is a unitary algebra, we can consider the proper L-codimension sequence and write

$$c_n^L(A) = \sum_{i=0}^{k-1} \binom{n}{i} \gamma_i^L(A),$$

with $\gamma_i^L(A) = 0$ for all $i \geq k$. In particular $\gamma_k^L(A) = 0$ and so $[x_1, \dots, x_k] \in \mathrm{Id}^L(A)$. Hence $\mathrm{Id}^L(N_k^\varepsilon) \subseteq \mathrm{Id}^L(A)$ and by Theorem 19, since $c_n^L(A) \approx qn^{k-1}$, it follows $A \sim_{T_L} N_k^\varepsilon$. ∎

We now prove some auxiliary lemmas very useful in the proof of the main theorem. We start by the following that allows us to reduce our problem to the study of a variety generated by an L-algebra with either $J_{01} = 0$ or $J_{10} = 0$.

Lemma 25. *Let $A = F + J \in var^L(UT_2^\varepsilon)$. Then $A \sim_{T_L} (F + J_{00} + J_{10} + J_{11}) \oplus (F + J_{00} + J_{01} + J_{11})$.*

Proof: Let $B_1 = F + J_{00} + J_{10} + J_{11}$ and $B_2 = F + J_{00} + J_{01} + J_{11}$. Since $F^\varepsilon = 0$ and $J_{ij}^\varepsilon \subseteq J_{ij}$ for all $i, j \in \{0, 1\}$, it is clear that B_1 and B_2 are L-subalgebras of A. Then $\mathrm{Id}^L(A) \subseteq \mathrm{Id}^L(B_1 \oplus B_2) = \mathrm{Id}^L(B_1) \cap \mathrm{Id}^L(B_2)$.

Moreover, since $J_{01}J_{10} = J_{10}J_{01} = 0$, it turns out that also $\mathrm{Id}^L(B_1 \oplus B_2) \subseteq \mathrm{Id}^L(A)$ holds. Thus $A \sim_{T_L} B_1 \oplus B_2$ as claimed. ∎

Lemma 26. *Let $A = F + J_{00} + J_{10} + J_{11} \in var^L(UT_2^\varepsilon)$ with $J_{10} \neq 0$ (resp. $A = F + J_{00} + J_{01} + J_{11} \in var^L(UT_2^\varepsilon)$ with $J_{01} \neq 0$).*

1. *If $J_{11}^\varepsilon = 0$, then $A \sim_{T_L} A_k^\varepsilon \oplus N$ (resp. $A \sim_{T_L} (A_k^\varepsilon)^* \oplus N$), for some $k \geq 2$ where N is a nilpotent L-algebra.*
2. *If $J_{11}^\varepsilon \neq 0$, then $A \sim_{T_L} A_k^\varepsilon \oplus N_u^\varepsilon \oplus N$ (resp. $A \sim_{T_L} (A_k^\varepsilon)^* \oplus N_u^\varepsilon \oplus N$), for some $u \geq 2$ and $k \geq 2$ where N is a nilpotent L-algebra.*

Proof: Let $A = F + J_{00} + J_{10} + J_{11} \in \mathrm{var}^L(UT_2^\varepsilon)$ with $J_{10} \neq 0$. The other case will follow with similar arguments.

Suppose first $J_{11}^\varepsilon = 0$ and let $t \geq 0$ be the greatest integer such that $J_{10}J_{00}^t \neq 0$. Notice that if $t = 0$ then $J_{10}J_{00} = 0$ and $A = (F + J_{10} + J_{11}) \oplus J_{00}$ as L-algebras. By Lemma 23 we get $F + J_{10} + J_{11} \sim_{T_L} A_2^\varepsilon$, hence $A \sim_{T_L} A_2^\varepsilon \oplus J_{00}$, where J_{00} is a nilpotent L-algebra.

So let assume $t > 0$, i.e., $J_{10}J_{00}^t \neq 0$ (that is in particular $J_{10}J_{00} \neq 0$) and $J_{10}J_{00}^{t+1} = 0$.

Suppose that $J_{00}^\varepsilon J_{00}^{t+1} = 0$. Then it is easy to check that $x_1^\varepsilon x_2 \cdots x_{t+2} \in \mathrm{Id}^L(A)$, thus $\mathrm{Id}^L(A_{t+2}^\varepsilon) \subseteq \mathrm{Id}^L(A)$. Furthermore, since $J_{10}J_{00}^t \neq 0$, there exist $a \in J_{10}$ and $b_1, \ldots, b_t \in J_{00}$ such that $ab_1 \cdots b_t \neq 0$. Therefore, as in the proof of Lemma 21, one can prove that $A \sim_{T_L} A_{t+2}^\varepsilon$.

Let suppose now $J_{00}^\varepsilon J_{00}^{t+1} \neq 0$. Remark that, since $J_{00}^\varepsilon \subseteq J_{00}$, $\varepsilon^2 = \varepsilon$ and Lemma 15 holds, J_{00}^ε is an L-ideal of A, thus we can consider $\bar{A} = A/J_{00}^\varepsilon$. As before, since $J_{10}J_{00}^{t+1} = 0$, it follows that $x_1^\varepsilon x_2 \cdots x_{t+2} \in \mathrm{Id}^L(\bar{A})$, $\mathrm{Id}^L(A_{t+2}^\varepsilon) \subseteq \mathrm{Id}^L(\bar{A})$ and so $\bar{A} \sim_{T_L} A_{t+2}^\varepsilon$.

Notice that $\mathrm{Id}^L(A) \subseteq \mathrm{Id}^L(\bar{A}) = \mathrm{Id}^L(A_{t+2}^\varepsilon)$ and, since J_{00} is an L-subalgebra of A, $\mathrm{Id}^L(A) \subseteq \mathrm{Id}^L(J_{00})$. Therefore $\mathrm{Id}^L(A) \subseteq \mathrm{Id}^L(A_{t+2}^\varepsilon \oplus J_{00})$.

Conversely, let $f \in \mathrm{Id}^L(A_{t+2}^\varepsilon \oplus J_{00})$ be a multilinear L-polynomial of degree n. We can write f as

$$\begin{aligned} f = \alpha x_1 \cdots x_n + \beta x_2 \cdots x_n x_1^\varepsilon + \sum_{l<t+1} \sum_{I,J} \alpha_{I,J} x_{i_1} \cdots x_{i_k} [x_i, x_j] x_{j_1} \cdots x_{j_l} \\ + \sum_{s<t+1} \sum_{P,Q} \beta_{P,Q} x_{p_1} \cdots x_{p_r} x_m^\varepsilon x_{q_1} \cdots x_{q_s} \sum_{l'>t} \sum_{I',J'} \alpha_{I',J'} x_{i'_1} \cdots x_{i'_k} [x_{i'}, x_{j'}] x_{j'_1} \cdots x_{j'_l} \\ + \sum_{s'>t} \sum_{P',Q'} \beta_{P',Q'} x_{p'_1} \cdots x_{p'_r} x_{m'}^\varepsilon x_{q'_1} \cdots x_{q'_s} + g, \end{aligned} \tag{6}$$

where $g \in \mathrm{Id}^L(UT_2^\varepsilon) \subseteq \mathrm{Id}^L(A)$ and the indices of the variables are subjected to the conditions as in Lemma 7.

Remark that g and the last two summand of f are L-identities of A_{t+2}^ε. Moreover, in Lemma 7 it was also proved that the first four summand of f are linearly independent modulo $\mathrm{Id}^L(A_{t+2}^\varepsilon)$, hence $\alpha = \beta = \alpha_{I,J} = \beta_{P,Q} = 0$ for all I, J, P and Q, and

$$\begin{aligned} f = \sum_{l'>t} \sum_{I',J'} \alpha_{I',J'} x_{i'_1} \cdots x_{i'_k} [x_{i'}, x_{j'}] x_{j'_1} \cdots x_{j'_l} \\ + \sum_{s'>t} \sum_{P',Q'} \beta_{P',Q'} x_{p'_1} \cdots x_{p'_r} x_{m'}^\varepsilon x_{q'_1} \cdots x_{q'_s} + g. \end{aligned} \tag{7}$$

Since $f \in \mathrm{Id}^L(J_{00})$, if we evaluate all the variables on J_{00}, we get zero. Now, since $J_{10}J_{00}^{t+1} = J_{11}^\varepsilon = [J_{11}, J_{11}] = 0$, every evaluation of f into elements of

A gives the zero value, hence $f \in \mathrm{Id}^L(A)$. So $\mathrm{Id}^L(A^\varepsilon_{t+2} \oplus J_{00}) \subseteq \mathrm{Id}^L(A)$ and $A \sim_{T_L} A^\varepsilon_{t+2} \oplus J_{00}$ follows.

Suppose now $J^\varepsilon_{11} \neq 0$.

Let $B = F + J_{00} + J_{10}$ and $D = F + J_{11}$. It is clear that B and D are L-subalgebras of A. Moreover, for the first part of the proof, $B \sim_{T_L} A^\varepsilon_{t+2} \oplus N$, for some $t \geq 0$, and by Lemma 24, $D \sim_{T_L} N^\varepsilon_u$ for some $u \geq 2$. Thus $\mathrm{Id}^L(A) \subseteq \mathrm{Id}^L(B \oplus D) = \mathrm{Id}^L(A^\varepsilon_{t+2} \oplus N^\varepsilon_u \oplus N)$.

Conversely, let $f \in \mathrm{Id}^L(A^\varepsilon_{t+2} \oplus N^\varepsilon_u \oplus N)$ be a multilinear polynomial of degree n and write f as in (6). As in the previous case, since $f \in \mathrm{Id}^L(A^\varepsilon_{t+2})$, we can reduce f as in (7). Notice that $f \in \mathrm{Id}^L(B) \cap \mathrm{Id}^L(D)$, thus any evaluation of f in B or in D gives zero. Furthermore, since $J_{10} J_{00}^{t+1} = J^\varepsilon_{11} J_{10} = 0$, we get that f vanishes under any evaluation on elements of A.

Thus $f \in \mathrm{Id}^L(A)$ and $\mathrm{Id}^L(A) = \mathrm{Id}^L(B \oplus D) = \mathrm{Id}^L(A^\varepsilon_{t+2} \oplus N^\varepsilon_u \oplus N)$. This immediately implies $A \sim_{T_L} A^\varepsilon_{t+2} \oplus N^\varepsilon_u \oplus N$ and we are done. ∎

By putting together the previous results, we get the following.

Lemma 27. *Let $A = F + J \in var^L(UT^\varepsilon_2)$ with $J_{10} \neq 0$ and $J_{01} \neq 0$. Then either $A \sim_{T_L} A^\varepsilon_k \oplus (A^\varepsilon_r)^* \oplus N$ or $A \sim_{T_L} A^\varepsilon_k \oplus (A^\varepsilon_r)^* \oplus N^\varepsilon_u \oplus N$, for some $k, r, u \geq 2$, where N is a nilpotent L-algebra.*

Proof: By Lemma 25, $A \sim_{T_L} B_1 \oplus B_2$ where $B_1 = F + J_{00} + J_{10} + J_{11}$ and $B_2 = F + J_{00} + J_{01} + J_{11}$. Moreover, by the previous Lemma, $B_1 \sim_{T_L} A^\varepsilon_k \oplus N$ or $B_1 \sim_{T_L} A^\varepsilon_k \oplus N^\varepsilon_u \oplus N$ and $B_2 \sim_{T_L} (A^\varepsilon_r)^* \oplus N$ or $B_2 \sim_{T_L} (A^\varepsilon_r)^* \oplus N^\varepsilon_u \oplus N$, for some $k, r, u \geq 2$ and N a nilpotent L-algebra. It readily follows that

$$A \sim_{T_L} A^\varepsilon_k \oplus (A^\varepsilon_r)^* \oplus N \quad \text{or}$$
$$A \sim_{T_L} A^\varepsilon_k \oplus (A^\varepsilon_r)^* \oplus N^\varepsilon_u \oplus N,$$

as claimed. ∎

We are now in a position to prove the main theorem of the paper.

Theorem 28. *If $A \in var^L(UT^\varepsilon_2)$ then A is T_L-equivalent to one of the following L-algebras: UT^ε_2, N, $N^\varepsilon_t \oplus N$, $A^\varepsilon_k \oplus N$, $(A^\varepsilon_r)^* \oplus N$, $A^\varepsilon_k \oplus N^\varepsilon_u \oplus N$, $(A^\varepsilon_r)^* \oplus N^\varepsilon_u \oplus N$, $A^\varepsilon_k \oplus (A^\varepsilon_r)^* \oplus N$, $A^\varepsilon_k \oplus (A^\varepsilon_r)^* \oplus N^\varepsilon_u \oplus N$, where N is a nilpotent algebra and $k, r, u \geq 2$, $t \geq 1$.*

Proof: If $A \sim_{T_L} UT^\varepsilon_2$ there is nothing to prove, so let suppose that A generates a proper L-subvariety of $\mathrm{var}^L(UT^\varepsilon_2)$. Thus, by Theorem 3, $c^L_n(A)$ is polynomially bounded and by [31, Theorem 8] we may assume

$$A = B_1 \oplus \cdots \oplus B_m,$$

where $B_1, \ldots, B_m$ are finite dimensional L-subalgebras of A such that $\dim_F \frac{B_i}{J(B_i)} \leq 1$, for all $1 \leq i \leq m$.

If for all i, $\dim_F \frac{B_i}{J(B_i)} = 0$, then $B_i = J(B_i)$ is a nilpotent L-algebra and $A \sim_{T_L} N$ where $N = B_1 \oplus \cdots \oplus B_m$.

Thus suppose that there exists i such that $\dim_F \frac{B_i}{J(B_i)} = 1$, that is $B_i = F + J(B_i)$. Write

$$J(B_i) = J_{00} \oplus J_{10} \oplus J_{01} \oplus J_{11}.$$

If $J_{10} = J_{01} = 0$, then by Lemma 24, $A \sim_{T_L} N^{\varepsilon}_{u_i} \oplus N$ for some $u_i \geq 1$, where N is a nilpotent L-algebra. If either $J_{10} \neq 0$ or $J_{01} \neq 0$, then by Lemmas 26 and 27, B_i is T_L-equivalent to one of the following L-algebras: $A^{\varepsilon}_{k_i} \oplus N$, $(A^{\varepsilon}_{r_i})^* \oplus N$, $A^{\varepsilon}_{k_i} \oplus N^{\varepsilon}_{u_i} \oplus N$, $(A^{\varepsilon}_{r_i})^* \oplus N^{\varepsilon}_{u_i} \oplus N$, $A^{\varepsilon}_{k_i} \oplus (A^{\varepsilon}_{r_i})^* \oplus N$ or $A^{\varepsilon}_{k_i} \oplus (A^{\varepsilon}_{r_i})^* \oplus N^{\varepsilon}_{u_i} \oplus N$, for some $k_i, r_i, u_i \geq 2$.

Since $A = B_1 \oplus \cdots \oplus B_m$, by taking into account the previous possibilities, we get the desired conclusion. ∎

As a direct consequence of the previous Theorem and Lemmas 19 and 22, we get the following corollary that classifies, up to T_L-equivalence, all L-algebras generating minimal varieties of polynomial growth inside $\mathrm{var}^L(UT_2^{\varepsilon})$.

Corollary 29. *Let $A \in var^L(UT_2^{\varepsilon})$. Then A generates a minimal L-variety if and only if either $A \sim_{T_L} N^{\varepsilon}_u$ or $A \sim_{T_L} A^{\varepsilon}_k$ or $A \sim_{T_L} (A^{\varepsilon}_k)^*$, for some $u \geq 1$, $k \geq 2$.*

7. Classifying subvarieties of $\mathrm{var}^L(UT_2)$

In this section we classify, up to T_L-equivalence, all the L-subvarieties of $\mathrm{var}^L(UT_2)$. As we remarked before, since L acts trivially on UT_2, this is equivalent to the classification of the algebras inside the variety generated by UT_2 in the ordinary case given in [22]. In what follows we present such results in the language of L-algebras for convenience of the reader.

For $k \geq 2$, let A_k, A_k^* and N_k be the algebras A^{ε}_k, $(A^{\varepsilon}_k)^*$ and N^{ε}_k constructed in Section 3, respectively, where L acts trivially on them.

Since $x^{\delta} \equiv 0$ for all $\delta \in L$, in this case we are dealing with ordinary identities. Thus we have the following results characterizing the L-identities and the growth of the L-codimensions of the above algebras.

Theorem 30 ([4], Lemma 3)**.**

1. $Id^L(A_2) = \langle [x_1, x_2]x_3 \rangle_{T_L}$ *and* $Id^L(A_2^*) = \langle x_1[x_2, x_3] \rangle_{T_L}$.
2. $c_n^L(A_2) = c_n^L(A_2^*) = n$.

Theorem 31 ([22], Lemma 3.1)**.** *Let* $k \geq 3$, *then:*

1. $Id^L(A_k) = \langle [x_1, x_2][x_3, x_4],\ [x_1, x_2]x_3 \cdots x_{k+1} \rangle_{T_L}$;
2. $c_n^L(A_k) = \displaystyle\sum_{l=0}^{k-2} \binom{n}{l}(n - l - 1) + 1 \approx qn^{k-1}$, *for some* $q > 0$.

Hence $Id^L(A_k^*) = \langle [x_1, x_2][x_3, x_4],\ x_1 \cdots x_{k-2}[x_{k-1}, x_k] \rangle_{T_L}$ *and* $c_n^L(A_k^*) = c_n^L(A_k) \approx qn^{k-1}$.

Theorem 32 ([5], Theorem 3.4,)**.** *Let* $k \geq 3$, *then:*

1. $Id^L(N_k) = \langle [x_1, x_2][x_3, x_4],\ [x_1, \dots, x_k] \rangle_{T_L}$;
2. $c_n^L(N_k) = 1 + \sum_{j=2}^{k-1} \binom{n}{j}(j - 1) \approx qn^{k-1}$, *for some* $q > 0$.

Moreover, $N_2 \sim_{T_L} F$.

The following result classifies the subvarieties of $\mathrm{var}^L(UT_2)$.

Theorem 33 ([22], Theorem 5.4)**.** *If* $A \in var^L(UT_2)$ *then* A *is* T_L*-equivalent to one of the following* L*-algebras:* UT_2, N, $N_u \oplus N$, $A_k \oplus N$, $A_r^* \oplus N$, $A_k \oplus N_u \oplus N$, $A_r^* \oplus N_u \oplus N$, $A_k \oplus A_r^* \oplus N$, $A_k \oplus A_r^* \oplus N_u \oplus N$, *where* N *is a nilpotent algebra and* $k, r, u \geq 2$.

As a consequence of the previous theorems, we can also get the classification of all L-algebras generating minimal varieties.

Corollary 34. *An* L*-algebra* $A \in var^L(UT_2)$ *generates a minimal variety of polynomial growth if and only if either* $A \sim_{T_L} N_u$ *or* $A \sim_{T_L} A_k$ *or* $A \sim_{T_L} A_k^*$, *for some* $u \geq 2$, $k \geq 2$.

References

[1] O. M. Di Vincenzo, V. Nardozza, *Differential Polynomial Identities of Upper Triangular Matrices Under the Action of the Two-Dimensional Metabelian Lie Algebra*, Algebr. Represent. Theory (2021), in Press.

[2] V. Drensky, *Free algebras and PI-algebras, graduate course in algebra*, Springer, Singapore (2000).

[3] A. Giambruno, A. Ioppolo, D. La Mattina, *Varieties of algebras with superinvolution of almost polynomial growth*, Algebr. Represent. Theory **19** (2016), no. 3, 599–611.

[4] A. Giambruno, D. La Mattina, *PI-algebras with slow codimension growth*, J. Algebra **284** (2005), 371–391.

[5] A. Giambruno, D. La Mattina, V. Petrogradsky, *Matrix algebras of polynomial codimension growth*, Israel J. Math **158** (2007), 367–378.

[6] A. Giambruno, D. La Mattina, M. Zaicev, *Classifying the minimal varities of polynomial growth*, Canad. J. Math. **66** (3) (2014), 625–640.

[7] A. Giambruno, S. Mishchenko, *On star–varieties with almost polynomial growth*, Algebra Colloq. **8** (2001), no. 1, 33–42.

[8] A. Giambruno, S. Mishchenko, M. Zaicev, *Polynomial identities on superalgebras and almost polynomial growth*, Comm. Algebra **29** (9) (2001), 3787–3800.

[9] A. Giambruno, C. Rizzo, *Differential identities,* 2×2 *upper triangular matrices and varieties of almost polynomial growth*, J. Pure Appl. Algebra **223** (2019), no. 4, 1710–1727.

[10] A. Giambruno, R. B. dos Santos, A. C. Vieira, *Identities of* $*$*-superalgebras and almost polynomial growth*, Linear Multilinear Algebra **64** (2016), no. 3, 484-501.

[11] A. Giambruno, M. Zaicev, *Asymptotics for the standard and the Capelli identities*, Israel J. Math. **135** (2003), 125–145.

[12] A. Giambruno, M. Zaicev, *Polynomial identities and asymptotic methods*, AMS, Math. Surv. Monogr. **122** (2005).

[13] A.S. Gordienko, *Asymptotics of H-identities for associative algebras with an H-invariant radical*, J. Algebra **393** (2013), 92–101.

[14] A.S. Gordienko, M.V. Kochetov, *Derivations, gradings, actions of algebraic groups, and codimension growth of polynomial identities*, Algebr. Represent. Th. **17** (2014), no. 2, 539–563.

[15] G. Hochschild. *Semi-simple algebras and generalized derivations*, Amer. J. Math. **64** (1942), no. 1, 677–694.

[16] Y. Karasik, *Kemer's theory for H-module algebras with application to the PI exponent*, J. Algebra **457** (2016), 194–227.

[17] V.K. Kharchenko, *Differential identities of semiprime rings*, Algebra Log. **18** (1979) 86-–119.

[18] A.R. Kemer, *T-ideals with power growth of the codimensions are Specht*, Sibirsk. Mat. Zh. **19** (1978) 54–69 (in Russian); English translation: Sib. Math. J. **19** (1978) 37–48.

[19] A.R. Kemer, *Varieties of finite rank*, Proc. 15-th All the Union Algebraic Conf., Krasnoyarsk. **2** (1979), p. 73 (in Russian).

[20] A. Ioppolo, D. La Mattina, *Polynomial codimension growth of algebras with involutions and superinvolutions*, J. Algebra **472** (2017), 519–545.

[21] A. Ioppolo, F. Martino, *Varieties of algebras with pseudoinvolution and polynomial growth*, Linear Multilinear Algebra **66** (2018) (11), 2286–2304.

[22] D. La Mattina, *Varieties of almost polynomial growth: classifying their subvarieties*, Manuscripta Math. **123** (2007), 185–203.

[23] D. La Mattina, *Polynomial codimension growth of graded algebras*, in: Group, Rings and Group Rings, in: Contemp. Math. vol. **499**, Amer. Math. Soc., Providence, RI, 2009, pp. 189–197.

[24] D. La Mattina *Varieties of superalgebras of almost polynomial growth*, J. Algebra **336** (2011), 209–226.

[25] D. La Mattina, F. Martino, *Polynomial growth and star-varieties*, J. Pure Appl. Algebra **220** (2016), 246–262.

[26] J.N. Malcev, *A basis for the identities of the algebra of upper triangular matrices*, Algebra i Logika **10** (1971), 393–400.

[27] F. Martino, *Varieties of special Jordan algebras of almost polynomial growth*, J. Algebra **531** (2019), 184–196.

[28] A. Regev, *Existence of identities in* $A \otimes B$, Israel J. Math. **11** (1972), 131–152.

[29] C. Rizzo, *The Grassmann algebra and its differential identities*, Algebr. Represent. Theory **23** (2020), n. 1, 125–134.

[30] C. Rizzo, *Growth of differential identities*, Polynomial Identities in Algebras, Springer INdAM Series **44** (2021), 383–400.

[31] C. Rizzo, R.B. dos Santos, A.C. Vieira, *Differential identities and polynomial growth of the codimensions*, preprint.

[32] A. Valenti, *Group graded algebras and almost polynomial growth*, J. Algebra **334** (2011), 247–254.

Fabrizio Martino
Dipartimento di Matematica e Informatica, Università degli Studi di Palermo, via Archirafi 34, 90123, Palermo, Italy
E-mail address: `fabrizio.martino@unipa.it`

Carla Rizzo
CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal
E-mail address: `carlarizzo@mat.uc.pt`