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A HESSIAN-DEPENDENT FUNCTIONAL WITH FREE BOUNDARIES AND APPLICATIONS TO MEAN-FIELD GAMES

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ABSTRACT: We study a Hessian-dependent functional driven by a fully nonlinear operator. The associated Euler-Lagrange equation is a fully nonlinear mean-field game with free boundaries. Our findings include the existence of solutions to the mean-field game, together with Hölder continuity of the value function and improved integrability of the density. In addition, we derive a free boundary condition and prove that the reduced free boundary is a set of finite perimeter. To conclude our analysis, we investigate a Γ-convergence scheme for the functional.

KEYWORDS: Hessian-dependent functionals, fully nonlinear mean-field games with free boundaries, regularity theory.
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1. Introduction

We examine Hessian-dependent functionals of the form

$$\mathcal{F}_{\Lambda,p}[u] := \int_{B_1} F(D^2u)^p dx + \Lambda|\{u > 0\} \cap B_1|, \tag{1}$$

where $F : S(d) \to \mathbb{R}$ is a uniformly elliptic operator, $\Lambda > 0$ is a fixed constant, and $p > d/2$. Our results include the existence of minimizers for (1), amounting to the existence of solutions to a fully nonlinear mean-field game with free boundaries. We prove Hölder-continuity of minimizers and improved integrability of the density. We also produce a free boundary condition. Finally, we establish a result on the Γ-convergence of $\mathcal{F}_{\Lambda,p}$ and examine its consequences.

The functional in (1) is inspired by the usual one-phase Bernoulli problem, driven by the Dirichlet energy. To a limited extent, we understand $\mathcal{F}_{\Lambda,p}$ as a Hessian-dependent counterpart of that problem. See [7]; see also [19].

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The analysis of (1) relates closely with the system

$$\begin{cases} F(D^2u) = m^{\frac{1}{p-1}} & \text{in} \quad B_1 \cap \{u > 0\} \\ \left(F_{i,j}(D^2u)m\right)_{x_i x_j} = 0 & \text{in} \quad B_1 \cap \{u > 0\}, \end{cases} \tag{2}$$

where $F_{i,j}(M)$ denotes the derivative of F with respect to the entry $m_{i,j}$ of M. Here, the unknown is a pair (u, m) solving the problem in a sense we make precise further.

The system in (2) amounts to the Euler-Lagrange equation associated with (1). We notice that (2) satisfies an *adjoint structure*. Its double-divergence equation is the formal adjoint, in the L^2-sense, of the linearized fully nonlinear problem. Due to such a distinctive pattern, we refer to (2) as a fully nonlinear mean-field game with free boundary. The profession has largely studied fully nonlinear elliptic operators as well as equations in the double-divergence form. An attempt to put together a comprehensive list of references on those topics is unrealistic. For that reason, we mention solely the monographs [16, 17] and the references therein.

Our analysis sits at the intersection of Hessian-dependent functionals, free boundary problems, and mean-field games systems. Hence we proceed with some context on those classes of problems. Hessian-dependent functionals play an essencial role in various contexts. From a purely mathematical viewpoint, they are useful to produce examples of conformally invariant energies. In dimension $d = 4$, this is the case of

$$J[u] := \int_{B_1} (\Delta u)^2 \, \mathrm{d}x,$$

whose Euler-Lagrange equation is the biharmonic operator; see [31, 30].

Another example concerns Hessian-dependent perturbations of nonconvex functionals. For instance, consider the nonconvex problem

$$I[u] := \int_{B_1} \left(|Du|^2 + 1\right)^2 \mathrm{d}x.$$

To obtain information on $I[\,\cdot\,]$ one can study the perturbed model

$$I_\varepsilon[u] := \int_{B_1} \left(|Du|^2 + 1\right)^2 + \varepsilon \|D^2u\| \mathrm{d}x,$$

for $0 < \varepsilon \ll 1$. Known as Aviles-Giga functional, $I_\varepsilon[\,\cdot\,]$ is convex with respect to higher-order terms; by taking the limit $\varepsilon \to 0$, one expects to derive information on the original model [10, 9]; see also [47]. An interesting feature of this limit is the appearance of further complexities, known as microstructures.

When it comes to applications, we mention the realm of mechanics of solids. In particular, the analysis of energy-driven pattern formation and nonlinear elasticity. For example, Hessian-dependent models play a role in studying the occurrence of wrinkles in a twisted ribbon [48]. The energy modeling the system depends on the thickness of the ribbon, denoted with h, and two symmetric tensors M and B. It has the form

$$\int_{B_1} |M(u,v)|^2 + h^2|B(u,v)|^2 \mathrm{d}x.$$

Although M depends on its arguments only through lower-order terms, the tensor B depends on $\|D^2u\|$. Another instance where Hessian-dependent functionals appear is the analysis of blister patterns in thin films on compliant substrates [12]. Here the phenomena are modeled in terms of lower-order quantities, a small Hessian-dependent perturbation, and a parameter $h > 0$. An important question concerning this class of problems is the limiting behavior $h \to 0$; in fact, one expects that the lower and upper bounds of the functional scale similarly. We refer the reader to [36, 37, 53]. In this context, (1) amounts to an energy penalized by the measure of the positive phase.

As noticed before, one can state the Euler-Lagrange equation associated with (1) in terms of the fully nonlinear mean-field game system with free boundaries (2). Mean-field games is a set of methods and techniques to model strategic interactions involving many players [49, 50, 51]; see also [52]. At the intersection of partial differential equations (PDE), stochastic analysis and numerical methods, this class of problems has attracted the attention of several authors, who have developed the theory in various directions.

A mean-field game can be characterized by its master equation. A Hamilton-Jacobi equation on the space of probability measures, this object encodes all the information on the problem under analysis. For recent developments concerning this equation, we refer the reader to [39, 40, 27, 14, 28, 22, 15, 42]. Under additional assumptions on the stochastic dynamics governing the problem (e.g., independence of the Brownian motions among the population of players), it is possible to write the master equation in terms of a coupling. It comprises a Hamilton-Jacobi equation, accounting for the value of the game, and a Fokker-Planck equation describing the evolution of the population. For the existence and uniqueness of solutions in this setting, we refer the reader to [24, 21, 25, 26, 23, 52, 44, 45, 43, 34, 35, 11, 29]; advances in the numerical approach have been reported in [1, 6, 2, 3, 5] and the references therein. We also mention the monographs [20, 13, 46, 4]. Fully nonlinear mean-field games are the subject of [8, 32].

The interesting aspect in (2) concerns the appearance of a free boundary. At least heuristically, the game is played only in the regions where the value function is strictly positive. Combined with the free boundary condition, (2) models a game in which players optimize in the region where the value function is positive and might face extinction according to a flux condition endogenously determined.

Our first contribution is to prove the existence of solutions for the mean-field game system in (2). We report our findings in the following theorem.

Theorem 1 (Existence and regularity of solutions)**.** *Suppose Assumptions A2, A3, and A4, to be detailed further, are in force. Then there exists a solution (u, m) to (2). In addition, fix $\alpha \in (0, 1)$. We have $u \in C^{\alpha}_{loc}(B_1)$ and there exists $C > 0$ such that*

$$\|u\|_{C^{\alpha}(B_{1/2})} \leq C\|g\|_{W^{2,p}(B_1)}.$$

The constant $C > 0$ depends on the exponent α.

If, in addition, F is strictly convex and $p > 2$, we can prove that m is not only integrable but is indeed an $L^{\frac{p}{p-1}}$-function, with estimates; c.f. [41]. To establish Theorem 1, we start with the direct method in the calculus of variations and the existence of minimizers for (1). Then we turn our attention to (2). First, we resort to the theory of weak solutions available for equations in the double-divergence form. Finally, elements in the L^p-viscosity theory lead to the existence of solutions to the system. To complete the proof, we resort to a delicate application of Sobolev inequalities.

Once we have established the existence of solutions for (2) and produced a regularity result, we examine the free boundary. We resort to a variation of the functional and derive a free boundary condition. Then regularity results for the solutions build upon ingredients of geometric measure theory to ensure the reduced free boundary is a set of finite perimeter. We summarize our findings in this direction in the following result.

Theorem 2 (Free boundary condition and finite perimeter)**.** *Let $u \in W^{2,p}_{loc}(B_1) \cap W^{1,p}_{g}(B_1)$ be a minimizer for (1), for $p > d/2$. Suppose Assumptions A3 and*

A4, to be detailed further, are in force. Then $\partial^*\{u>0\}$ is a set of finite perimeter. Suppose in addition $u \in C^2(B_1)$; then

$$\int_{\partial\{u>0\}} \left(F(D^2u)^{p-1} F_{ij}(D^2u)_{x_i} u_{x_j} - \frac{\Lambda}{2p} \right) \langle \xi, \nu \rangle \, d\mathcal{H}^{d-1} = 0 \tag{3}$$

for every $\xi \in C_c^\infty(B_1, \mathbb{R}^d)$.

The remainder of this paper is organized as follows. Section 2.1 details our main assumptions, whereas Section 2.2 gathers preliminary material and results. Section 3 presents the proof of Theorem 1. In Section 4, we examine the free boundary and put forward the proof of Theorem 2. A final section closes the paper with a Γ-convergence result and some consequences.

2. Preliminaries

This section presents the main assumptions under which we work and collects some preliminary notions and results.

2.1. Main assumptions. We proceed with a condition on the uniform ellipticity of the operator F.

A 1 (Uniform ellipticity)**.** *We suppose the operator $F : S(d) \to \mathbb{R}$ is λ-elliptic for some $\lambda \geq 1$. That is,*

$$\frac{1}{\lambda}\|N\| \leq F(M+N) - F(M) \leq \lambda\|N\| \tag{4}$$

for every $M, N \in S(d)$, with $N \geq 0$. We also suppose $F(0) = 0$

Remark 1. *Note that A1 implies a coercivity condition on F over non-negative matrices. By taking $M \equiv 0$, the inequalities in (4) yield*

$$\frac{1}{\lambda}\|N\| \leq F(N) \leq \lambda\|N\|$$

for every $N \geq 0$.

Next, we impose a convexity condition on the operator F.

A 2 (Convexity of the operator F)**.** *We suppose the operator $F = F(M)$ to be convex with respect to M.*

Part of our arguments requires F to satisfy a coercivity condition in the entire $S(d)$. To that end, we strength A1 as follows.

A 3 (Growth condition)**.** *We suppose there exists* $\lambda \geq 1$ *such that the operator* F *satisfies*

$$\frac{1}{\lambda}\|M\| \leq F(M) \leq \lambda\|M\|$$

for every $M \in S(d)$*. In addition,* $F(0) = 0$*.*

We conclude this section with an assumption on the boundary data.

A 4 (Boundary data)**.** *We suppose the function* $g \in W^{2,p}_{loc}(B_1)$ *is non-negative and non-trivial.*

2.2. Preliminary notions and results. For the sake of completeness, we recall definitions and former results we use throughout the manuscript. We continue with a definition

Definition 1 (Affine Sobolev spaces)**.** *Let* $p > d/2$ *and* $g \in W^{2,p}_{loc}(B_1)$*. We say that*

$$u \in W^{2,p}_{loc}(B_1) \cap W^{1,p}_g(B_1)$$

if $u \in W^{2,p}_{loc}(B_1)$ *and* $u - g \in W^{1,p}_0(B_1)$*.*

From a PDE perspective, having $u \in W^{2,p}_{loc}(B_1) \cap W^{1,p}_g(B_1)$ is tantamount to prescribe $u = g$ on ∂B_1 in the Sobolev sense. This interpretation will be helpful when relating (1) and (2).

As usual in the literature on mean-field games [49, 50, 51], a solution to (2) relies on two distinct definitions – namely, the notions of viscosity and weak (distributional) solutions. We proceed by recalling the definition of L^p-viscosity solution of a fully nonlinear elliptic equation; see [18, Definition 2.1].

Definition 2 (L^p-viscosity solutions)**.** *Let* $F : S(d) \to \mathbb{R}$ *be a fully nonlinear operator satisfying A1 and* $f \in L^p_{loc}(B_1)$*, for* $p > d/2$*. A function* $u \in C(B_1)$ *is an* L^p*-viscosity sub-solution (resp. super-solution) of*

$$F(D^2u) = f \quad in \quad B_1$$

if, for all $\varphi \in W^{2,p}_{\text{loc}}(B_1)$*, whenever* $\varepsilon > 0$*,* $U \subset B_1$ *is open, and*

$$F\left(D^2\varphi(x)\right) - f(x) \geq +\varepsilon \quad a.e. - x \in U$$
$$(resp.\ F\left(D^2\varphi(x)\right) - f(x) \geq -\varepsilon \quad a.e. - x \in U),$$

then $u - \varphi$ *cannot have a local maximum (resp. minimum) in* U*. Moreover, if* u *is both an* L^p*-viscosity sub-solution and an* L^p*-viscosity super-solution,* u *is said to be an* L^p*-viscosity solution.*

The definition of L^p-viscosity solution is necessary since L^p-functions might not be defined at the points where the usual conditions must be tested. For a comprehensive account of this notion, we refer the reader to [18]. We continue with the definition of weak solutions for double-divergence equations.

Definition 3 (Weak solution). *Let $A \in L^\infty(B_1, S(d))$ and denote $A(x) =: [a_{i,j}(x)]_{i,j=1}^d$. Suppose*

$$\frac{1}{\lambda} I \leq A(x) \leq \lambda I \quad a.e. - x \in B_1.$$

We say $m \in L^1(B_1)$ is a weak solution to

$$\left(a_{i,j}(x)m\right)_{x_i x_j} = 0 \quad in \quad B_1$$

if, for every $\phi \in C_c^\infty(B_1)$ we have

$$\int_{B_1} \left(a_{i,j} m\right) \phi_{x_i x_j} \mathrm{d}x = 0.$$

A solution to the mean-field game in (2) combines Definitions 2 and 3.

Definition 4 (Solution for the MFG system). *The pair (u, m) is a weak solution to (2) if the following hold:*

(1) We have $u \in C(B_1) \cap W_g^{1,p}$ and $m \in L^1(B_1)$, with $m \geq 0$;
(2) The function u is an L^p-viscosity solution to

$$F(D^2 u) = m^{\frac{1}{p-1}} \quad in \quad B_1 \cap \{u > 0\};$$

(3) The function m is a weak solution to

$$\left(F_{ij}(D^2 u) m\right)_{x_i x_j} = 0 \quad in \quad B_1 \cap \{u > 0\}.$$

Next, we recall the Poincaré's inequality for functions lacking compact support. In particular, we are interested in $u \in W_g^{1,p}(B_1)$.

Lemma 1 (Poincaré's inequality). *Let $u \in W_g^{1,p}(B_1)$ and $C_p > 0$ be the Poincaré's constant associated with $L^p(B_1)$ and the dimension d. Then for every $C < C_p$, there exists $C_1(C, C_p) > 0$ and $C_2 \geq 0$ such that*

$$\int_{B_1} |Du|^p dx - C \int_{B_1} |u|^p dx + C_2 \geq C_1 \left(\int_{B_1} |Du|^p dx + \int_{B_1} |u|^p dx \right).$$

For the detailed proof of this fact, we refer the reader to [38, Lemma 2.7, p. 22]. It follows from $u-g\in W^{1,p}_0(B_1)$ and the usual Poincaré's inequality.

In Section 3, we deal with the existence of minimizers for $\mathcal{F}_{\Lambda,p}$ in $W^{2,p}_{loc}(B_1)\cap W^{1,p}_g(B_1)$. Our reasoning uses the weak lower-semicontinuity of the functional

$$u\mapsto\mathcal{F}_{0,p}[u]:=\int_{B_1}\left(F(D^2u)\right)^p\mathrm{d}x;$$

this is the content of the following lemma.

Lemma 2. *Let $p>d/2$ and suppose A2, A3 and A4 hold true. Let $(u_n)_{n\in\mathbb{N}}\subset W^{2,p}_{loc}(B_1)\cap W^{1,p}_g(B_1)$ be such that*

$$D^2u_n\rightharpoonup D^2u_\infty\quad in\quad L^p\left(B_1,S(d)\right),$$

Then,

$$\int_{B_1}\left(F(D^2u_\infty)\right)^p\mathrm{d}x\le\liminf_{n\to\infty}\int_{B_1}\left(F(D^2u_n)\right)^p\mathrm{d}x$$

For the proof of Lemma 2, we refer to [8, Proposition 3]. In what follows, we detail the proof of Theorem 1.

3. Existence of solutions

In this section, we present the proof of Theorem 1; we start by establishing the existence of minimizers for (1).

Proposition 1 (Existence of minimizers)**.** *Suppose Assumptions A2, A3, and A4 are in force and fix $p>d/2$, arbitrary. Then there exists $u^*\in W^{2,p}_{loc}(B_1)\cap W^{1,p}_g(B_1)$ such that*

$$\mathcal{F}_{\lambda,p}[u^*]\le\mathcal{F}_{\lambda,p}[u],$$

for all $u\in W^{2,p}_{loc}(B_1)\cap W^{1,p}_g(B_1)$.

Proof: Under Assumptions A2 and A3, the existence of minimizers follows from the direct method in the calculus of variations. We split the argument into three steps.

Step 1 - We first examine

$$\gamma:=\inf\left\{\mathcal{F}_{\Lambda,p}[u]:u\in W^{2,p}_{loc}(B_1)\cap W^{1,p}_g(B_1)\right\}.$$

In view of the Remark 1, $\gamma \geq 0$. Furthermore, since $g \in W^{2,p}_{loc}(B_1)$,

$$\begin{aligned} \gamma &\leq \mathcal{F}_{\Lambda,p}[g] \\ &\leq \int_{B_1} \big(F(D^2 g)\big)^p \,\mathrm{d}x + \Lambda\,|B_1| \\ &\leq \lambda^p \|D^2 g\|^p_{L^p(B_1)} + \Lambda\,|B_1|\,. \end{aligned}$$

Hence, $0 \leq \gamma \leq C(g,\Lambda) < \infty$. Let $(u_n)_{n\in\mathbb{N}} \subset W^{2,p}_{loc}(B_1) \cap W^{1,p}_g(B_1)$ be a minimizing sequence; there exists $N \in \mathbb{N}$ such that

$$\mathcal{F}_{\Lambda,p}[u_n] \leq \gamma + 1,$$

for every $n \geq N$. Therefore, for all $n \geq N$,

$$\begin{aligned} \left\|D^2 u_n\right\|_{L^p(B_1)} &\leq \lambda \left(\int_{B_1} \big(F(D^2 u)\big)^p \,\mathrm{d}x\right)^{\frac{1}{p}} \\ &\leq \lambda \left(\int_{B_1} \left[F\left(D^2 u_n\right)\right]^p dx + \Lambda\,|\{u > 0\} \cap B_1|\right)^{\frac{1}{p}} \\ &\leq C(\gamma, p). \end{aligned}$$

In the next step the upper bound for $D^2 u_n$ builds upon properties of the functional.

Step 2 - As a consequence of the former inequality, we infer that $\left(D^2 u_n\right)_{n\in\mathbb{N}}$ is uniformly bounded in $L^p(B_1)$. Since $p > d/2$, the embedding $W^{2,p}(B_1) \hookrightarrow W^{1,p}(B_1)$ is compact. Furthermore, we conclude that $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $W^{2,p}_{loc}(B_1) \cap W^{1,p}_g(B_1)$; it follows from Lemma 1 combined with general facts [33]. Hence, there exists $u_\infty \in W^{2,p}_{loc}(B_1) \cap W^{1,p}_g(B_1)$ such that

$$u_n \rightharpoonup u_\infty \text{ in } W^{2,p}_{loc}(B_1) \cap W^{1,p}_g(B_1) \tag{5}$$

and

$$u_n \to u_\infty \text{ strongly in } L^p(B_1). \tag{6}$$

The result follows at once if we ensure that

$$\int_{B_1} \big(F(D^2 u_\infty)\big)^p \,\mathrm{d}x \leq \liminf_{n\to\infty} \int_{B_1} \big(F(D^2 u_n)\big)^p \,\mathrm{d}x \tag{7}$$

and

$$|\{u_\infty > 0\} \cap B_1| \leq \liminf_{n\to\infty} |\{u_n > 0\} \cap B_1| \tag{8}$$

hold. Notice that Lemma 2 combines the convergence mode in (5) to yield (7). In the sequel, we establish (8).

Step 3 - Because of the strong convergence (6), there exists a subsequence, also denoted with $(u_n)_{n\in\mathbb{N}}$, such that $u_n(x) \to u_\infty(x)$ for almost every x in B_1. As a consequence, we get

$$\chi_{\{u_\infty>0\}}(x) \leq \liminf_{n\to\infty} \chi_{\{u_n>0\}}(x) \tag{9}$$

for almost every $x \in B_1$. If (9) fails to hold, there exists $x^* \in B_1$ such that $u_\infty(x^*) > 0$ and $u_n(x^*) \leq 0$, for $n \gg 1$. This fact contradicts the pointwise convergence. Hence,

$$\begin{aligned}|\{u_\infty > 0\} \cap B_1| &= \int_{B_1} \chi_{\{u_\infty>0\}}\mathrm{d}x \\ &\leq \liminf_{n\to\infty} \int_{B_1} \chi_{\{u_n>0\}}\mathrm{d}x \\ &= \liminf_{n\to\infty} |\{u_n > 0\} \cap B_1|,\end{aligned}$$

which completes the proof. ■

We close this section with the proof of Theorem 1.

Proof of Theorem 1: We split the proof into four steps.

Step 1 - Let $u^* \in W^{2,p}_{loc}(B_1) \cap W^{1,p}_g(B_1)$ be the minimizer for (1) whose existence follows from Proposition 1. There exists $\mathcal{N} \subset B_1$ such that $D^2u^*(x)$ is well-defined for every $x \in B_1 \setminus \mathcal{N}$, with $|\mathcal{N}| = 0$. This fact, combined with A3, implies that $F(D^2u^*(x)) \geq 0$ for almost every $x \in B_1$. Therefore, u^* satisfies $F(D^2u^*) \geq 0$ in the L^p-viscosity sense; see [18, Lemma 2.6]. Because $g \geq 0$, it follows that $u^* \geq 0$ in B_1.

Step 2 - By considering a variation of u^* compactly supported in $B_1 \cap \{u > 0\}$, we obtain

$$\int_{B_1\cap\{u>0\}} \left(F_{ij}(D^2u^*)F(D^2u^*)^{p-1}\right) \varphi_{x_ix_j}\mathrm{d}x = 0 \tag{10}$$

for every $\varphi \in C^\infty_c(B_1 \cap \{u > 0\})$. Set $F(D^2u^*) =: m^{\frac{1}{p-1}}$; we infer that $m(x)$ is well-defined and satisfies $m(x) \geq 0$ for almost every $x \in B_1$. In addition,

$$\int_{B_1} m(x)\mathrm{d}x \leq C + \int_{B_1} F(D^2g)^p\mathrm{d}x;$$

that is, $m \in L^1_{loc}(B_1)$. Finally, we notice the integral in (10) is well-defined and leads to

$$\int_{B_1 \cap \{u>0\}} \big(F_{ij}(D^2u^*)m\big)\, \varphi_{x_ix_j} \mathrm{d}x = 0,$$

for every $\varphi \in C^\infty_c(B_1 \cap \{u > 0\})$.

Step 3 - It remains to check that u^* is an L^p-viscosity solution to the first equation in (2). The definition of m implies that u^* satisfies

$$F(D^2u^*(x)) = m(x)^{\frac{1}{p-1}}$$

for almost every $x \in B_1 \cap \{u > 0\}$. As before, an application of [18, Lemma 2.6] ends the proof.

Step 4 - We prove that $Du \in L^r(B_1)$ for every $1 < r < \infty$. We start by recalling the Gagliardo-Nirenberg inequality for bounded domains. If $u \in W^{2,p}_{loc}(B_1) \cap W^{1,p}_g(B_1)$ is a minimizer for (1), there exists $C_1, C_2 > 0$ such that

$$\|Du\|_{L^r(B_1)} \leq C_1(\Lambda, d)\left[\left(1 + \|D^2g\|^{\alpha}_{L^p(B_1)}\right)\|u\|^{1-\alpha}_{L^{q_1}(B_1)}\right] + C_2\|u\|_{L^{q_2}(B_1)}, \quad (11)$$

provided

$$\frac{1}{r} = \frac{1}{d} + \left(\frac{1}{p} - \frac{2}{d}\right)\alpha + \frac{1-\alpha}{q_1} \quad (12)$$

for some $1/2 < \alpha < 1$ and $q_2 > 0$. Given $d \geq 2$, $p > d/2$, and $1 < r < \infty$. it is always possible to find $\alpha \in (1/2, 1)$ and $q_1 > 1$ such that (12) is satisfied. Because $F(D^2u) \geq 0$, the Harnack inequality implies that u is essentially bounded in the unit ball. Hence, (11) becomes

$$\|Du\|_{L^r(B_1)} \leq C(\lambda, d, \Lambda, g)$$

and a straightforward application of Morrey's Theorem completes the proof. ■

Remark 2 (Improved integrability for m)**.** *In case F is strictly convex and $p > 2$, we claim that $m \in L^{\frac{p}{p-1}}(B_1)$. In fact, m is defined almost everywhere in B_1 as $m = F(D^2u)^{p-1}$. Under the strict convexity of F and $p > 2$, solutions to the Euler-Lagrange equation are minimizers for the functional* (1)*. Hence, A3 transmits the integrability of $D^2u \in L^p(B_1)$ to m, and the claim follows. Compare with* [41]*; see also* [16]*. Re-writing the exponent above as $1+1/(p-1)$ we quantify the improved integrability of m in face of the L^1-regime.*

Remark 3 (Improved regularity for the value function)**.** *The value function is α-Hölder-continuous, for every $\alpha \in (0,1)$. Hence, the regularity established in the former argument amounts to an improvement of the usual Krylov-Safonov regularity theory implied by uniform ellipticity.*

4. Information on the free boundary

In the sequel, we examine some properties of the free boundary $\partial\{u > 0\}$ and present the proof of Theorem 2. The following corollary connects the regularity of minimizers with information on the free boundary. We refer to it when proving the first part of Theorem 2.

Corollary 1. *Let $u \in W^{2,p}_{loc}(B_1) \cap W^{1,p}_g(B_1)$ be a minimizer to $\mathcal{F}_{\Lambda,p}$, where $p > d/2$. Suppose A3 holds true. For every $\varepsilon > 0$ there exists $C > 0$ such that*

$$\int_0^\varepsilon \mathcal{H}^{d-1}\left(\partial^*\{u > t\}\right) \mathrm{d}t \leq \varepsilon C.$$

Moreover, $C = C(\lambda, \Lambda, p)$.

Proof: We split the argument into two steps and begin by proving that

$$\int_{B_1 \cap \{0 \leq u \leq \varepsilon\}} \left(F(D^2u)\right)^p \mathrm{d}x < \varepsilon. \tag{13}$$

Step 1 - Given $\varepsilon > 0$ define

$$u_\varepsilon := \max(u - \varepsilon, 0).$$

Then

$$\int_{B_1 \cap \{u>0\}} \left(F(D^2u)\right)^p \mathrm{d}x + \Lambda \left|\{u > 0\} \cap B_1\right| \leq \mathcal{F}_{\Lambda,p}[u] \leq \mathcal{F}_{\Lambda,p}[u_\varepsilon].$$

Hence, for every $\varepsilon > 0$,

$$\int_{B_1 \cap \{u>0\}} \left(F(D^2u)\right)^p \mathrm{d}x - \int_{B_1} \left(F(D^2u_\varepsilon)\right)^p \mathrm{d}x + \Lambda \left|\{0 \leq u \leq \varepsilon\} \cap B_1\right| \leq 0 < \varepsilon.$$

Because of A1, $F(0) = 0$; as a consequence,

$$F(D^2u_\varepsilon) := \begin{cases} F(D^2u) & \text{in} \quad \{u > \varepsilon\} \\ 0 & \text{in} \quad \{0 < u \leq \varepsilon\}. \end{cases}$$

Therefore

$$
\begin{aligned}
\int_{B_1\cap\{0\le u\le\varepsilon\}} \left(F(D^2u)\right)^p \mathrm{d}x &= \int_{B_1\cap\{u>0\}} \left(F(D^2u)\right)^p \mathrm{d}x \\
&\quad - \int_{B_1\cap\{u>\varepsilon\}} \left(F(D^2u_\varepsilon)\right)^p \mathrm{d}x \\
&\le \int_{B_1\cap\{u>0\}} \left(F(D^2u)\right)^p \mathrm{d}x - \int_{B_1} \left(F(D^2u_\varepsilon)\right)^p \mathrm{d}x,
\end{aligned}
$$

and (13) follows.

Step 2 - Now, Assumption A1 yields

$$
\frac{1}{\lambda^p}\int_{B_1\cap\{0\le u\le\varepsilon\}} \left|D^2u\right|^p \mathrm{d}x \le \int_{B_1\cap\{0\le u\le\varepsilon\}} \left(F(D^2u)\right)^p \mathrm{d}x.
$$

Since $p>d/2$, there exists a universal constant $C_1 := C_1(p,d)$ such that

$$
\int_{B_1\cap\{0\le u\le\varepsilon\}} |Du|^p \,\mathrm{d}x \le C_1 \int_{B_1\cap\{0\le u\le\varepsilon\}} \left|D^2u\right|^p \mathrm{d}x.
$$

Also,

$$
\left(\int_{B_1\cap\{0\le u\le\varepsilon\}} |Du|\,\mathrm{d}x\right)^p \le \left|\{0\le u\le\varepsilon\}\cap B_1\right|^{p(p-1)} \int_{B_1\cap\{0\le u\le\varepsilon\}} |Du|^p \,\mathrm{d}x.
$$

By combining the former inequalities, we get

$$
\int_{B_1\cap\{0\le u\le\varepsilon\}} |Du|\,\mathrm{d}x < \varepsilon C\left|\{0\le u\le\varepsilon\}\cap B_1\right|^{p-1},
$$

where $C := C_1^p\lambda\Lambda^{\frac{1}{p}}$. A straightforward application of the area formula yields

$$
\int_0^\varepsilon \mathcal{H}^{d-1}\left(\partial^*(\{u>t\})\right)\mathrm{d}t \le \varepsilon C\left|\{0\le u\le\varepsilon\}\cap B_1\right|^{p-1}
$$

and finishes the proof. ■

In what follows, we examine the first variation of the functional (1).

4.1. First variation of the energy. To derive the free boundary condition, we consider a variation of $\mathcal{F}_{\Lambda,p}$. Given a compactly supported and smooth vector field $\xi : B_1 \to \mathbb{R}^d$ and $0 < t \ll 1$, we consider two structures: the diffeomorphism $\Psi_t(x) := x + t\xi(x)$ and the test function $u_t := u \circ \Psi_t^{-1}$. The stationarity of u implies

$$0 = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{F}_{\Lambda,p}[u_t]. \tag{14}$$

The free boundary condition follows from (14). Next, we detail its building blocks.

Lemma 3. *Let $u \in W^{2,p}_{loc}(B_1) \cap W^{1,p}_g(B_1)$ be stationary for (1), for $p > d/2$. Let $\xi \in C^\infty_c(B_1, \mathbb{R}^d)$ and define $\Psi_t : B_1 \to \mathbb{R}^d$ as*

$$\Psi_t(x) := x + t\xi(x).$$

Then

(1) For small enough t, $\Psi_t : B_1 \to B_1$ is a diffeomorphism and by setting $\Phi_t := \Psi_t^{-1}$, the function $u_t = u \circ \Phi_t$ is well defined and belongs to $W^{2,p}_{loc}(B_1)$.

(2) In addition,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_{B_1} \left(F(D^2u_t(x))\right)^p \mathrm{d}x$$
$$= -d \int_{B_1} \left(F(D^2u(x))\right)^{p-1} F_{ij}\left(D^2u(x)\right) \left(\langle Du(x), \xi(x)\rangle\right)_{x_ix_j} \mathrm{d}x.$$

Proof: Since ξ is smooth and compactly supported in B_1, it follows that, for t small enough, Ψ_t is a diffeomorphism and $u_t \in W^{2,p}_{loc}(B_1)$. This leads to the first claim. To prove (ii), we first compute

$$\int_{B_1} \left(F(D^2u_t(x))\right)^p \mathrm{d}x,$$

where $u_t(x)$ is defined as in (i). Notice that

$$D^2u_t(x) = \left(D\Phi_t(x)\right)^T D^2u(\Phi_t)\left(D\Phi_t(x)\right) + Du(\Phi_t(x))D^2\Phi_t(x).$$

Now, denote

$$P(x) := \left(D\Phi_t(x)\right)^T D^2u(\Phi_t(x))\left(D\Phi_t(x)\right)$$

and

$$Q(x) := Du(\Phi_t(x))D^2\Phi_t(x).$$

Consider the change of variables $\Phi_t(x) = y$, which is tantamount to $x = \Psi_t(y)$. Hence

$$\int_{B_1} \left(F(D^2 u_t(x))\right)^p \mathrm{d}x = \int_{B_1} \left(F\left(P(\Psi_t(y)) + Q(\Psi_t(y))\right)\right)^p |\det \Psi_t(y)|\, dy.$$

For $0 < t \ll 1$,

$$\det \Psi_t(y) = \det(I(y) + t\xi(y)) = 1 + t \operatorname{div} \xi(y) + o(t).$$

Thus

$$\begin{aligned}\int_{B_1} \left(F(D^2 u_t(x))\right)^p \mathrm{d}x = &\int_{B_1} (F(P(\Psi_t(y)) + Q(\Psi_t(y))))^p \,\mathrm{d}y \\ &+ \int_{B_1} (F(P(\Psi_t(y)) + Q(\Psi_t(y))))^p \, t \operatorname{div} \xi(y) \mathrm{d}y \\ &+ o(t).\end{aligned} \tag{15}$$

By differentiating (15) with respect to t and evaluating at $t = 0$, we get

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_{B_1} \left(F(D^2 u_t(x))\right)^p \mathrm{d}x = &\int_{B_1} \left.\frac{\partial}{\partial t}\right|_{t=0} (F(P(\Psi_t(y)) + Q(\Psi_t(y))))^p \,\mathrm{d}y \\ &+ \int_{B_1} \left(F(D^2 u(y))\right)^p \operatorname{div} \xi(y) \mathrm{d}y.\end{aligned} \tag{16}$$

Since the first term on the right-hand side of (16) depends on $P(\Psi_t(y))$ and $Q(\Psi_t(y))$, we compute them directly. We begin with $P(\Psi_t(y))$; we have

$$P(\Psi_t(y)) = \left[(D\Psi_t(y))^{-1}\right]^T D^2 u(y) \left[(D\Psi_t(y))^{-1}\right]. \tag{17}$$

Moreover, notice that

$$\begin{aligned} Q(\Psi_t(y)) &= Du\left(\Phi_t(\Psi_t(y)) D^2 \Phi_t(\Psi_t(y))\right) \\ &= Du(y) D^2 \Phi_t(\Psi_t(y)) \end{aligned}$$

and

$$D^2\left(\Phi_t\left(\Psi(y)\right)\right) = -\left(D\Psi_t(y)\right)^{-1} D^2 \Psi_t(y) \left(D\Psi_t(y)\right)^{-2};$$

Then

$$Q(\Psi_t(y)) = -Du(y) \left(\left[D\Psi_t(y)\right]^{-1} D^2 \Psi_t(y) \left[D\Psi_t(y)\right]^{-2}\right). \tag{18}$$

Because $\Psi_t \equiv I + t\xi$, we get

$$D\Psi_t(x) = I + tD\xi(x) \quad \text{and} \quad D^2 \Psi_t(x) = tD^2 \xi(x).$$

As before, for $0 < t \ll 1$,

$$(I + tD\xi(y))^{-1} = I - tD\xi(y) + o(t).$$

Therefore,

$$\begin{aligned}
P(\Psi_t(y)) &= (I - tD\xi(y) + o(t))^T D^2u(y) (I - tD\xi(y) + o(t)) \\
&= D^2u(y) - t\left(D^2u(y)D\xi(y) + (D\xi(y))^T D^2u(y)\right) \\
&\quad + t^2 (D\xi(y))^T D^2u(y) (D\xi(y)) + o(t) \\
&= D^2u(y) - t\left(D^2u(y)D\xi(y) + (D\xi(y))^T D^2u(y)\right) + o(t).
\end{aligned}$$

Also,

$$\begin{aligned}
-Q(\Psi_t(y)) &= Du(y)\left((I - tD\xi(y) + o(t))\left(tD^2\xi(y)\right)(I - tD\xi(y) + o(t))^2\right) \\
&= Du(y)\left(\left(tD^2\xi(y) + o(t^2)\right)(I - 2tD\xi(y) + o(t))\right) \\
&= Du(y)\left(tD^2\xi(y) + o(t^2)\right) \\
&= tDu(y)D^2\xi(y) + o(t^2).
\end{aligned}$$

As a consequence,

$$(F(P(\Psi_t(y)) + Q(\Psi_t(y))))^p = \left(F\left(D^2u(y) - tM(y) + o(t)\right)\right)^p,$$

where

$$M(y) := D^2u(y)D\xi(y) + (D\xi(y))^T D^2u(y) + Du(y)D^2\xi(y).$$

By differentiating with respect to t, we get

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \left(F\left(D^2u_t(\Psi_t(y))\right)\right)^p = -p\left(F(D^2u(y))\right)^{p-1} F_{ij}\left(D^2u(y)\right)(M(y))_{ij}. \tag{19}$$

Combining (19) and (16), we obtain

$$\begin{aligned}
&\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_{B_1} \left(F(D^2u_t(x))\right)^p \mathrm{d}x \\
&\qquad = -p\int_{B_1} \left(F(D^2u(x))\right)^{p-1} F_{ij}\left(D^2u(x)\right)(M(x))_{ij}\,\mathrm{d}x \qquad (20)\\
&\qquad\quad + \int_{B_1} \left(F\left(D^2u(x)\right)\right)^p \operatorname{div}\xi(x)\mathrm{d}x,
\end{aligned}$$

Finally, notice that

$$
\begin{aligned}
(M(x))_{ij} &= \sum_{\ell=1}^{d} \left(u_{x_j x_\ell} \xi^i_\ell + u_{x_i x_\ell} \xi^j_{x_\ell} + u_{x_\ell} \xi^\ell_{x_i x_j} \right) \\
&= \sum_{\ell=1}^{d} \left(u_{x_j x_\ell} \xi^i_\ell + u_{x_i x_\ell} \xi^j_{x_\ell} + u_{x_\ell} \xi^\ell_{x_i x_j} + \left(u_{x_\ell x_i}\right)_{x_j} \xi^\ell - \left(u_{x_\ell x_i}\right)_{x_j} \xi^\ell \right) \\
&= \left(\langle Du, \xi \rangle\right)_{x_i x_j} - \sum_{\ell=1}^{d} \left(u_{x_\ell x_i}\right)_{x_j} \xi^\ell .
\end{aligned}
$$

Integrating by parts the second term in the right-hand side of (20), we recover

$$
\begin{aligned}
&\int_{B_1} \left(F\left(D^2 u(x)\right)\right)^p \operatorname{div} \xi(x) \mathrm{d}x \\
&\qquad = -p \int_{B_1} \left(F\left(D^2 u(x)\right)\right)^{p-1} F_{ij}\left(D^2 u(x)\right) \sum_{\ell=1}^{d} \left(u_{x_\ell x_i}\right)_{x_j} \xi^\ell \mathrm{d}x .
\end{aligned}
$$

Therefore, (20) becomes

$$
\begin{aligned}
&\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_{B_1} \left(F(D^2 u_t(x))\right)^p \mathrm{d}x \\
&\qquad = -d \int_{B_1} \left(F(D^2 u(x))\right)^{p-1} F_{ij}\left(D^2 u(x)\right) \left(\langle Du(x), \xi(x) \rangle\right)_{x_i x_j} \mathrm{d}x,
\end{aligned}
$$

which implies assertion (ii) and completes the proof. ■

Lemma 4. *Let $u \in W^{2,p}_{loc}(B_1) \cap W^{1,p}_g(B_1)$ be a minimizer for (1), with $p > d/2$. Consider*

$$
Y := \sum_{i,j=1}^{d} \left(\left(F(D^2 u)\right)^{d-1} F_{ij}(D^2 u) \right)_{x_i} \partial_j ,
$$

where $\partial_1, \ldots, \partial_d$ is an orthonormal basis of $T_x \partial\{u > 0\}$. If $u \in C^2(\{u > 0\})$, then

$$
\int_{B_1} \left(F(D^2 u)\right)^{p-1} F_{ij}(D^2 u) \langle Du, \xi \rangle_{x_i x_j} \mathrm{d}x = -2 \int_{\partial\{u>0\}} \langle \langle Du, \xi \rangle Y, \nu \rangle \, d\mathcal{H}^{d-1},
$$

where $\nu(x)$ is the outward normal vector to $\partial\{u > 0\}$ at x.

Proof: By applying integration by parts twice, we get

$$\begin{aligned}\int_{B_1}\left(F(D^2u)\right)^{p-1}F_{ij}(D^2u)\left(\langle Du,\xi\rangle_{x_ix_j}\right)\mathrm{d}x\\ =\int_{\partial\{u>0\}}\left(F(D^2u)\right)^{p-1}F_{ij}(D^2u)\left\langle Du,\xi\right\rangle_{x_i}\langle\partial_j,\nu\rangle\,d\mathcal{H}^{d-1}\\ -\int_{\partial\{u>0\}}\left(\left(F(D^2u)\right)^{p-1}F_{ij}(D^2u)\right)_{x_j}\langle Du,\xi\rangle_{x_i}\langle\partial_i,\nu\rangle\,d\mathcal{H}^{d-1}\\ +\int_{B_1}\left(\left(F(D^2u)\right)^{p-1}F_{ij}(D^2u)\right)_{x_ix_j}\langle Du,\xi\rangle\,\mathrm{d}x.\end{aligned}$$

Because u is regular enough,

$$\left(\left(F(D^2u)\right)^{p}F_{ij}(D^2u)\right)_{x_ix_j}=0.$$

Hence

$$\begin{aligned}\int_{B_1}\left(F(D^2u)\right)^{p-1}F_{ij}(D^2u)\left\langle Du,\xi\right\rangle_{x_ix_j}\mathrm{d}x=\int_{\partial\{u>0\}}\left\langle\langle Du,\xi\rangle_{x_i}X,\nu\right\rangle d\mathcal{H}^{d-1}\\ -\int_{\partial\{u>0\}}\langle\langle Du,\xi\rangle\,Y,\nu\rangle\,d\mathcal{H}^{d-1},\end{aligned}$$

where

$$X:=\sum_{i,j=1}^{d}\left(\left(F(D^2u)\right)^{p-1}F_{ij}(D^2u)\right)\partial_j.$$

As a consequence, the result follows if we ensure that

$$-\int_{\partial\{u>0\}}\left\langle\langle Du,\xi\rangle_{x_i}X,\nu\right\rangle d\mathcal{H}^{d-1}=\int_{\partial\{u>0\}}\langle\langle Du,\xi\rangle\,Y,\nu\rangle\,d\mathcal{H}^{d-1}. \tag{21}$$

To prove (21), notice that Y induces the exact form

$$\omega:=\sum_{ij=1}^{d}\left(\left(F(D^2u)\right)^{p-1}F_{ij}(D^2u)\right)_{x_i}\mathrm{d}x^1\wedge\cdots\wedge\hat{dx^j}\wedge\mathrm{d}x^d,$$

In fact, we can define

$$\eta:=\sum_{I\in\mathcal{J}_{d-2,d}}\left(F(D^u)\right)^{p-1}F_{ij}(D^2u)\mathrm{d}x^I,$$

where $\mathcal{J}_{d-2,d} := \{I = (i_1, \ldots, i_{d-2}) : 1 \leq i_1 < i_2 < \cdots < i_{d-2} \leq n\}$. Hence,

$$\int_\Gamma \langle Du, \xi\rangle \, d\eta = -\int_{\partial\{u>0\}} p\left(\langle Du, \xi\rangle\right) \wedge \eta.$$

We conclude that (21) holds and complete the proof. ■

Lemma 5. *Let $u \in W^{2,p}_{loc}(B_1) \cap W^{1,p}_g(B_1)$ be a minimizer for* (1)*, with $p > d/2$. Let $\xi \in C^\infty_c(B_1, \mathbb{R}^d)$ and define $\Psi_t(x) := x + t\xi(x)$ in B_1. Then the function $t \mapsto |\{u_t > 0\} \cap D|$ is differentiable at $t = 0$ and*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} |\{u_t > 0\} \cap B_1| = \int_{\{u>0\}\cap B_1} \operatorname{div}\xi \mathrm{d}x.$$

Proof: Because ξ is smooth and compactly supported in B_1, we immediately infer that $t \mapsto |\{u_t > 0\} \cap B_1|$ is differentiable at $t = 0$. It is also clear that $x \in \{u_t > 0\}$ if and only if $\Phi_t(x) \in \{u > 0\}$; hence $\chi_{\{u_t>0\}} = \chi_{\{u>0\}} \circ \Phi_t$. As a result,

$$|\{u_t > 0\}| = \int_{B_1} \chi_{\{u>0\}} \left(\Phi_t(x)\right) dx = \int_{B_1} \chi_{\{u>0\}}(y) \left|\det D\Psi_t(y)\right| dy.$$

Using the expression available for Ψ_t, we recover

$$\begin{aligned} |\{u_t > 0\}| &= \int_{\{u>0\}} (1 + t \operatorname{div}\xi(y) + o(t))\mathrm{d}y \\ &= |\{u > 0\}| + t\int_{\{u>0\}} \operatorname{div}\xi \mathrm{d}x + o(t), \end{aligned}$$

and the proof is complete. ■

4.2. Proof of Theorem 2. In what follows, we organize the previous results and present the proof of Theorem 2. The Sobolev regularity of minimizers and its corollary leads to the finite perimeter of the reduced free boundary. Furthermore, the first variation of the functional yields the free boundary condition.

Proof of Theorem 2: For convenience, we split the proof into two steps. We start with the finite perimeter of the reduced free boundary.

Step 1 - Because of Corollary 1, there exists a sequence $(\delta_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ of real numbers, with $\delta_n \to 0$, satisfying

$$\mathcal{H}^{d-1}(\partial^*(u > \delta_n)) \leq C,$$

for every $n \in \mathbb{N}$. Standard convergence results ensure that

$$\lim_{n\to\infty} \int_{B_1} \chi_{\{u>\delta_n\}} \mathrm{d}x = \int_{B_1} \chi_{\{u>0\}} \mathrm{d}x.$$

Finally, the lower semi-continuity of the perimeter implies

$$\mathcal{H}^{d-1}(\partial^*(\{u>0\})) \leq C$$

and yields the conclusion. We continue with the derivation of the free boundary condition.

Step 2 - To establish the free boundary condition we start by noticing that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{F}_{\Lambda,p}[u_t] = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_{B_1} \left(F(D^2u_t)\right)^p \mathrm{d}x + \Lambda \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left|\{u_t > 0\} \cap B_1\right|.$$

Putting Lemma 3 and (4.1) together we obtain

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{F}_{\Lambda,p}[u_t] = -p \int_{B_1} \left(F(D^2u(x))\right)^{p-1} F_{ij}\left(D^2u(x)\right)\left(\langle Du(x), \xi(x)\rangle\right)_{x_ix_j} \mathrm{d}x \\ + \Lambda \int_{\{u>0\}} \operatorname{div}\xi(x)\mathrm{d}x.\end{aligned}$$

Note also that

$$\int_{\{u>0\}} \operatorname{div}\xi(x)\mathrm{d}x = \int_{\partial\{u>0\}} \langle \xi, \nu\rangle \, d\mathcal{H}^{d-1}.$$

Due to Lemma 4,

$$\begin{aligned}&\int_{B_1} \left(F(D^2u)\right)^{p-1} F_{ij}(D^2u) \langle Du, \xi\rangle_{x_ix_j} \mathrm{d}x \\ &\qquad = -2\int_{\partial\{u>0\}} \left(\left(F(D^2u)\right)^{p-1} F_{ij}(D^2u)\right)_{x_i} \langle Du, \xi\rangle \langle \partial_j, \nu\rangle \, d\mathcal{H}^{d-1},\end{aligned}$$

where $\{\partial_1, \ldots, \partial_d\}$ is an orthonormal frame of $T(\partial\{u > 0\})$. Because $u = 0$ on $\partial\{u > 0\}$ and $u > 0$ in $\{u > 0\}$, we have that $Du = \nu|Du|$. Moreover,

$$\begin{aligned}
&\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{F}_{\Lambda,p}[u_t] \\
&\quad= \int_{\partial\{u>0\}} \left(2p\left(\left(F(D^2u)\right)^{p-1} F_{ij}(D^2u)\right)_{x_i} \langle \partial_j, Du\rangle + \Lambda\right) \langle \xi, \nu\rangle \, d\mathcal{H}^{d-1} \\
&\quad= \int_{\partial\{u>0\}} \left(2p\left(\left(F(D^2u)\right)^{p-1} F_{ij}(D^2u)\right)_{x_i} u_{x_j} + \Lambda\right) \langle \xi, \nu\rangle \, d\mathcal{H}^{d-1}.
\end{aligned}$$

Since u is a minimizer for (1) and ξ is arbitrary, the proof is complete. ■

5. Perturbation analysis via Γ-convergence

This section specializes the operator F to be the norm and considers small values of the parameter Λ in (1). We regard the functional

$$\mathcal{G}_{\Lambda,p}[v] := \int_{B_1\cap\{u>0\}} \|D^2v\|^p \mathrm{d}x + \Lambda|\{u > 0\} \cap B_1| \tag{22}$$

as a free boundary perturbation of

$$\mathcal{G}_{0,p}[v] := \int_{B_1} \|D^2v\|^p \mathrm{d}x. \tag{23}$$

Denote with u_Λ a minimizer for (22) and with u_0 the minimizer for (23). We are interested in the behavior of $(u_\Lambda)_{\Lambda>0}$, as $\Lambda \to 0$. In particular, we search for the topologies where the convergence $u_\Lambda \to u_0$ is available. Our starting point is a Γ-convergence result. Namely, we first prove that $\mathcal{G}_{\Lambda,p} \xrightarrow{\Gamma} \mathcal{G}_{0,p}$ as $\Lambda \to 0$. We proceed with some auxiliary lemmas.

Lemma 6 (Equicoerciveness). *Let $p > 1$ be fixed and $(\Lambda_n)_{n\in\mathbb{N}}$ be a sequence such that $\Lambda_n \to 0$, as $n \to \infty$. Define the functional $\mathcal{G}_{n,p} : L^p(B_1) \to \mathbb{R}$ as*

$$\mathcal{G}_{n,p}[v] := \int_{B_1} \|D^2v\|^p \mathrm{d}x + \Lambda_n|\{v > 0\} \cap B_1|$$

if $v \in W^{2,p}_{loc}(B_1)$, and $\mathcal{G}_{n,p}[v] := +\infty$ in case $v \in L^p(B_1) \setminus W^{2,p}_{loc}(B_1)$. Let $(u_m)_{m\in\mathbb{N}} \subset L^p(B_1)$ be such that

$$\mathcal{G}_{n,p}[u_m] \le C, \tag{24}$$

for every $m \in \mathbb{N}$ and some $C > 0$. Then $\|u_m\|_{W^{2,p}(B_1)} \le C$, uniformly in $m \in \mathbb{N}$, for some $C > 0$.

Proof: It follows from (24) that

$$\int_{B_1} \|D^2 u_m\|^p \mathrm{d}x \leq \mathcal{G}_{n,p}[u_m] \leq C.$$

By Lemma 1 and standard inequalities available for Sobolev spaces [33], there exists $C > 0$ such that

$$\|u_m\|_{W^{2,p}(B_1)} \leq C,$$

uniformly in $m \in \mathbb{N}$. ■

Before continuing, we introduce the functional $\mathcal{G}_{0,p} : L^p(B_1) \to \mathbb{R}$, given by

$$\mathcal{G}_{0,p}[v] := \int_{B_1} \|D^2 v\|^p \mathrm{d}x$$

if $v \in W^{2,p}_{loc}(B_1)$, and $\mathcal{G}_{0,p}[v] := +\infty$ if $v \in L^p(B_1) \setminus W^{2,p}_{loc}(B_1)$. The next lemma relates $\mathcal{G}_{n,p}$ and $\mathcal{G}_{0,p}$.

Lemma 7. *Let $p > 1$ be fixed and $(\Lambda_n)_{n\in\mathbb{N}}$ be a sequence of real numbers so that $\Lambda_n \to 0$, as $n \to \infty$. For each $u \in L^p(B_1)$ there exists a sequence $(u_n)_{n\in\mathbb{N}} \in L^p(B_1)$ converging strongly to u in $L^p(B_1)$, such that*

$$\lim_{n\to\infty} \mathcal{G}_{n,p}[u_n] = \mathcal{G}_{0,p}[u]. \tag{25}$$

Proof: Let $u \in L^p(B_1)$ be given and $u_n := u$, for every $n \in \mathbb{N}$. If $u \in L^p(B_1) \setminus W^{2,p}(B_1)$, we get

$$\mathcal{G}_{n,p}[u_n] = +\infty \quad \text{and} \quad \mathcal{G}_{0,p}[u] = +\infty,$$

and (25) is immediately satisfied. Conversely, suppose $u \in W^{2,p}_{loc}(B_1)$. In that case, we have

$$\lim_{n\to\infty} \mathcal{G}_{n,p}[u_n] = \int_{B_1} \|D^2 u\|^p \mathrm{d}x + \lim_{n\to\infty} \Lambda_n |\{u > 0\} \cap B_1| = \mathcal{G}_{0,p}[u]. \qquad ■$$

Lemma 8. *Let $p > 1$ be fixed and $(\Lambda_n)_{n\in\mathbb{N}}$ be a sequence of real numbers so that $\Lambda_n \to 0$, as $n \to \infty$. Given $(u_n)_{n\in\mathbb{N}} \subset L^p(B_1)$ and $u \in L^p(B_1)$, with $u_n \to u$ strongly in $L^p(B_1)$, we have*

$$\mathcal{G}_{0,p}[u] \leq \liminf_{n\to\infty} \mathcal{G}_{n,p}[u_n]. \tag{26}$$

Proof: To deduce (26) from the strong convergence, suppose first $(u_n)_{n\in\mathbb{N}} \subset L^p(B_1) \setminus W^{2,p}(B_1)$. Then

$$\int_{B_1} \|D^2 u_n\|^p \mathrm{d}x = +\infty$$

and (26) follows. Otherwise, suppose $(u_n)_{n\in\mathbb{N}} \subset W^{2,p}(B_1)$. Then

$$\mathcal{G}_{n,p}[u_n] \leq C,$$

for some $C > 0$. As a consequence, $\|D^2 u_n\|_{L^p(B_1)}$ is uniformly bounded; evoking once again standard inequalities for Sobolev functions, one infers the existence of a constant $C > 0$ such that

$$\|u_n\|_{W^{2,p}_{loc}(B_1)} \leq C.$$

The weakly lower semi-continuity of the L^p-norm yields

$$\mathcal{G}_{0,p}[u] = \int_{B_1} \|D^2 u\|^p \mathrm{d}x \leq \liminf_{n\to\infty} \int_{B_1} \|D^2 u_n\|^p \mathrm{d}x \leq \liminf_{n\to\infty} \mathcal{G}_{n,p}[u_n]$$

and completes the proof. ■

By combining Lemmas 6, 7, and 8, we derive the following theorem.

Theorem 3 (Gamma Convergence). *Let $p > d/2$ be fixed and $(\Lambda_n)_{n\in\mathbb{N}}$ be a sequence of real numbers so that $\Lambda_n \to 0$, as $n \to \infty$. Then $\mathcal{G}_{n,p} \xrightarrow{\Gamma} \mathcal{G}_{0,p}$.*

In the sequel, we explore a consequence of the Γ-convergence result. It consists of an approximation result by $C^{1,\alpha}$-regular functions.

5.1. Regular approximations. We have proved that minimizers for (1) are Hölder-continuous. However, the use of Γ-convergence allows us to arbitrarily approximate minimizers by $C^{1,\alpha}$-regular functions. This is the content of the following proposition

Proposition 2 ($C^{1,\alpha}$-approximation). *Let $p > d/2$ be fixed. Given $\delta > 0$, there exists $\varepsilon > 0$ such that if $\Lambda < \varepsilon$ one can find $h \in C^{1,\alpha}_{loc}(B_1)$ satisfying*

$$\|u - h\|_{W^{1,p}_g(B_1)} < \delta$$

Proof: We use a contradiction argument. Suppose the statement of the proposition is false. In this case, there exist a real number $\delta_0 > 0$ and sequences $(u_n)_{n\in\mathbb{N}}$ and $(\Lambda_n)_{n\in\mathbb{N}}$ such that

$$\Lambda_n \to 0$$

as $n \to \infty$,

$$\mathcal{G}_{n,p}[u_n] \leq \mathcal{G}_{n,p}[u]$$

for every $u \in W^{2,p}_{loc}(B_1) \cap W^{1,p}_g(B_1)$ and every $n \in \mathbb{N}$, but

$$\|u_n - h\|_{W^{1,p}_g(B_1)} > \delta_0, \tag{27}$$

for every $h \in C^{1,\alpha}_{loc}(B_1)$.

However,

$$\|u_n\|_{W^{2,p}(B_1)} \leq C \left(\|g\|_{W^{2,p}(B_1)} + 1 \right),$$

for some $C > 0$. Hence, there exists

$$u_\infty \in W^{2,p}_{loc}(B_1) \cap W^{1,p}_g(B_1)$$

such that u_n converges u_∞, weakly in $W^{2,p}(B_1)$ and strongly in $W^{1,p}_g(B_1)$. That is tantamount to say that u_∞ is an accumulation point for the sequence $(u_n)_{n\in\mathbb{N}}$.

Because of Theorem 3, we conclude that u_∞ is a minimizer for $\mathcal{G}_{0,p}$. Previous results in the literature ensure that $u_\infty \in C^{1,\alpha}_{loc}(B_1)$ [8]. By taking $h := u_\infty$ in (27), we get a contradiction and complete the proof. ∎

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