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ON THE CATEGORICAL BEHAVIOUR OF LOCALES AND *D*-LOCALIC MAPS

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Abstract: It was shown by Banaschewski and Pultr that the classical adjunction between Top and Loc restricts to an adjunction between the category Top_D of T_D-spaces and their continuous maps, and the category Loc_D of all locales and localic maps which preserve coveredness of primes. Despite the fact that Loc_D plays an important role in the T_D-duality, not much is known about its categorical structure, and it is the aim of this paper to fill this gap. In particular, we show that Loc_D is closed under finite products in Loc and moreover we characterize the existence of equalizers. As a consequence, it is proved that regular monomorphisms in Loc_D are precisely the D-sublocales — the notion analogue to sublocale in the T_D-duality — a situation akin to the standard fact that sublocales are precisely regular monomorphisms in Loc. The results are then applied to obtain the T_D-analogues of some familiar results for sober spaces and some new characterizations of T_D-spatiality of localic squares in terms of certain discrete covers of locales.

Keywords: Locale, covered prime, T_D-space, T_D-spatial locale, D-localic map, D-sublocale, product, equalizer.
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1. Introduction

It is well known that sobriety and the T_D-axiom somehow mirror each other (cf. [5, 18]). In fact, besides the classical adjunction

$$\begin{tikzcd} \mathsf{Top} \arrow[r, bend left, "\Omega"] \arrow[r, phantom, "\perp"] & \mathsf{Loc} \arrow[l, bend left, "\Sigma"] \end{tikzcd}$$

that yields the equivalence between sober spaces and spatial locales, there is also the adjunction

$$\begin{tikzcd} \mathsf{Top}_\mathsf{D} \arrow[r, bend left, "\Omega"] \arrow[r, phantom, "\perp"] & \mathsf{Loc}_D \arrow[l, bend left, "\Sigma'"] \end{tikzcd}$$

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introduced more recently by Banaschewki and Pultr [5]. Here, Top_D denotes the full subcategory of Top consisting of T_D-spaces and Loc_D is certain non-full subcategory of locales whose morphisms are D-localic maps (namely, localic maps sending covered primes into covered primes). The latter adjunction yields an equivalence between Top_D and the category of T_D-spatial locales and D-localic maps between them. This adjunction encodes important aspects of T_D-spaces, e.g., the fact that, similarly to sober spaces, T_D-spaces can also reconstructed from their lattices of open sets.

Despite the usefulness of the Banaschewski-Pultr adjunction in describing such phenomena concerning T_D-spatiality and related topics (see, e.g., [5, 6, 1, 2]), not much is known about the categorical structure of Loc_D. In this paper, we prove that finite products in Loc_D exist and are computed as in Loc, though infinite products may fail to exist. Moreover, we fully characterize the existence of equalizers thereby establishing relations with the D-sublocales from [2]. More precisely, we show that the D-sublocales are precisely the regular subobjects of Loc_D. Subsequently, we apply these results in order to obtain some T_D-analogues of well-known results for sober spaces. As another application, we give some new criteria of T_D-spatiality of localic squares in connection with the system of smooth sublocales of a locale — a system that recently has attracted attention in point-free topology [17, 9, 4, 3, 1].

2. Preliminaries

The main references on point-free topology are Johnstone [12] and the more recent Picado and Pultr [14]. The standard notation and terminology in the present paper follows that of [14].

Recall that a topological space X is T_D if for each $x \in X$ there is an open neighborhood U of x such that $U - \{x\}$ is open. Moreover, a locale L is T_D-spatial if $L \cong \Omega(X)$ for some T_D-space X (see [5]).

If L is a locale, for each $a \in L$ we denote by $\mathfrak{b}(a)$ the least sublocale of L containing a. If p is a prime in L, then $\mathfrak{b}(p) = \{p, 1\}$.

A $\bigvee$*-base* of a locale is a subset $B \subseteq L$ such that every $a \in L$ can be expressed as $a = \bigvee B'$ with $B' \subseteq B$.

We follow the description of localic products (i.e., frame coproducts) as explained in [14]; in particular $L \oplus M$ stands for the localic product of L and M. The localic product projections are denoted by $\pi_1 \colon L \oplus M \to L$

and $\pi_2 \colon L \oplus M \to M$, while their left adjoints, the coproduct injections, are denoted by $\iota_1 \colon L \to L \oplus M$ and $\iota_2 \colon M \to L \oplus M$.

In what follows, we provide some more specific preliminaries.

2.1. Primes and covered primes. The set of prime elements of a locale L will be denoted by $\mathsf{pt}(L)$. Moreover, an element $p \in L$ is said to be a *covered prime* if for every $\{a_i\}_{i\in I} \subseteq L$ with $p = \bigwedge_{i\in I} a_i$, there is an $i \in I$ with $p = a_i$. The subset $\mathsf{pt}_D(L) \subseteq \mathsf{pt}(L)$ will denote the set of covered primes of L.

Remark 2.1. Elements $p \in L$ such that $p = \bigwedge_{i\in I} a_i$ implies the existence of an $i \in I$ with $p = a_i$ were also referred to as *completely prime* elements in [5]. However, this terminology was corrected in [6] because that term usually means that $\bigwedge_{i\in I} a_i \leq p$ implies the existence of an $i \in I$ with $a_i \leq p$. In a general locale, the notions are not equivalent (see [6, Rem. 1]).

An alternative characterization of covered primes was given in [5, Prop. 2.1.1]; it is the equivalence between (i) and (ii) of the following proposition. For our purposes it will be convenient to present a modification of this characterization for $\bigvee$-bases:

Proposition 2.2. *Let L be a locale and $B \subseteq L$ a $\bigvee$-base of L. If $p \in \mathsf{pt}(L)$, then the following are equivalent*:

(i) *p is covered;*
(ii) *there is an $a \in L$ with $p < a$ such that $p \leq b \leq a$ implies $b = p$ or $b = a$;*
(iii) *there is an $a \in L$ with $p < a$ such that for all $b \in B$ with $b \leq a$, either $b \leq p$ or $a \leq b \vee p$.*

Proof: (i) $\Longrightarrow$ (ii). Let $a = \bigwedge\{ b \in L \mid p < b \}$. Since p is a covered prime, we have $p < a$. If $p < b \leq a$, we have $a \leq b$ and hence $b = a$.
(ii) $\Longrightarrow$ (iii). Let $b \in B$ and set $b' = (b \vee p) \wedge a$. Then $p \leq b' \leq a$, so either $b' \leq p$ or $a \leq b'$. In the former case, since p is prime we have $b \leq b \vee p \leq p$, and the latter case is equivalent to $a \leq b \vee p$.
(iii) $\Longrightarrow$ (i). Let $a \in L$ with $p < a$ such that for all $b \in B$ with $b \leq a$, either $b \leq p$ or $a \leq b \vee p$ and suppose that $p = \bigwedge_{i\in I} a_i$. Then there is an $i_0 \in I$ such that $a \nleq a_{i_0}$. Now write $a \wedge a_{i_0} = \bigvee B_{i_0}$ with $B_{i_0} \subseteq B$ and let $b \in B_{i_0}$. Then $b \leq a$ and $a \nleq b \vee p$ (if $a \leq b \vee p$ then $a \leq (a \wedge a_{i_0}) \vee p \leq a_{i_0}$) and so $b \leq p$. Consequently $a \wedge a_{i_0} = \bigvee B_{i_0} \leq p$. By primality of p, we then necessarily have $a_{i_0} \leq p$ and so $p = a_{i_0}$. ■

If p is a covered prime, it is not difficult to show that the element $a > p$ in (ii) and (iii) of Proposition 2.2 must coincide with $\bigwedge\{ b \in L \mid p < b \}$ and hence it is uniquely determined. We shall therefore refer to it as the *cover* of p and we denote it by p^+.

Note that an element p in L is maximal if and only if it is a covered prime with cover $p^+ = 1$.

Covered primes have the following very useful characterization in terms of one-point sublocales:

Lemma 2.3 ([11, Prop. 10.2]). *A prime p is covered in a locale L if and only if $\mathfrak{b}(p)$ is a complemented sublocale of L.*

Moreover, coveredness of primes captures the T_D-property:

Lemma 2.4 ([5, Prop. 2.3.2]). *A T_0-space X is T_D if and only if $X - \overline{\{x\}}$ is a covered prime in $\Omega(X)$ for every $x \in X$.*

As is well known, localic maps always send prime elements into prime elements. However the analogous assertion for covered primes is not generally true (cf. [5, 6]). We shall say that a localic map $f\colon L \to M$ is *D-localic* if $f(p) \in \mathsf{pt}_D(L)$ for each $p \in \mathsf{pt}_D(M)$ — i.e., if it sends covered primes into covered primes. Following [5] we shall also say that a frame homomorphism is a *D-homomorphism* if its right adjoint is a D-localic map.

Lemma 2.5 ([5, 3.2]). *If X and Y are T_D-spaces and $f\colon X \to Y$ is a continuous map, then $\Omega(f)\colon \Omega(X) \to \Omega(Y)$ is a D-localic map.*

2.2. The T_D-duality. The material in this subsection is due to Banaschewski and Pultr [5]. For every $a \in L$, we set $\Sigma'_a = \{ p \in \mathsf{pt}_D(L) \mid a \not\leq p \}$. It turns out that the family $\{ \Sigma'_a \mid a \in L \}$ is a topology on $\mathsf{pt}_D(L)$. This topology is denoted by $\Sigma'(L)$ and referred to as the *T_D-spectrum* of L. It is not difficult to show that $\Sigma'(L)$ is always a T_D-space (see [5, Prop. 3.3.2]). One defines the following categories:

- Frm_D is the category consisting of frames and D-homomorphisms between them. Loc_D is by definition the dual of Frm_D — i.e., $\mathsf{Loc}_D = \mathsf{Frm}_D^{op}$. We regard Loc_D as a concrete category whose morphisms are D-localic maps;
- Top_D is the full subcategory of Top consisting of T_D-spaces.

Because of Lemma 2.5, the functor $\Omega\colon \mathsf{Top} \to \mathsf{Loc}$ can be restricted to a functor $\Omega\colon \mathsf{Top}_D \to \mathsf{Loc}_D$. If $f\colon L \to M$ is a D-localic map, it may be restricted and co-restricted to a map $\mathsf{pt}_D(L) \to \mathsf{pt}_D(M)$ which is easily seen to be continuous with respect the topologies of the T_D-spectra, and so one obtains a morphism $\Sigma'(f)\colon \Sigma'(L) \to \Sigma'(M)$ in Top_D and a functor $\Sigma'\colon \mathsf{Loc}_D \to \mathsf{Top}_D$. Moreover, there is an adjunction

$$\begin{tikzcd} \mathsf{Top}_D \arrow[r, bend left, "\Omega"] \arrow[r, phantom, "\perp"] & \mathsf{Loc}_D \arrow[l, bend left, "\Sigma'"] \end{tikzcd}$$

Furthermore, the unit η of the adjunction is a natural isomorphism (and therefore Ω is full and faithful). Specifically, η has components $\eta_X\colon X \to \Sigma'(\Omega(X))$ which are homeomorphisms and send $x \in X$ to $X - \overline{\{x\}}$.

The counit of the adjunction has components ϵ_L which are injective D-localic maps $\epsilon_L\colon \Omega(\Sigma'(L)) \rightarrowtail L$ sending Σ'_a to $\bigwedge\{p \in \mathsf{pt}_D(L) \mid a \leq p\}$. The map ϵ_L is called the *T_D-spatialization* of L. We shall need the following easy consequence:

Lemma 2.6. *A locale is T_D-spatial if and only if every element is a meet of covered primes.*

Proof: If X is a T_D-space and U an open set, then $U = \bigwedge_{x \notin U} X - \overline{\{x\}}$ with each $X - \overline{\{x\}}$ covered by Lemma 2.4. For the converse, assume that every element in L is a meet of covered primes. Then obviously the map ϵ_L defined above is also surjective and thus an isomorphism. Thus $L \cong \Omega(\Sigma'(L))$ with $\Sigma'(L)$ a T_D-space. ■

Therefore, ϵ_L is an isomorphism if and only if L is T_D-spatial and so the adjunction restricts to an equivalence between Top_D and the category consisting of T_D-spatial locales and D-localic maps between them.

Corollary 2.7. *Let L and M be locales and $f\colon L \to M$ be a surjective D-localic map. If L is T_D-spatial then so is M.*

2.3. A few properties of primes in finite products of locales. If L and M are locales and $a \in L$ and $b \in M$, we denote $a \parr b = (a \oplus 1) \vee (1 \oplus b) \in L \oplus M$.

Lemma 2.8. *Let L and M be locales, $L \oplus M$ the localic product of L and M, $\pi_1\colon L \oplus M \to L$ and $\pi_2\colon L \oplus M \to M$ the projections, $a \in L$ and $b \in M$, $\{a_i\}_{i \in I} \subseteq L$ and $\{b_j\}_{j \in J} \subseteq M$. Then:*

(1) *$a \parr b = \{ (x,y) \in L \times M \mid x \le a$ or $y \le b \}$.*
(2) *$\left(\bigwedge_{i\in I} a_i\right) \parr \left(\bigwedge_{j\in J} b_j\right) = \bigwedge_{i\in I,\, j\in J} a_i \parr b_j$.*
(3) *If $b \neq 1$ then $\pi_1(a \parr b) = a$ and if $a \neq 1$ then $\pi_2(a \parr b) = b$.*
(4) *If a is prime in L and b is prime in M then $a \parr b$ is prime in $L \oplus M$.*
(5) *If a is a covered prime in L with cover a^+ and b is a covered prime in M with cover b^+ then $a \parr b$ is covered in $L \oplus M$ with cover $(a \parr b)^+ = (a^+ \parr b) \wedge (a \parr b^+)$.*

Proof: (1) follows easily from the fact that $\{ (x,y) \in L \times M \mid x \le a \text{ or } y \le b \}$ is a *cp*-ideal.

The inequality $\le$ in (2) is trivial so let us show the reverse one. Let $(x,y) \in \bigwedge_{i\in i, j\in J} a_i \parr b_j$. By (1), we have to show that $x \le \bigwedge_{i\in i} a_i$ or $y \le \bigwedge_{j\in J} b_j$. Assume that $x \nleq \bigwedge_{i\in I} a_i$. Then there is an $i_0 \in I$ with $x \nleq a_{i_0}$. But $(x,y) \in \bigwedge_{i\in i, j\in J} a_i \parr b_j$ and so $(x,y) \in a_{i_0} \parr b_j$ for all $j \in J$. By (1), we have $y \le b_j$ for all $j \in J$ — i.e., $y \le \bigwedge_{j\in J} b_j$.

For (3), we use the adjunction $\iota_1 \dashv \pi_1$ to compute

$$\pi_1(a \parr b) = \bigvee \{ x \in L \mid x \oplus 1 \le a \parr b \} = \bigvee \{ x \in L \mid x \le a \text{ or } b = 1 \}.$$

Similarly $\pi_2(a \parr b) = \bigvee \{ y \in M \mid a = 1 \text{ or } y \le b \}$.

(4) can be shown using the fact that $\Sigma\colon \mathsf{Loc} \to \mathsf{Top}$ is a right adjoint and hence it preserves limits. For the sake of completeness, we give a direct proof. First, let $a \in \mathsf{pt}(L)$ and $b \in \mathsf{pt}(M)$. Let $U_1, U_2 \in L \oplus M$ with $U_1 \wedge U_2 \le a \parr b$ and suppose that $U_1 \nleq a \parr b$. Then there is an $(x_1, y_1) \in U_1$ with $x_1 \nleq a$ and $y_1 \nleq b$. For each $(x_2, y_2) \in U_2$, one has $(x_1 \wedge x_2, y_1 \wedge y_2) \in U_1 \wedge U_2 \le a \parr b$, and so either $x_1 \wedge x_2 \le a$ or $y_1 \wedge y_2 \le b$. By primality of a and b, it follows that either $x_2 \le a$ or $y_2 \le b$. Thus $U_2 \le a \parr b$ and $a \parr b \in \mathsf{pt}(L \oplus M)$.

For (5), let $a \in \mathsf{pt}_D(L)$ and $b \in \mathsf{pt}_D(M)$. Since $\{ x \oplus y \mid x \in L,\ y \in M \}$ is a $\bigvee$-base of $L \oplus M$, we shall use Proposition 2.2 (iii) for proving that $a \parr b$ is covered with

$$(a \parr b)^+ := (a^+ \parr b) \wedge (a \parr b^+) = (a^+ \oplus b^+) \vee (a \parr b).$$

Obviously, $a \parr b < (a \parr b)^+$ (if the equality holds then we would have $(a^+, b^+) \in (a^+ \parr b) \wedge (a \parr b^+) = a \parr b$ and so either $a^+ \le a$ or $b^+ \le b$, a contradiction). Now let $x \in L$ and $y \in M$ with $x \oplus y \le (a \parr b)^+$. If $x \le a$ or $y \le b$, then $x \oplus y \le a \parr b$ and we are done. Hence suppose that $x \nleq a$ and $y \nleq b$. Then $a^+ \le x \vee a$ and $b^+ \le y \vee b$ and so $(a \parr b)^+ \le ((x \vee a) \oplus (y \vee b)) \vee (a \parr b) = (x \oplus y) \vee (a \parr b)$, as required. Hence $a \parr b \in \mathsf{pt}_D(L \oplus M)$. ■

Corollary 2.9. *The map* $\varphi_{L,M}\colon \mathsf{pt}(L)\times\mathsf{pt}(M) \to \mathsf{pt}(L\oplus M)$ *given by* $\varphi_{L,M}(p,q) = p \parr q$ *is a bijection.*

Proof: $\varphi_{L,M}$ is well-defined by Lemma 2.8 (4) and it is obviously injective. Moreover, given a prime $U \in \mathsf{pt}(L \oplus M)$, since localic maps send primes into primes, one has $p = \pi_1(U) \in \mathsf{pt}(L)$ and $q = \pi_2(U) \in \mathsf{pt}(M)$ and clearly $p \parr q \leq U$. On the other hand, if $(a, b) \in U$ then $a \oplus b = (a \parr 0) \wedge (0 \parr b) \leq U$, and since U is prime, either $a \parr 0 \leq U$ or $0 \parr b \leq U$. Assume without loss of generality the former. Then $\iota_1(a) = a \oplus 1 = a \parr 0 \leq U$, i.e., $a \leq \pi_1(U) = p$ and thus $(a, b) \in p \parr q$. Consequently $p \parr q = U$ and $\varphi_{L,M}$ is surjective. ■

3. Finite products in Loc_D

We begin by showing that the localic product projections live in Loc_D.

Lemma 3.1. *The localic product projections* $\pi_1\colon L\oplus M \to L$ *and* $\pi_2\colon L\oplus M \to M$ *are D-localic maps.*

Proof: Let $U \in \mathsf{pt}_D(L \oplus M)$. Since in particular U is a prime in $L \oplus M$, by Corollary 2.9 there are $p \in \mathsf{pt}(L)$ and $q \in \mathsf{pt}(M)$ with $U = p \parr q$. By Lemma 2.8 (3) we have to show that $p \in \mathsf{pt}_D(L)$ and $q \in \mathsf{pt}_D(M)$. We shall only show that $p \in \mathsf{pt}_D(L)$ since the other case is similar. Assume that $p = \bigwedge_i a_i$ with $\{a_i\}_{i\in I} \subseteq L$ and let $(a, b) \in \bigwedge_i (a_i \parr q)$. If $b \leq q$, then obviously $(a, b) \in p \parr q$. On the other hand, if $b \nleq q$, then $a \leq a_i$ for all $i \in I$, and so $a \leq \bigwedge_i a_i = p$. Thus $(a, b) \in p \parr q$. This shows that $\bigwedge_i (a_i \parr q) \leq p \parr q$, whereas the reverse inequality is trivial. Since $U = p \parr q \in \mathsf{pt}_D(L \oplus M)$, there is an $i_0 \in I$ with $a_{i_0} \parr q = p \parr q$. Since $q \neq 1$, it follows that $p = a_{i_0}$. ■

Corollary 3.2. *The map* $\psi_{L,M}\colon \mathsf{pt}_D(L) \times \mathsf{pt}_D(M) \to \mathsf{pt}_D(L \oplus M)$ *given by* $\psi_{L,M}(p,q) = p \parr q$ *is a bijection.*

Proof: $\psi_{L,M}$ is well-defined by Lemma 2.8 (5) and it is obviously injective. Moreover, it is surjective by the proof of Lemma 3.1, because we showed that if $U \in \mathsf{pt}_D(L \oplus M)$, then $U = p \parr q$ with $p \in \mathsf{pt}_D(L)$ and $q \in \mathsf{pt}_D(M)$. ■

Corollary 3.3. *Let L and M be locales. Then the following are equivalent*:

(i) $L \oplus M$ *is* T_D*-spatial;*

(ii) $L \oplus M$ *is spatial and both L and M are* T_D*-spatial.*

Proof: (i) $\Longrightarrow$ (ii) follows immediately from Lemma 3.1 and Corollary 2.7.
(ii) $\Longrightarrow$ (i): Let $U \in L \oplus M$. Since $L \oplus M$ is spatial, then one can write $U = \bigwedge_{i\in I} p_i \parr q_i$ with $\{p_i\}_{i\in I} \subseteq \mathsf{pt}(L)$ and $\{q_i\}_{i\in I} \subseteq \mathsf{pt}(M)$ by Corollary 2.9.

Now, L is T_D-spatial, so by Lemma 2.6, for each $i \in I$ there is a family $\{p^i_j\}_{j\in I_i} \subseteq \mathsf{pt}_D(L)$ with $p_i = \bigwedge_{j\in I_i} p^i_j$. Similarly, for each $i \in I$ there is a family $\{q^i_k\}_{k\in J_i} \subseteq \mathsf{pt}_D(M)$ with $q_i = \bigwedge_{k\in J_i} q^i_k$. By Lemma 2.8 (2) and (5), it follows that $U = \bigwedge_{i\in I,\, j\in I_i,\, k\in J_i} p^i_j \parr q^i_k$ with each $p^i_j \parr q^i_k$ being covered in $L \oplus M$, so the assertion now follows from Lemma 2.6. ■

We can now show the main result of this section.

Proposition 3.4. *Let L and M be locales. Then the system* $(L \oplus M, \pi_1, \pi_1)$ *is a product in* Loc_D*. Consequently, the category* Loc_D *has finite products and the inclusion functor* $I\colon \mathsf{Loc}_D \hookrightarrow \mathsf{Loc}$ *preserves them.*

Proof: The localic product projections π_1 and π_2 are D-localic maps by Lemma 3.1. It remains to be proved that if $f\colon N \to L$ and $g\colon N \to M$ are D-localic maps, then the induced map $\langle f, g\rangle\colon N \to L \oplus M$ is also D-localic. Hence let $p \in \mathsf{pt}_D(N)$. Then

$$\begin{aligned}\langle f, g\rangle(p) &= \bigvee\{\, a \oplus b \mid f^*(a) \wedge g^*(b) \le p \,\} = \bigvee\{\, a \oplus b \mid f^*(a) \le p \text{ or } g^*(b) \le p \,\}\\ &= \bigvee\{\, a \oplus b \mid a \le f(p) \text{ or } b \le g(p) \,\} = (f(p) \oplus 1) \vee (1 \oplus g(p))\\ &= f(p) \parr g(p).\end{aligned}$$

Since f and g are D-localic, $f(p) \in \mathsf{pt}_D(L)$ and $g(q) \in \mathsf{pt}_D(M)$, so the conclusion now follows from Lemma 2.8 (5). ■

Since $\Sigma'\colon \mathsf{Loc}_D \to \mathsf{Top}_D$ is a right adjoint, it preserves products, so by the previous corollary we can improve the bijection in Corollary 3.2 to a homeomorphism (observe that finite products of T_D-spaces are T_D, hence finite products in Top_D are just products in Top):

Corollary 3.5. *Let L and M be locales. Then the canonical map*

$$(\Sigma'(\pi_1), \Sigma'(\pi_2))\colon \Sigma'(L \oplus M) \to \Sigma'(L) \times \Sigma'(M)$$

is a homeomorphism.

As an application of the above, we obtain a *finite* T_D-analogue of a well-known result for the classical spectrum, namely the fact that for *sober* spaces X_i, if $\oplus_{i\in I}\Omega(X_i)$ is spatial, then $\oplus_{i\in I}\Omega(X_i) \cong \Omega(\oplus_{i\in I} X_i)$ (see [14, IV 5.4.2]).

Corollary 3.6. *Let X and Y be* T_D*-spaces. Then the the following are equivalent*:

(i) $\Omega(X) \oplus \Omega(Y) \cong \Omega(X \times Y)$;

(ii) $\Omega(X) \oplus \Omega(Y)$ *is spatial*;

(iii) $\Omega(X) \oplus \Omega(Y)$ *is* T_D*-spatial.*

Proof: (i) $\Longrightarrow$ (ii) is trivial and the equivalence between (ii) and (iii) follows from Corollary 3.3. Finally, assume that $\Omega(X) \oplus \Omega(Y)$ is T_D-spatial. Then, $\Omega(X) \oplus \Omega(Y) \cong \Omega(\Sigma'(\Omega(X) \oplus \Omega(Y))$ via the counit of the adjunction $\Omega \vdash \Sigma'$ which is an isomorphism by T_D-spatiality. Finally,

$$\Omega(\Sigma'(\Omega(X) \oplus \Omega(Y)) \cong \Omega(\Sigma'(\Omega(X)) \times \Sigma'(\Omega(Y))) \cong \Omega(X \times Y),$$

where the first isomorphism follows by applying Ω to the homeomorphism in Corollary 3.5, and the second isomorphism follows from the fact that the unit of the adjunction $\Omega \vdash \Sigma'$ is always an isomorphism. Hence $\Omega(X) \oplus \Omega(Y) \cong \Omega(X \times Y)$. ■

3.1. Infinite products in Loc_D. In order to state the fact that

the category Loc_D *does not have infinite products*,

we shall need the fact that infinite products of the Sierpinski space in Top_D fail to exist.

It is well known that infinite products of Sierpinski spaces are not T_D (see [8, 10]). Actually, infinite products of T_D spaces that are not T_1 are never T_D ([19]). However, some more work has to be done in order to assert that such products do not exist in Top_D. Since we have not found this result in the literature, we present it Appendix A.

Now, for each $n \in \mathbb{N}$, let $L_n = \Omega(\mathbb{S})$ be the Sierpinski locale. If $(L_n)_{n\in\mathbb{N}}$ had a product in Loc_D, we would have a countable product of $\Sigma'(\Omega(\mathbb{S})) \cong \mathbb{S}$ in Top_D because $\Sigma'\colon \mathsf{Loc}_D \to \mathsf{Top}_D$ preserves products (as a right adjoint). However, such product does not exist by Fact A.1.

4. D-sublocales and equalizers

We recall from [2] that a sublocale $S \subseteq L$ is a D-sublocale if the embedding $S \hookrightarrow L$ is a D-localic map — i.e., if $\mathsf{pt}_D(S) \subseteq \mathsf{pt}_D(L)$. Plenty of sublocales are actually D-sublocales (e.g. every join of complemented sublocales is a D-sublocale, and so is every sublocale without covered primes, cf. [2, p. 11]). In fact, the system of D-sublocales of a locale plays an important role in the T_D-duality (see for example [2, Thm. 3.21] for a Niefield-Rosenthal type theorem which characterizes locales whose sublocales are T_D-spatial).

Now, it is clear that if the equalizer of two D-localic maps computed in Loc is a D-sublocale, then it is also the equalizer in Loc_D. Further, it follows from the results in [2] that there is always the largest D-sublocale contained in a given sublocale, and it can therefore be tempting to conjecture that the equalizer of any pair of D-localic maps is given by the largest D-sublocale contained in their Loc-equalizer. However, it does not satisfy the appropriate universal property because the embedding part of the factorization of a D-localic map is generally not a D-sublocale. In fact, equalizers in Loc_D may fail to exist at all.

We start with the following:

Lemma 4.1. *Let L and M be locales and $f, g\colon L \to M$ be localic maps. If $e\colon E \to L$ is an equalizer of f and g in Loc_D then $e[E]$ is a D-sublocale of L.*

Proof: Let $p \in \mathsf{pt}_D(e[E])$. By Lemma 2.3, $\mathfrak{b}(p)$ is a complemented sublocale of $e[E]$ and it follows that $e_{-1}[\mathfrak{b}(p)]$ is a complemented sublocale of E. Indeed, consider the factorization of e, namely

$$E \xrightarrow{\;j\;} \!\!\!\!\twoheadrightarrow e[E] \xhookrightarrow{\;\iota\;} L \ .$$

Then $e_{-1}[\mathfrak{b}(p)] = j_{-1}[\iota_{-1}[\mathfrak{b}(p)]] = j_{-1}[\mathfrak{b}(p) \cap e[E]] = j_{-1}[\mathfrak{b}(p)]$ and recall that coframe homomorphisms preserve complements.

We distinguish two cases:

(1) Suppose first that there is some $q \in \mathsf{pt}_D(e_{-1}[\mathfrak{b}(p)])$. Then $\mathfrak{b}(q) \subseteq e_{-1}[\mathfrak{b}(p)]$ so by adjunction $\mathfrak{b}(e(q)) = e[\mathfrak{b}(q)] \subseteq \mathfrak{b}(p)$ — i.e., $e(q) = p$. But $e_{-1}[\mathfrak{b}(p)]$ is a complemented sublocale of E, so in particular it is a D-sublocale (see [2, 2.4]). Thus $q \in \mathsf{pt}_D(E)$, and since e is a D-localic map it follows that $p = e(q) \in \mathsf{pt}_D(L)$, as required.

(2) Assume now that $\mathsf{pt}_D(e_{-1}[\mathfrak{b}(p)]) = \varnothing$ and select a locale M such that $\mathsf{pt}(M) \neq \varnothing$ and $\mathsf{pt}_D(M) = \varnothing$ (for example $M = [0, 1]$) and let h be the composite

$$M \twoheadrightarrow \mathfrak{b}(p) \hookrightarrow L$$

where the first map is the unique surjection onto the terminal locale, hence $h(1) = 1$ and $h(a) = p$ for all $a < 1$. Since $\mathsf{pt}_D(M) = \varnothing$, h is a D-localic map, and it equalizes f and g because so does e and $p \in e[E]$. Hence there is a unique D-localic map $k\colon M \to E$ such that $ek = h$. Let $p_0 \in \mathsf{pt}(M)$ and $q_0 := k(p_0)$. Since localic maps send primes into primes, we have $q_0 \in \mathsf{pt}(E)$, and $e(q_0) = h(p_0) = p$.

Finally, let ℓ be the composite

$$e_{-1}[\mathfrak{b}(p)] \twoheadrightarrow \mathfrak{b}(q_0) \hookrightarrow E \ .$$

Then $e\circ\ell = e\circ\iota$ where $\iota\colon e_{-1}[\mathfrak{b}(p)] \hookrightarrow E$ is the inclusion. Since $\mathsf{pt}_D(e_{-1}[\mathfrak{b}(p)]) = \varnothing$, ℓ and ι are trivially D-localic maps and by the uniqueness clause of the equalizer we must then have $\ell = \iota$. But then $\mathfrak{b}(q_0) = e_{-1}[\mathfrak{b}(p)]$ is a complemented sublocale of E — i.e., $q_0 \in \mathsf{pt}_D(E)$. Since e is a D-localic map, $p = e(q_0) \in \mathsf{pt}_D(L)$, as required. ∎

Proposition 4.2. *Let L and M be locales and let $f, g\colon L \to M$ be D-localic maps. If the equalizer of f and g exists in Loc_D then their Loc-equalizer is a D-sublocale.*

Proof: Assume that

$$E \xrightarrow{\ e\ } L \underset{g}{\overset{f}{\rightrightarrows}} M$$

is an equalizer in Loc_D and let $S \subseteq L$ be the equalizer of f and g in Loc, hence $e[E] \subseteq S$ by the universal property of the equalizer, and by the previous lemma we know that $e[E]$ is a D-sublocale of L. Let $p \in \mathsf{pt}_D(S)$. Select a nontrivial pointless Boolean algebra B and let h be the composite

$$B \twoheadrightarrow \mathfrak{b}(p) \hookrightarrow L$$

where the first map is the unique surjection onto the terminal locale, hence $h(1) = 1$ and $h(a) = p$ for all $a < 1$. Since $\mathsf{pt}_D(B) \subseteq \mathsf{pt}(B) = \varnothing$, h is a D-localic map, and it equalizes f and g because $p \in S$. Hence there is a unique D-localic map $k\colon B \to E$ such that $ek = h$. Then $p = h(0) = e(k(0)) \in e[E] \subseteq S$ and since $e[E]$ is a D-sublocale, $p \in \mathsf{pt}_D(S) \cap e[E] \subseteq \mathsf{pt}_D(e[E]) \subseteq \mathsf{pt}_D(L)$. Hence S is a D-sublocale. ∎

Example 4.3. Let $L = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ and $f\colon L \to L$ given by $f(0) = 0$, $f(1) = 1$ and $f(\frac{1}{n}) = \frac{1}{n+1}$. One readily verifies that f is a D-localic map. It follows that

$$\mathfrak{b}(0) = \{0, 1\} \hookrightarrow L \underset{f}{\overset{1_L}{\rightrightarrows}} L$$

is the equalizer in Loc and hence the equalizer of f and 1_L in Loc_D does not exist because $\mathfrak{b}(0)$ is not a D-sublocale.

By the comment at the beginning of this section we have:

Corollary 4.4. *Let L and M be locales and let $f, g\colon L \to M$ be D-localic maps. Then the equalizer of f and g in Loc_D exists if and only if the equalizer of f and g in Loc is a D-sublocale (and so their equalizers in Loc and Loc_D coincide).*

A sufficient condition for the situation in Proposition 4.2 is as follows:

Proposition 4.5. *If M has a complemented diagonal (e.g., if M is locally strongly Hausdorff), then, for every pair of morphisms $f, g\colon L \to M$ in Loc_D, their equalizer in Loc_D exists and is given by their equalizer in Loc.*

Proof: It follows from general category theory that the equalizer of f and g in Loc can be computed as the preimage (=pullback) of the diagonal along the map $\langle f, g\rangle\colon L \to M \oplus M$. But the preimage operator is a coframe homomorphism [16, 14] and so it sends complemented sublocales into complemented sublocales. It follows that the equalizer in Loc is a complemented sublocale of L. But complemented sublocales are D-sublocales (see [2, Cor. 2.4]). ∎

In [2] we claimed that D-sublocales play the role of plain sublocales in the duality of T_D-spaces. In what follows, we provide some more evidence of this assertion by showing that they are precisely the regular monomorphisms in Loc_D (cf. the fact that sublocales are precisely regular monomorphisms in Loc).

Proposition 4.6. *Any D-sublocale is a regular monomorphism in Loc_D.*

Proof: Let $S \subseteq L$ be a D-sublocale. As is well known, sublocale embeddings are regular monomorphisms in Loc, and hence $S \hookrightarrow L$ is the equalizer of its cokernel pair in Loc — i.e., there is an equalizer diagram

$$S \hookrightarrow L \underset{g}{\overset{f}{\rightrightarrows}} P$$

in Loc, where $P = \{ (x, y) \in L \times L \mid \nu_S(x) = \nu_S(y) \}$ (ν_S denotes the nucleus associated to the sublocale S),

$$f(a) = \bigvee \{ (b, c) \in P \mid b \leq a \} = (a, \nu_S(a))$$

and

$$g(a) = \bigvee \{ (b, c) \in P \mid c \leq a \} = (\nu_S(a), a)$$

for each $a \in L$. Observe that f (resp. g) is the right adjoint of the coordinate projection $P \to L$ sending (b, c) to b (resp. (b, c) to c).

Clearly, it suffices to show that f and g are D-localic maps, as then the equalizer diagram above will be a equalizer diagram in Loc_D. By symmetry, we shall just prove it for f. Let $p \in \mathsf{pt}_D(L)$ and denote by p^+ the cover of p in L. We distinguish two cases:

(1) If $p \in S$ then $\nu_S(p) = p$, and since $p < p^+$ it follows that $f(p) = (p, p) < (p^+, p^+)$. Let $(b, c) \in P$ with

$$f(p) = (p, p) \le (b, c) \le (p^+, p^+).$$

Then $p \le b \wedge c \le p^+$ and we again distinguish two cases:
(1.1) If $b \wedge c = p$ then $b \le \nu_S(b) = \nu_S(b \wedge c) = \nu_S(p) = p$ and hence $(b, c) \le f(p)$.
(1.2) If $b \wedge c = p^+$ then $(p^+, p^+) \le (b, c)$.
Consequently (p^+, p^+) is the cover of $f(p)$ in P.
(2) If $p \notin S$ — i.e., $p < \nu_S(p)$ then since $p \le \nu_S(p) \wedge p^+ \le p^+$ and p is prime, we must have $\nu_S(p) \wedge p^+ = p^+$. It follows that $\nu_S(p^+) = \nu_S(p)$ and so we have $f(p) = (p, \nu_S(p)) = (p, \nu_S(p^+)) < (p^+, \nu_S(p^+)) = f(p^+)$. Let $(b, c) \in P$ with

$$f(p) = (p, \nu_S(p^+)) \le (b, c) \le (p^+, \nu_S(p^+)) = f(p^+).$$

Then $p \le b \le p^+$ and $c = \nu_S(p^+)$. If $b = p$ then clearly $(b, c) = f(p)$, and if $b = p^+$ then $(b, c) = f(p^+)$. Consequently $f(p^+)$ is the cover of $f(p)$ in P. ■

Corollary 4.7. *The D-sublocales are precisely the regular subobjects in* Loc_D.

Proof: The "only if" implication follows from Proposition 4.6 whereas the "if" implication follows from Proposition 4.2. ■

5. T_D-spatiality of squares and the system of smooth sublocales

We now give a further characterization of T_D-spatiality of localic squares in terms of functoriality properties of certain subcolocale of $\mathcal{S}(L)$. Let us recall from [1] that the system

$$\mathcal{S}_b(L) = \Big\{ \bigvee_{a \in A, b \in B} \mathfrak{c}(a) \cap \mathfrak{o}(b) \mid A, B \subseteq L \Big\}$$

is a subcolocale of $\mathcal{S}(L)$ which is a Boolean algebra — indeed, it is precisely the Booleanization of $\mathcal{S}(L)$. Therefore, one may regard $\mathcal{S}_b(L)$ as a discretization of L, similar to $\mathcal{S}(L)^{op}$, but more discrete (as $\mathcal{S}(L)^{op}$ is zero-dimensional but it is seldom Boolean). Sublocales contained in $\mathcal{S}_b(L)$ are often referred to as *smooth* sublocales.

Whenever L is subfit, $\mathcal{S}_b(L)$ coincides with the system of joins of closed sublocales from [17]. This has recently attracted attention in point-free topology; for instance, the naturality of the construction as a maximal essential extension in the category of frames [4], its role as a discretization of

L for modeling not necessarily continuous real-valued functions (conservatively in the class of T_1-spaces) [15], its (non-) functoriality properties [3], or as a useful tool for studying several (conservative) point-free extensions of classical topological properties [9].

We first recall the following result, which reveals also a strong connection of the Boolean algebra $\mathcal{S}_b(L)$ with T_D-spatiality of L:

Theorem 5.1 ([1, Thm. 3.4]). *The following are equivalent for a locale L:*

(i) *L is T_D-spatial;*
(ii) *The map $\mathfrak{m}\colon \mathcal{P}(\mathsf{pt}_D(L)) \to \mathcal{S}_b(L)$ which sends $Y \subseteq \mathsf{pt}_D(L)$ to $\bigvee_{p\in Y} \mathfrak{b}(p)$ is an isomorphism (whose inverse $\mathsf{pt}_D\colon \mathcal{S}_b(L) \to \mathcal{P}(\mathsf{pt}_D(L))$ sends $S \in \mathcal{S}_b(L)$ to $\mathsf{pt}_D(S)$);*
(iii) *There exists an isomorphism $\mathcal{S}_b(L) \cong \mathcal{P}(\mathsf{pt}_D(L))$;*
(iv) *$\mathcal{S}_b(L)$ is atomic (i.e., it is spatial).*

We are now in position to prove the main result, which connects preservation of localic squares by $\mathcal{S}_b(-)$ and their T_D-spatiality.

Theorem 5.2. *Let L be a locale. Then the localic product $L \oplus L$ in Loc_D is T_D-spatial if and only if there exists an isomorphism $\mathcal{S}_b(L) \oplus \mathcal{S}_b(L) \cong \mathcal{S}_b(L \oplus L)$.*

Proof: If $L \oplus L$ is T_D-spatial then, so is L and it follows from Theorem 5.1 that

$$\mathfrak{m}\colon \mathcal{P}(\mathsf{pt}_D(L \oplus L)) \to \mathcal{S}_b(L \oplus L)$$

and

$$\mathsf{pt}_D \oplus \mathsf{pt}_D\colon \mathcal{S}_b(L) \oplus \mathcal{S}_b(L) \to \mathcal{P}(\mathsf{pt}_D(L)) \oplus \mathcal{P}(\mathsf{pt}_D(L))$$

are isomorphisms. On the other hand, the map

$$\psi_{L,L}\colon \mathsf{pt}_D(L) \times \mathsf{pt}_D(L) \to \mathsf{pt}_D(L \oplus L)$$

from Corollary 3.2 is a bijection, and hence $\mathcal{P}(\psi_{L,L})$ is an isomorphism (where $\mathcal{P}$ is the covariant power set functor). Finally, it is well-known (see for example [16, 1.6.4] for a direct proof) that for any set X, the map $\mathcal{P}(X) \oplus \mathcal{P}(X) \to \mathcal{P}(X \times X)$ that sends $A \oplus B \in \mathcal{P}(X) \oplus \mathcal{P}(X)$ to $A \times B$ is an isomorphism. Consequently, the composite

$$\begin{tikzcd}
\mathcal{S}_b(L) \oplus \mathcal{S}_b(L) \arrow[rr, dotted] \arrow[d, "\mathsf{pt}_D \oplus \mathsf{pt}_D"'] & & \mathcal{S}_b(L \oplus L) \\
\mathcal{P}(\mathsf{pt}_D(L)) \oplus \mathcal{P}(\mathsf{pt}_D(L)) \arrow[r] & \mathcal{P}(\mathsf{pt}_D(L) \times \mathsf{pt}_D(L)) \arrow[r, "\mathcal{P}(\psi_{L,L})"] & \mathcal{P}(\mathsf{pt}_D(L \oplus L)) \arrow[u, "\mathfrak{m}"']
\end{tikzcd}$$

is an isomorphism.

Assume now that $\mathcal{S}_b(L) \oplus \mathcal{S}_b(L) \cong \mathcal{S}_b(L \oplus L)$ holds. Since $\mathcal{S}_b(L \oplus L)$ is Boolean, so is $\mathcal{S}_b(L) \oplus \mathcal{S}_b(L)$. In particular, the diagonal in $\mathcal{S}_b(L) \oplus \mathcal{S}_b(L)$ is open and hence $\mathcal{S}_b(L)$ is atomic (see [13]). Now, since $\mathcal{S}_b(L)$ is atomic, the product $\mathcal{S}_b(L) \oplus \mathcal{S}_b(L)$ is atomic as well (recall as we mentioned above that $\mathcal{P}(X \times X) \cong \mathcal{P}(X) \oplus \mathcal{P}(X)$ for any set X). Thus $\mathcal{S}_b(L \oplus L)$ is atomic, and by Theorem 5.1 it follows that $L \oplus L$ is T_D-spatial. ■

Appendix A. Infinite products of T_D-spaces

Let $\mathbb{S}$ denote the Sierpinski space — i.e., $\{0, 1\}$ with the topology $\{\varnothing, \{1\}, \{0, 1\}\}$.

Fact A.1. *The countable power of $\mathbb{S}$ does not exist in the category Top_D.*

Proof: Suppose that a countable power of $\mathbb{S}$ exists in Top_D, say $(p_n \colon X \to \mathbb{S})_{n\in\mathbb{N}}$. Clearly, the forgetful functor $U\colon \mathsf{Top}_D \to \mathsf{Set}$ is representable (the singleton space is T_D, hence $U \cong \mathsf{Top}_D(\{*\}, -)$) and so it preserves limits, in particular we may assume that, as sets, $U(X) = \prod_{n\in\mathbb{N}} \mathbb{S}$ is the cartesian product and p_n is just the n-th coordinate projection. Since the projections $p_n \colon X \to \mathbb{S}$ are continuous, by the universal property of the product in Top, the identity $X \to (\prod_{n\in\mathbb{N}} \mathbb{S}, \tau_{Tych})$ is continuous — i.e., $\tau_{Tych} \subseteq \Omega(X)$ (where τ_{Tych} denotes the product (Tychonoff) topology).

However, since a countable product of Sierpinski spaces is not T_D (see [8]) we conclude that $\tau_{Tych} \subsetneq \Omega(X)$ and hence there is an $V_0 \in \Omega(X) - \tau_{Tych}$.

Consider now the space

$$(\prod_{n\in\mathbb{N}} \mathbb{S}, \tau_{Box})$$

where τ_{Box} is the box topology and denote by β_{Tych} (resp. β_{Box}) the usual bases of τ_{Tych} (resp. τ_{Box}) consisting of products of opens where all but finitely many are proper (resp. all products of opens).

Observe that $(\prod_{n\in\mathbb{N}} \mathbb{S}, \tau_{Box})$ is a T_D-space. Indeed, let $x = (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathbb{S}$. Set $B := \prod_{n\in\mathbb{N}} V_n \in \beta_{Box}$ with $V_n = \{1\}$ if $x_n = 1$ and $V_n = \{0, 1\}$ if $x_n = 0$. Moreover, let $F := \prod_{n\in\mathbb{N}} F_n$ with $F_n = \{0\}$ if $x_n = 0$ and $F_n = \{0, 1\}$ if $x_n = 1$. Then F is closed in the box topology. Thus B is an open neighborhood of x and $B - \{x\} = B - F$ is open and so $(\prod_{n\in\mathbb{N}} \mathbb{S}, \tau_{Box})$ is T_D. Since the projections $p_n \colon (\prod_{n\in\mathbb{N}} \mathbb{S}, \tau_{Box}) \to \mathbb{S}$ are continuous, by the universal property of the product in Top_D, the identity $(\prod_{n\in\mathbb{N}} \mathbb{S}, \tau_{Box}) \to X$

is continuous — i.e., $\Omega(X) \subseteq \tau_{Box}$. Consequently $V_0 \in \tau_{Box}$ and we can write

$$V_0 = \bigcup_{j \in J} B_j \cup \bigcup_{i \in I} B_i$$

where $B_i \in \beta_{Tych}$ and $B_j \in \beta_{Box} - \beta_{Tych}$ for all $i \in I$, $j \in J$. (Note that J must be non-empty since $V \notin \tau_{Tych}$ and we can assume that $B_i, B_j \neq \varnothing$). Then

$$B_i = \prod_{n \in \mathbb{N}} V_n^i, \quad B_j = \prod_{n \in \mathbb{N}} V_n^j,$$

where $V_n^i, V_n^j \in \{\{1\}, \{0,1\}\}$, and for $j \in J$, the set $A_j := \{n \in \mathbb{N} \mid V_n^j = \{1\}\}$ is infinite, while for $i \in I$, $V_n^i = \{1\}$ only for finitely many $n \in \mathbb{N}$.

Let us fix an $j_0 \in J$ such that $B_{j_0} \nsubseteq \bigcup_{i \in I} B_i$ (if $B_j \subseteq \bigcup_{i \in I} B_i$ for all $j \in J$ then $V_0 = \bigcup_{i \in I} B_i \in \tau_{Tych}$). Further, let $\mathbb{R}$ be the real line endowed with the usual topology and $f_n \colon \mathbb{R} \to \mathbb{S}$ be the characteristic function of the open interval $(-\frac{1}{n}, \frac{1}{n})$ whenever $V_n^{j_0} = \{1\}$ and the constant function with value 0 whenever $V_n^{j_0} = \{0,1\}$. Then f_n is obviously continuous for all n and since $\mathbb{R}$ is T_D, by the universal property of the product in Top_D, the map $f := (f_n)_{n \in \mathbb{N}} \colon \mathbb{R} \to X$ is continuous.

We now compute $f^{-1}(B_i)$ and $f^{-1}(B_j)$ for each $i \in I$ and $j \in J$.

Let $i \in I$. Since $B_{j_0} \nsubseteq B_i$, there is an $n_0 \in \mathbb{N}$ with $V_{n_0}^{j_0} = \{0,1\}$ and $V_{n_0}^i = \{1\}$. Then $f_{n_0} = 0$ and so

$$f^{-1}(B_i) = \bigcap_{n \in \mathbb{N}} f_n^{-1}(V_n^i) \subseteq f_{n_0}^{-1}(V_{n_0}^i) = \varnothing.$$

Let $j \in J$. We distinguish two cases:

(1) If $B_{j_0} \nsubseteq B_j$ then, as in the previous case, $f^{-1}(B_j) = \varnothing$.

(2) If $B_{j_0} \subseteq B_j$ and $n \in A_j$ then $V_n^j = \{1\} = V_n^{j_0}$ and f_n is the characteristic function of the open interval $(-\frac{1}{n}, \frac{1}{n})$. Hence

$$f^{-1}(B_j) = \bigcap_{n \in \mathbb{N}} f_n^{-1}(V_n^j) = \bigcap_{n \in A_j} f_n^{-1}(\{1\}) = \bigcap_{n \in A_j} (-1/n, 1/n) = \{0\}$$

(because A_j is infinite).

Hence,

$$f^{-1}(V_0) = \bigcup_{B_{j_0} \subseteq B_j} f^{-1}(V_j) \cup \bigcup_{B_{j_0} \nsubseteq B_j} f^{-1}(B_j) \cup \bigcup_{i \in I} f^{-1}(B_i) = \{0\},$$

which is not open in $\mathbb{R}$. This is a contradiction and we conclude that the countable power of $\mathbb{S}$ does not exist. ∎

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