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UNIFORMITIES AND A QUANTALE STRUCTURE ON LOCALIC GROUPS

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ABSTRACT: Similarly like classical topological groups, the point-free counterparts, localic groups, possess natural uniformities (see e.g. [4, 2, 10]) obtained from an involutive binary operation on L (roughly corresponding to the algebra of subsets of a classical group). It is the operation that naturally induces the uniformities (even if it would not result from a group) and a study of this aspect of the construction is the main topic of this article. We have here a functor associating with the localic groups quantales of a special type (and with frame group homomorphisms quantale homomorphisms) which are shown to create the uniformities, in fact as a special case of the natural uniformities connected with metric structures. Also, we present a condition under which the quantale allows a reconstruction of the localic group.

KEYWORDS: Frame, locale, localic group, frame entourage, localic group uniformities, ordered semigroup, quantale, involutive quantale.

MATHEMATICS SUBJECT CLASSIFICATION (2020): 06D22, 06F07, 18F70, 54H11.

Introduction

Recall that when studying spaces we can more often than not forget about points, and consider a structure mimicking the behavior of classical open sets. Thus we obtain an extension of the theory of topological spaces, usually referred to as point-free topology. The classical topological concepts and facts can be typically naturally extended, often to advantage in the sense that the results are more satisfactory, or offering new insights into the phenomena.

In this paper we wish to discuss some aspects of the natural uniformities of localic groups corresponding to the natural uniform structures induced on topological groups by the algebraic one. Like in the classical case, a uniformity on a frame (locale) can be defined as a suitable system of covers

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(Tukey's approach) or as a system of entourages modelling neighborhoods of the diagonal (Weil's approach), see [14, 9, 10]. These approaches turn to be, again, equivalent, although the products of locales (and consequently the entourages) are not a conservative extension of the classical products of spaces and neighborhoods of diagonals (see [8, 9, 13]).

The algebraic structure on a classical topological group very naturally induces uniformities on the underlying space, and this is also the case in the more general localic groups. Considering, however, that a localic group may lack points (with the exception of the unit) and still be very large we see that one cannot use the intuition of canonical shifting the neighborhoods of the unit e to general points a of G by the homeomorphisms $\varphi_a = (x \mapsto xa)$ creating uniform covers $\mathcal{C}(U) = \{\varphi_a[U] \mid a \in G\}$ for neighborhoods U of e, and we have to be careful with the intuition of neighborhoods of the diagonals that are now in fact something else. Nevertheless there are again naturally induced uniformities (both of the Tukey and Weil type, and equivalent – see [4, 2, 10]). The entourage construction is based on a suitably defined (associative) binary operation on the underlying frame L, let us denote it by $*$. It turns out that we have here a quantale $(L, *)$ naturally associated with the group $(L, \mu, \gamma, \varepsilon)$. We analyse this quantale and characterize the quantales obtained this way, showing which properties of (enriched) quantales allow to reconstruct the group structure. The (localic) group homomorphisms are shown to correspond to quantale homomorphisms, and discussed in some detail.

From another point of view, $(L, *)$ is simply an ordered semigroup and as such can be used as a set of values of a generalized metric. The natural group uniformities can then be shown to be metric uniformities of thus generalized metrics, quite analogous to the canonical uniformities of metric spaces.

The paper is organized as follows. After necessary Preliminaries we recall in Section 2 a few technical facts about localic groups. Then, in Section 3, we introduce the concept of a G-quantale, a special case of an involutive quantale, and discuss some of its properties. The following section is then devoted to the G-quantale induced on a localic group; in particular we show that under certain continuity properties of the quantic arrows the group structure can be reconstructed back. Next, in Section 5, the uniformities on G-quantales are discussed (which implies the slightly more special uniformizing localic groups); a.o. we show that they can be viewed as uniformities of metric type. We conclude the story in Section 6 with a short discussion of homomorphisms.

1. Preliminaries

1.1. We use the standard notation for meets (infima) and joins (suprema) in posets $a \wedge b$, $\bigwedge A$ or $\bigwedge_{a\in A} a$, $a \vee b$, $\bigvee A$ or $\bigvee_{a\in A} a$. Our posets will typically be complete lattices, but we will use the symbols also for the infima and suprema in more general posets in case they exist.

The least resp. largest element (if it exists) will be denoted by 0 resp. 1. Further, we write

$$\downarrow a \text{ for } \{x \mid x \le a\} \quad\text{and}\quad \downarrow A = \{x \mid \exists\ a \in A,\ x \le a\}.$$

The subsets $A \subseteq (X, \le)$ such that $\downarrow A = A$ will be referred to as *down-sets*.

1.2. Monotone maps $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow X$ between posets are (*Galois*) *adjoint, f to the left and g to the right*, if

$$f(x) \le y \iff x \le g(y),$$

equivalently, if $fg \le \mathrm{id}$ and $gf \ge \mathrm{id}$. It is standard that

(1) left adjoints preserve all existing suprema and right adjoints preserve all existing infima,
(2) and on the other hand, if X, Y are complete lattices then each $f\colon X \longrightarrow Y$ preserving all suprema is a left adjoint (has a right adjoint), and each $g\colon Y \longrightarrow X$ preserving all infima is a right adjoint.

1.3. A *frame* is a complete lattice L satisfying the distributivity law

$$\Big(\bigvee A\Big) \wedge b = \bigvee\{a \wedge b \mid a \in A\} \qquad \text{(frm)}$$

for all $A \subseteq L$ and $b \in L$. A *frame homomorphism* preserves all joins and all finite meets. The resulting category will be denoted by

Frm.

A typical frame is the lattice $\Omega(X)$ of open sets of a topological space, and if $f\colon X \longrightarrow Y$ is a continuous map we have a frame homomorphism $\Omega(f)\colon \Omega(Y) \longrightarrow \Omega(X)$ defined by $\Omega(f)(U) = f^{-1}[U]$. Thus, Ω is a contravariant functor **Top** $\longrightarrow$ **Frm**. For an important part of **Top** this is a full embedding — up to the contravariance; to mend that, one introduces the category of *locales*

$$\mathbf{Loc} = \mathbf{Frm}^{\mathrm{op}}$$

for which one usually uses to advantage the concrete representation by right adjoints of frame homomorphisms.

1.4. Coproducts in Frm. The category **Frm** is complete and cocomplete. In this paper we will frequently use the coproduct of two frames. Because it is not quite obvious let us present it in some detail.

The set of all non-empty down-sets of a frame L, ordered by inclusion, denoted $\mathfrak{D}(L)$, is obviously a frame. Given frames L, M we speak of $U \in \mathfrak{D}(L \times M)$ as of a *saturated* $U \subseteq L \times M$ (sometimes one speaks of a *cp-ideal*) if

$$A \times \{b\} \subseteq U \;\Rightarrow\; (\bigvee A, b) \in U \quad \text{and} \quad \{a\} \times B \subseteq U \;\Rightarrow\; (a, \bigvee B) \in U.$$

Because of the void A, B, every saturated U contains

$$\mathsf{n} = \{(0, b), (a, 0) \mid a \in L, b \in M\}$$

and it is easy to see that e.g. each of the sets

$$a \oplus b = {\downarrow}(a, b) \cup \mathsf{n}$$

is saturated. Then

$$L \oplus M = \{U \in \mathfrak{D}(L \times M) \mid U \text{ saturated}\}$$

with the injections

$$\iota_L = (a \mapsto a \oplus 1) \colon L \longrightarrow L \oplus M \quad \text{and} \quad \iota_M = (b \mapsto 1 \oplus b) \colon M \longrightarrow L \oplus M$$

constitutes a coproduct of L and M (see e.g. [5, 11]).

1.4.1. For every $U \in \mathfrak{D}(L \times M)$ there is the smallest saturated V containing U; it will be denoted by

$$\kappa(U).$$

$\kappa(U)$ can be constructed by a transfinite process defining

$$\kappa_0(U) = \{(\bigvee A, b), (a, \bigvee B)) \mid A \times \{b\} \subseteq U, \{a\} \times B \subseteq U\},$$

then $\kappa_{\alpha+1}(U) = \kappa_0(\kappa_\alpha(U))$ and $\kappa_\lambda(U) = \bigcup_{\alpha<\lambda} \kappa_\alpha(U)$ and finally $\kappa(U) = \kappa_\alpha(U)$ if $\kappa_{\alpha+1}(U) = \kappa_\alpha(U)$.

1.4.2. $L \oplus M$ as a tensor product. The same construction can be used to obtain a *tensor product* in the category of sup-lattices (complete lattices with morphisms preserving suprema), see [6] (also e.g. the Appendix in [12]). Thus, for any mapping $f\colon L \times M \longrightarrow K$ such that all $f(a,-)\colon M \longrightarrow K$ and all $f(-,b)\colon L \longrightarrow K$ preserve suprema there is precisely one $\widetilde{f}\colon L \oplus M \longrightarrow K$ preserving all suprema in $L \oplus M$ such that $\widetilde{f}(a \oplus b) = f(a,b)$.

1.5. Composition of saturated sets. Now let $L = M$. For $U, V \in \mathfrak{D}(L \oplus L)$ set

$$U \cdot V = \{(a,c) \mid \exists b \neq 0 \text{ with } (a,b) \in U, (b,c) \in V\} \text{ and}$$
$$U \circ V = \bigvee \{a \oplus c \mid \exists b \neq 0 \text{ with } (a,b) \in U, (b,c) \in V\} \in L \oplus L.$$

Obviously $U \circ V = \kappa(U \cdot V)$ and for U, V saturated

$$U \circ V = \bigvee \{a \oplus c \mid a \oplus b \in U,\ b \oplus c \in V \text{ for some } b \neq 0\} =$$
$$= \kappa\big(\bigcup \{a \oplus c \mid a \oplus b \in U,\ b \oplus c \in V \text{ for some } b \neq 0\}\big).$$

For a $U \in L \oplus L$ set

$$U^{-1} = \bigvee \{a \oplus b \mid b \oplus a \subseteq U\}.$$

1.6. Quantales. Recall that a *quantale* (see e.g. [15]) is a complete lattice L with a binary operation $*$ such that

(Q1) $*$ is associative and
(Q2) $\big(\bigvee A\big) * b = \bigvee\{a * b \mid a \in A\}$ and $b * \big(\bigvee A\big) = \bigvee\{b * a \mid a \in A\}$ for all $A \subseteq L$ and $b \in L$.

2. A few facts about localic groups

2.1. Recall that a localic group is a group object in the category **Loc**. We will work, however, with its dual form in the category **Frm** of frames, that is, with the collection of data $(L, \mu, \gamma, \varepsilon)$ with

$$\mu\colon L \longrightarrow L \oplus L, \quad \gamma\colon L \longrightarrow L \quad \text{and} \quad \varepsilon\colon L \longrightarrow \mathbf{2}$$

satisfying (∇ stands for the codiagonal frame homomorphism $L \oplus L \to L$ and $\sigma = \sigma_L\colon \mathbf{2} \longrightarrow L$ is the trivial embedding)

(G1) $(\mu \oplus \mathrm{id})\mu = (\mathrm{id} \oplus \mu)\mu$
(G2) $(\varepsilon \oplus \mathrm{id})\mu = (\mathrm{id} \oplus \varepsilon)\mu = \mathrm{id}$ and
(G3) $\nabla(\gamma \oplus \mathrm{id})\mu = \nabla(\mathrm{id} \oplus \gamma)\mu = \sigma\varepsilon$.

2.2. It is well known (see e.g. [7]) that any identity that can be classically deduced in a variety of algebras also holds for the corresponding categorical algebras. Thus we have in particular derived the identities

(G4) $\gamma\gamma = \mathrm{id}$,
(G5) $\varepsilon\gamma = \varepsilon$,
(G6) $(\gamma \oplus \gamma)\mu = \tau\mu\tau$ for the isomorphism $\tau\colon L \oplus L \longrightarrow L \oplus L$ defined by $\tau\iota_i = \iota_{3-i}$, and
(Gα) $\alpha\alpha = \mathrm{id}$ and $\alpha\iota_1 = \mu$.

((G4) corresponding to $(x^{-1})^{-1} = x$, (G5) corresponding to $e^{-1} = e$ for the unit e, (G6) corresponding to $(xy)^{-1} = x^{-1}y^{-1}$, and (Gα) following from the behaviour of the map $(x,y) \mapsto (xy, y^{-1})$).

2.3. Lemma. *If $\bigvee$-homomorphisms φ, ψ have left adjoints $\varphi^\#$, $\psi^\#$, then $\varphi \oplus \psi$ has a left adjoint $\varphi^\# \oplus \psi^\#$.*

Proof: First we will prove that for a saturated U,

$$V = \bigcup\{\, \downarrow(\varphi(x), \psi(y)) \mid (x,y) \in U\}$$

is saturated and hence equal to $(\varphi \oplus \psi)(U)$. Let, say $(a_i, b) \in V$, hence $(a_i, b) \leq (\varphi(x_i), \psi(y_i))$ with $(x_i, y_i) \in U$. Then $b \leq \psi(y_i)$ and hence $\psi^\#(b) \leq y_i$ for all i, hence $(x_i, \psi^\#(b)) \in U$ for all i. As U is saturated, $(\bigvee x_i, \psi^\#(b)) \in U$, and finally $(\bigvee a_i, b) \leq (\bigvee \varphi(x_i), \psi\psi^\#(b)) = (\varphi(\bigvee x_i), \psi(\psi^\#(b)))$.

Now we have that $W \subseteq (\varphi \oplus \psi)(U)$ iff for every $(a,b) \in W$ there is an $(x,y) \in U$ such that $a \leq \varphi(x)$ and $b \leq \psi(y)$, that is, $\varphi^\#(a) \leq x$ and $\psi^\#(b) \leq y$, and hence $(\varphi^\# \oplus \psi^\#)(a \oplus b) \leq (x,y)$ which says that $(\varphi^\# \oplus \psi^\#)(W) \subseteq U$. ■

The following result is stated slightly differently in [13, Prop. 6.4.1]. Here we establish it with different argumentation.

2.4. Theorem. *μ has a left adjoint*

$$\mu^\#\colon L \oplus L \longrightarrow L$$

and the maps $\mu \oplus id$ and $id \oplus \mu$ have left adjoints

$$(\mu \oplus id)^\# = \mu^\# \oplus id \quad \textit{and} \quad (id \oplus \mu)^\# = id \oplus \mu^\#.$$

Proof: The former follows from (Gα): $\mu = \alpha\iota_1$ with α an isomorphism and ι_1 a right adjoint to $u \mapsto \bigvee\{x \mid \exists y \neq 0,\ x \oplus y \subseteq U\}$. Then we can infer the latter from the lemma. ■

3. G-quantales

3.1. An *involutive quantale* $(L,*,(-)^{-1})$ is a quantale $(L,*)$ with a map $(-)^{-1}\colon L \longrightarrow L$ satisfying

(Q3) $(a^{-1})^{-1} = a$, $(a*b)^{-1} = b^{-1}*a^{-1}$ and $(\bigvee a_i)^{-1} = \bigvee a_i^{-1}$ (in particular, $(-)^{-1}$ is monotone and $0^{-1} = 0$).

3.2. In the literature, frames with a quantale structure are called *quantal frames*. We will be concerned with a special case of involutive quantal frames $(L,*,(-)^{-1},N)$ endowed with a completely prime filter $N \subseteq L$ satisfying, moreover, the following axioms:

(A1) For every x, $x = \bigvee\{b \mid \exists a \in N, a*b \le x\} = \bigvee\{a \mid \exists b \in N, a*b \le x\}$.
(A2) For every $a \in N$, $\bigvee\{x \mid x^{-1}*x \le a\} = \bigvee\{x \mid x*x^{-1} \le a\} = 1$.
(A3) $a \in N$ implies $a^{-1} \in N$.
(A4) If $a \wedge b \ne 0$ then $a*b^{-1} \in N$.
(A5) N is completely prime, that is, $\bigvee a_i \in N$ implies $a_i \in N$ for some i.

We will speak of *G-quantales*.

3.2.1. Observation. From (A3), (Q3) and (A1) we immediately see that also

$$x = \bigvee\{b \mid \exists a \in N, a^{-1}*b \le x\} = \bigvee\{a \mid \exists b \in N, a*b^{-1} \le x\}.$$

3.2.2. Lemma. *If $y \in N$ then for every x, $x \le x*y$ and $x \le y*x$.*

(Use (A1) for $a = x*y$ resp. $y*x$.)

3.2.3. Lemma. *For every a, $a = \bigvee\{x*y \mid x*y \le a\}$.*

(We have $a = \bigvee\{x \mid \exists y \in N,\ x*y \le a\} \le \bigvee\{x*y \mid \exists y \in N,\ x*y \le a\} \le \bigvee\{x*y \mid x*y \le a\} \le a$.)

3.3. Proposition. *If $a \in N$ then there are $b,c \in N$ such that $b*b \le a$ and $c*c^{-1} \le a$.*

Proof: Use (A1). If $a = \bigvee\{x \mid \exists y \in N,\ x*y \le a\} \in N$ there has to be an $x \in N$ and a $y \in N$ such that $x*y \le a$. Set $b = x \wedge y$. Then $b \in N$ and by monotonicity $b*b \le x*y \le a$.

Similarly, using the equality $a = \bigvee\{x \mid \exists y^{-1} \in N,\ x*y \le a\}$ we obtain $x \in N$ and $y \in N$ such that $x*y^{-1} \le a$, and setting $c = x \wedge y$ we obtain the other statement. ∎

3.4. Lemma. *For every a we have $\{x \wedge y \mid x^{-1} * y \le a\} = \{x \mid x^{-1} * x \le a\}$.*

Proof: $\supseteq$ is trivial.
$\subseteq$: Set $z = x \wedge y$. Then $z^{-1} \le x^{-1}$ and $z \le x$ and hence $z^{-1} * x \le x^{-1} * y \le a$. ■

3.5. Proposition. *If $c * b \le a$, $u * u^{-1} \le c$ and $u \wedge b \ne 0$, then $u \le a$.*

Proof: By (A4), $u^{-1} * b \in N$ and hence, by 3.2.2, $u \le u * u^{-1} * b \le c * b \le a$. ■

3.5.1. Corollary. *Let $b \in N$. Denote by $C(b)$ the cover $\{x \mid x * x^{-1} \le b\}$ from (A2). Then $C(b)b \le b * b$ and hence $b \prec b * b$ (for the rather below relation $\prec$ in L [11]).*

(Apply 3.5 for $a = b * b$ and $c = b$. If $x \in C(b)$, that is, $x * x^{-1} \le b$, and $x \wedge b \ne 0$, then $x \le b * b$.)

3.6. Lemma. *If $a \ne 0$ then $a * a \in N$, $a * a^{-1} \in N$ and $a^{-1} * a \in N$.*

Proof: The first statement follows from (A4). The other two follow from (A4) as well since $a \ne 0$ implies $a^{-1} \ne 0$ (indeed, $a^{-1} = 0$ implies $a = (a^{-1})^{-1} = 0^{-1} = 0$ by (Q3)). ■

3.7. More about the quantale structure. By (Q2) we have quantale adjunctions with quantale implications $\nearrow$ and $\swarrow$

$$a * b \le c \text{ iff } a \le b\nearrow c \quad \text{and} \quad a * b \le c \text{ iff } b \le a\swarrow c \qquad \text{(qadj}*\text{)}$$

and modi ponentes

$$(c\nearrow b) * b \le c, \quad \text{and} \quad a * (c\swarrow a) \le c.$$

We have

$$c\nearrow(a * b) = (c\nearrow b)\nearrow a) \quad \text{and} \quad c\swarrow(a * b) = (c\swarrow a)\swarrow b.$$

From (A1) we immediately obtain

$$\begin{aligned} a &= \bigvee\{a\nearrow x \mid x \in N\} = \bigvee\{a\nearrow x^{-1} \mid x \in N\} = \\ &= \bigvee\{a\swarrow x \mid x \in N\} = \bigvee\{a\swarrow x^{-1} \mid x \in N\}. \end{aligned}$$

Consequently,

$$\text{for all } x \in N, \quad a\nearrow x \le a \ \text{ and } \ a\swarrow x \le a.$$

From (A4) (in the form $a * b^{-1} \le n \;\Rightarrow\; a \wedge b = 0$) we obtain that

$$(n\nearrow a^{-1}) \le a^* \quad \text{and} \quad (n\swarrow a)^{-1} \le a^*.$$

3.8. The quantale $(L \oplus L, \circ)$. Recall the $\circ$ from 1.5. We will show that it is a quantale operation on $L \oplus L$. The L in question is now not enriched by any further structure. Nevertheless, we will see later in 5.3.2 that in the case of a localic group it is connected with our $(L, *)$.

3.8.1. The following fundamental fact about the composition $\circ$ was proved in [8]. Since the original proof is not quite easily accessible we reproduce it here for convenience of the reader.

Lemma. *For any $A, B \in \mathfrak{D}(L \times L)$,*

$$\kappa(A) \circ \kappa(B) = A \circ B.$$

Proof: It suffices to show that $\kappa(A) \cdot \kappa(B) \subseteq \kappa(A \cdot B)$. For this, consider the nonempty set

$$\mathfrak{E} = \{E \in \mathfrak{D}(L \times L) \mid A \subseteq E \subseteq \kappa(A),\ E \cdot B \subseteq \kappa(A \cdot B)\}$$

and let us show that $\kappa_0(E) \in \mathfrak{E}$ whenever $E \in \mathfrak{E}$.

Consider $(x, y) \in \kappa_0(E) \cdot B$ (recall 1.4.1) and $z \neq 0$ such that $(x, z) \in \kappa_0(E)$ and $(z, y) \in B$. If $(x, z) = (x, \bigvee S)$ for some S with $\{x\} \times S \subseteq E$, then there is a nonzero $s \in S$ such that $(x, s) \in E$ and $(s, y) \in B$ and, therefore, $(x, y) \in E \cdot B \subseteq \kappa(A \cdot B)$. On the other hand, if $(x, z) = (\bigvee S, z)$ for some S with $S \times \{z\} \subseteq E$, then $(s, y) \in E \cdot B$ for every $s \in S$ and, thus, $(x, y) \in \kappa_0(E \cdot B) \subseteq \kappa(A \cdot B)$.

Moreover, $\bigcup_{F \in \mathfrak{F}} F \in \mathfrak{E}$ for any nonempty $\mathfrak{F} \subseteq \mathfrak{E}$, since

$$\Big(\bigcup_{F \in \mathfrak{F}} F\Big) \cdot B \subseteq \bigcup_{F \in \mathfrak{F}} (F \cdot B).$$

Therefore $S = \bigcup_{E \in \mathfrak{E}} E$ belongs to $\mathfrak{E}$, i.e., $\mathfrak{E}$ has a largest element S. But $\kappa_0(S) \in \mathfrak{E}$ so $S = \kappa_0(S)$, i.e., S is saturated. Hence $\kappa(A) = S \in \mathfrak{E}$ and, consequently, $\kappa(A) \cdot B \subseteq \kappa(A \cdot B)$. By symmetry, $A \cdot \kappa(B) \subseteq \kappa(A \cdot B)$.

In conclusion, we have $\kappa(A) \cdot \kappa(B) \subseteq \kappa(A \cdot \kappa(B)) \subseteq \kappa^2(A \cdot B) = \kappa(A \cdot B)$, as desired. ∎

3.8.2. Proposition. *For any $A_i, B \in L \oplus L$, $(\bigvee A_i) \circ B = \bigvee (A_i \circ B)$.*

Proof: The inclusion '$\supseteq$' is obvious. To prove the converse, it suffices to observe that, by the Lemma, $(\bigvee A_i) \circ B = (\bigcup A_i) \circ B$, and, furthermore, that $(\bigcup A_i) \circ B \subseteq \bigvee (A_i \circ B)$: indeed, if $(x, y) \in \bigcup A_i$ and $(y, z) \in B$ with $y \neq 0$, then (x, y) belongs to some A_j and $(x, z) \in A_j \circ B \subseteq \bigcup (A_i \circ B)$. ∎

4. A G-quantale on a localic group; reconstruction of the group structure

4.1. For a localic group $(L,\mu,\gamma,\varepsilon)$ define $a^{-1}=\gamma(a)$, $N=\varepsilon^{-1}[\{1\}]$. By 2.4, μ has a left adjoint $\mu^{\#}$; we set

$$a*b=\mu^{\#}(a\oplus b).$$

4.2. The operation $*$ and the mapping $\boldsymbol{\mu}$. In any quantale $(L,*)$, (Q2) claims that $*\colon L\times L\longrightarrow L$ is a bi-$\bigvee$-morphism. Since $L\oplus L$ is a tensor product in $\vee$**Lat** (recall 1.4.2) we have the unique lifting

$$\begin{tikzcd} L\oplus L \arrow[r, "\varphi"] & L \\ L\times L \arrow[u, "((x,y)\mapsto x\oplus y)"] \arrow[ur, "*"'] & \end{tikzcd}$$

with φ a $\bigvee$-morphism, which then has a right adjoint φ_*. We have

$$x\oplus y\le\varphi_*(a)\quad\text{iff}\quad\varphi(x\oplus y)=x*y\le a$$

and obtain the formula

$$\varphi_*(a)=\bigvee\{x\oplus y\mid x*y\le a\}=\{(x,y)\mid x*y\le a\}$$

(since the last set is obviously saturated). Thus in particular

if $(L,*)$ was obtained from a localic group $(L,\mu,\gamma,\varepsilon)$, then μ is uniquely determined by $*$, namely as

$$\mu(a)=\bigvee\{x\oplus y\mid x*y\le a\}=\{(x,y)\mid x*y\le a\}. \tag{4.2.1}$$

4.3. Lemma. *We have, for every x,*

$$x=\bigvee\{a\mid \exists b\in N, a*b\le x\}=\bigvee\{b\mid \exists a\in N, a*b\le x\}.$$

Proof: By (G2) and (4.2.1) we have

$$x=(\mathrm{id}\oplus\varepsilon)\mu(x)=(\mathrm{id}\oplus\varepsilon)\left(\bigvee\{a\oplus b\mid a*b\le x\}\right)=\bigvee\{a\oplus\varepsilon(b)\mid a*b\le x\}.$$

Now, representing $L\oplus\mathbf{2}$ as L (understanding $a\oplus 1$ as a and $a\oplus 0$ as 0), $a\oplus\varepsilon(b)$ is a if $\varepsilon(b)=1$, that is, if $b\in N$, otherwise it is 0; hence the first equality. The second one is obtained analogously using $(\varepsilon\oplus\mathrm{id})\mu$ instead of $(\mathrm{id}\oplus\varepsilon)\mu$. ■

4.3.1. Theorem. *$(L,*,(-)^{-1},N)$ is a G-quantale.*

Proof: (Q2), (Q3) and (A5) are immediate, (A3) follows from (G3) and (A1) is in 4.3.

(Q1): Since, by (G1), $(\mu \oplus \mathrm{id})\mu = (\mathrm{id} \oplus \mu)\mu$, we have, by 2.4, $\mu^{\#}(\mu^{\#} \oplus \mathrm{id}) = \mu^{\#}(\mathrm{id} \oplus \mu^{\#})$ so that

$$a * (b * c) = \mu^{\#}(a \oplus \mu^{\#}(b \oplus c)) = \mu^{\#}(\mathrm{id} \oplus \mu^{\#})(a \oplus b \oplus c) = \\ = \mu^{\#}(\mu^{\#} \oplus \mathrm{id})(a \oplus b \oplus c) = (a * b) * c.$$

(A2): By (G3), (4.2.1) and (A1)

$$1 = \sigma\varepsilon(a) = \nabla(\gamma \oplus \mathrm{id})\mu(a) = \nabla(\gamma \oplus \mathrm{id})(\bigvee\{x \oplus y \mid x * y \le a\}) = \\ = \nabla(\gamma \oplus \mathrm{id})(\bigvee\{x^{-1} \oplus y \mid x^{-1} * y \le a\} = \\ = \bigvee\{\nabla(x \oplus y) \mid x^{-1} * y \le a\} = \bigvee\{x \wedge y \mid x^{-1} * y \le a\} = \\ = \bigvee\{x \mid x^{-1} * x \le a\}.$$

(A4): Let $a * b^{-1} \notin N$. We have $a \oplus b^{-1} \le \mu(a * b^{-1})$ and hence, by (G3),

$$a \wedge b = \nabla(a \oplus b) = \nabla(1 \oplus \gamma)(a \oplus b^{-1}) \le \\ \le \nabla(1 \oplus \gamma)\mu(a * b^{-1}) = \sigma\varepsilon(a * b^{-1}) = 0. \quad \blacksquare$$

4.4. Lemma. *For (localic group) homomorphisms $h\colon L \longrightarrow M$ we have*

$$h(a) * h(b) \le h(a * b). \tag{$*$hom}$$

Proof: From $\mu^{\#}(a \oplus b) = a * b$ we obtain $a \oplus b \le \mu(a * b)$ and further

$$h(a) \oplus h(b) \le (h \oplus h)\mu(a * b) = \mu(h(a * b))$$

and ($*$hom) follows by adjunction. $\blacksquare$

4.4.1. Note. The inequality ($*$hom) is standardly used for the definition of a quantale homomorphism. Note that here we cannot have more. Consider (discrete) $\mathbb{Z}$ (it is locally compact and scattered; hence everything is the same classically and localically) and $h = \Omega(f)$ for $f = (n \mapsto 2n)\colon \mathbb{Z} \longrightarrow \mathbb{Z}$.

Take $U = \{0, 1\}$. Then $U{+}U = \{0, 1, 2\}$, $f^{-1}[U] = \{0\}$ and $f^{-1}[U{+}U] = \{0, 2\}$ so that

$$h(U) + h(U) = \{0\} \subset \{0, 1\} = h(U + U).$$

4.4.2. Theorem. *A map $h\colon (L, \mu, \gamma, \varepsilon) \longrightarrow (L', \mu', \gamma', \varepsilon')$ is a localic group homomorphism iff in the associated involutive quantales (with $N = \{a \mid \varepsilon(a) = 1\}$) holds*

$$h(a) * h(b) \le h(a * b), \quad h(a^{-1}) = h(a)^{-1} \quad \textit{and} \quad h[N] \subseteq N'.$$

Proof: To simplify notation we will omit the dashes in "L'" etc..

The second and the third part of the formula is trivial, and in the first one the implication $\Rightarrow$ is in Lemma above. Thus, let $h(a) * h(b) \le h(a * b)$. Then $\mu^{\#}(h(a) \oplus h(b)) \le h(a * b)$ and hence

$$h(a) \oplus h(b) \le \mu h(a * b). \tag{!}$$

We have $\mu(x) = \bigvee\{a \oplus b \mid a * b \le x\}$ and hence by (!)

$$\begin{aligned}(h \oplus h)\mu(x) &= \bigvee\{h(a) \oplus h(b) \mid a * b \le x\} \le \\ &\le \bigvee\{\mu h(a * b) \mid a * b \le x\} \le \mu h(x),\end{aligned}$$

that is, $(h \oplus h)\mu \le \mu h$, and since any localic group is regular ([11, XV.1.5.5]) and hence T_U [1], $(h \oplus h)\mu = \mu h$. ■

4.5. Reconstruction of the group. We will now show that under some circumstances the group structure can be reconstructed from the quantale one.

4.5.1. Proposition. *Let $(L, *, (-)^{-1}, N)$ be a G-quantale and define $\gamma\colon L \longrightarrow L$ by $\gamma(x) = x^{-1}$, $\varepsilon(x) = 1$ iff $x \in N$, and μ as in (4.2.1). If μ preserves joins then $(L, \mu, \gamma, \varepsilon)$ is the localic group inducing $(L, *, (-)^{-1}, N)$.*

Proof: By assumption (and by the adjunction), μ is a (complete) frame homomorphism, by (A5) ε is a frame homomorphism, and by (Q3) and (A3), γ is an isomorphism.

We have, by (Q1),

$$\begin{aligned}(\mu \oplus \mathrm{id})(\mu(a)) &= (\mu \oplus \mathrm{id})(\bigvee\{x \oplus y \mid x * y \le a\}) = \\ &= \bigvee\{\mu(x) \oplus y \mid x * y \le a\} = \\ &= \bigvee\{\bigvee\{u \oplus v \mid u * v \le x\} \oplus y \mid x * y \le a\} = \\ &= \bigvee\{u \oplus v \oplus y \mid u * v \le x,\ x * y \le a\} = \\ &= \bigvee\{u \oplus v \oplus y \mid u * v * y \le a\}\end{aligned}$$

and the same is obtained computing $(\mathrm{id} \oplus \mu)(\mu(a))$.

Next, by (A1), representing $\mathbf{2} \oplus L$ as L understanding $1 \oplus a$ as a and $0 \oplus a$ as 0, we obtain

$$(\varepsilon \oplus \mathrm{id})\mu(a) = \bigvee\{\varepsilon(x) \oplus y \mid x * y \le a\} = \bigvee\{y \mid x \in N, x * y \le a\} = a.$$

[1] The axiom T_U claims for a frame L that homomorphisms $h_1, h_2\colon L \longrightarrow M$ such that $h_1 \le h_2$ coincide. It was introduced by Isbell in [3], and it is implied even by weaker axioms than regularity — e.g. fitness, or strong Hausdorff property. See also [5, 12].

Finally, by (A2) (and the monotonicity from (Q1) and (Q3)) we obtain

$$\begin{aligned}\nabla(\gamma\oplus\mathrm{id})\mu(a) &= \bigvee\{x^{-1}\wedge y \mid x*y\le a\} = \\ &= \bigvee\{x\wedge y \mid x^{-1}*y\le a\} = \bigvee\{z \mid z^{-1}*z\le a\}\end{aligned}$$

(if we set $z = x\wedge y$ we have $z^{-1}\le x^{-1}$ and $z\le y$, hence $z^{-1}*z\le x^{-1}*y$). ■

4.5.2. The formula $\mu(a) = \bigvee\{x\oplus y \mid x*y\le a\} = \{(x,y) \mid x*y\le a\}$ can be by (qadj$*$) further rewritten as

$$\mu(a) = \bigcup_{y\in L}\downarrow(y\nearrow a, y) \quad \text{resp.} \quad \mu(a) = \bigcup_{x\in L}\downarrow(x, x\swarrow a)$$

and hence the condition for μ to preserve joins can be formulated as follows:

Let $\bigvee a_i = a$. Then, for every saturated U, we have the implications

$$\begin{aligned}&\forall x\ \forall i,\ (x\nearrow a_i, x)\in U \quad\Rightarrow\quad \forall x,\ (x\nearrow a, x)\in U, \quad \text{and}\\ &\forall x\ \forall i,\ (x, x\swarrow a_i)\in U \quad\Rightarrow\quad \forall x,\ (x, x\swarrow a)\in U.\end{aligned}$$

Thus, from 4.5.1 we obtain

4.5.3. Theorem. *The construction from 4.5.1 provides a one-to-one correspondence* $(L,\mu,\gamma,\varepsilon) \leftrightarrow (L,*,(-)^{-1},N)$ *between localic groups and G-quantales satisfying for saturated* U *the implications*

$$\begin{aligned}&\forall x\ \forall i,\ (x\nearrow a_i, x)\in U \quad\Rightarrow\quad \forall x,\ (x\nearrow \bigvee_i a_i, x)\in U, \quad \text{and}\\ &\forall x\ \forall i,\ (x, x\swarrow a_i)\in U \quad\Rightarrow\quad \forall x,\ (x, x\swarrow \bigvee_i a_i)\in U.\end{aligned}$$

5. The quantale and uniformities

5.1. Entourage (Weil) uniformities. An *entourage* in L is an element $E\in L\oplus L$ such that

$$\{u \mid u\oplus u\subseteq E\}$$

is a cover[2].

For a system $\mathcal{E}$ of entourages in L we write

$$b \lhd_{\mathcal{E}} a \quad \text{if there is an } E\in\mathcal{E} \text{ such that } E\circ(b\oplus b)\subseteq a\oplus a.$$

A *uniformity* on L is a system $\mathcal{E}$ of entourages such that

[2] This simple condition expresses precisely the fact that the corresponding open sublocale contains the diagonal of $L\oplus L$ (see [11, XII.1.4], or more explicitly [1, 3.4]).

(E1) if $E \in \mathcal{E}$ and $E \subseteq F$ then $F \in \mathcal{E}$,
(E2) if $E, F \in \mathcal{E}$ then $E \cap F \in \mathcal{E}$,
(E3) if $E \in \mathcal{E}$ then $E^{-1} \in \mathcal{E}$,
(E4) for every $E \in \mathcal{E}$ there is an $F \in \mathcal{E}$ such that $F \circ F \subseteq E$, and
(E5) for every $a \in L$, $a = \bigvee\{b \mid b \lhd_{\mathcal{E}} a\}$.

Without (E5) one speaks of a *pre-uniformity*, without (E1) one speaks of a *basis of a (pre-) uniformity* (in the latter case one obtains, of course, a (pre-) uniformity adding all the F with $F \supseteq E \in \mathcal{E}$).

5.2. Cover (Tukey) uniformities. The uniformities above model the classical uniformities as defined by Weil ((E5) takes care for the uniformity to be in agreement with the space structure which is, in the classical case, automatic). Equivalently one defines a *cover uniformity* (briefly, a *c-uniformity*), following the classical approach by Tukey, as a system $\mathcal{A}$ of covers of L such that

(C1) if $A \in \mathcal{A}$ and $A \leq B$ then $B \in \mathcal{A}$,
(C2) if $A, B \in \mathcal{A}$ then $A \wedge B = \{a \wedge b \mid a \in A, b \in B\} \in \mathcal{A}$,
(C4) for every $A \in \mathcal{A}$ there is a $B \in \mathcal{A}$ such that $BB \leq A$, and
(C5) for every $a \in L$, $a = \bigvee\{b \mid b \lhd_{\mathcal{A}} a\}$

where

- $A \leq B$ indicates that the cover A *refines* B, that is, for each $a \in A$ there is a $b \in B$ with $a \leq b$,
- one writes for a cover A and an element a, $Aa = \bigvee\{x \in A \mid x \wedge a \neq 0\}$ and for two covers A, B, $AB = \{Ab \mid b \in B\}$, and
- we write $b \lhd_{\mathcal{A}} a$ if there is an $A \in \mathcal{A}$ such that $Ab \leq a$.

5.2.1. Notes. 1. The equivalence is obtained by replacing an entourage uniformity $\mathcal{E}$ by the system of covers $\{\{u \mid u \oplus u \subseteq E\} \mid E \in \mathcal{E}\}$. In the classical situation, the fact that this leads to equivalent theories is easy. In the point-free extension it is in general a non-trivial fact that came as a pleasant surprise: note that the $L \oplus L$ does not precisely correspond with the product of spaces. In concrete cases (one of which will be discussed below) the equivalence is more straightforward.

2. Note the absence of a counterpart to the (E3) in the c-uniformities: they are symmetric automatically. A certain advantage of the entourage approach is that it allows a direct generalization modelling the non-symmetric uniformities.

5.3. Right and left uniformities on a G-quantale. For each $a \in N$ let

$$E_r(a) = \bigvee\{x \oplus y \mid x * y^{-1} \leq a\} \in L \oplus L \text{ and}$$
$$E_l(a) = \bigvee\{x \oplus y \mid x^{-1} * y \leq a\} \in L \oplus L.$$

We will discuss the right one generated by the $E_r(a)$, the left one is quite analogous.

5.3.1. Lemma. *Each $E_r(a)$ in an entourage of L.*

Proof: By (A2), $\bigvee\{x \mid x \oplus x \subseteq E_r(a)\} \geq \bigvee\{x \mid x * x^{-1} \leq a\} = 1$. ■

5.3.2. Lemma. *For every $a, b \in N$, $E_r(a) \circ E_r(b) \subseteq E_r(a * b)$.*

Proof: It suffices to check that $x * y^{-1} \leq a$ and $y * z^{-1} \leq b$, with $y \neq 0$, implies $x * z^{-1} \leq a * b$: by 3.6, if $y \neq 0$ then $y^{-1} * y \in N$; then, by 3.2.2, $x * z^{-1} \leq x * y^{-1} * y * z^{-1} \leq a * b$. ■

Remark. Recall 3.8. Note that the inclusion states that we have here a quantale homomorphism

$$E_r \colon (L \oplus L, \circ) \longrightarrow (L, *).$$

Similarly with E_l.

5.3.3. Proposition. *The system $E_r(a)$ $(a \in N)$ is a base for a preuniformity $\mathcal{U}_r$ on L.*

Proof:

- $E_r(a) \cap E_r(b) = \bigvee\{(x \oplus y) \cap (z \oplus w) \mid x * y^{-1} \leq a, z * w^{-1} \leq b\} = \bigvee\{(x \wedge z) \oplus (y \wedge w) \mid x * y^{-1} \leq a, z * w^{-1} \leq b\} \supseteq E_r(a \wedge b)$ and $a \wedge b \in N$.
- By (Q3), $x * y^{-1} \leq a$ iff $y * x^{-1} \leq a^{-1}$. Hence $E_r(a)^{-1} = E_r(a^{-1})$ and $a^{-1} \in N$ by (A3).
- By 3.3 and 5.3.2, for every $a \in N$ there is some $b \in N$ such that $E_r(b) \circ E_r(b) \subseteq E_r(b * b) \subseteq E_r(a)$. ■

5.3.4. Lemma. *Let $a \in N$. If $a * y \leq x$ then $E_r(a) \circ (y \oplus y) \subseteq x \oplus x$.*

Proof: By 3.8.1 we know that

$$E_r(a) \circ (y \oplus y) = \bigcup\{x \oplus y \mid x * y^{-1} \leq a\} \circ (y \oplus y).$$

Let (α, β) such that $\beta \neq 0$ and $\alpha * \beta^{-1} \leq a$, and let $(\beta, \gamma) \in y \oplus y$. By 3.2.2, $\gamma \leq y \leq x$. On the other hand, $\beta \neq 0$ implies $\beta^{-1} * \beta \in N$ (by 3.6) and thus, using 3.2.2 again, $\alpha \leq \alpha * \beta^{-1} * \beta \leq a * \beta \leq a * y \leq x$. Hence $(\alpha, \gamma) \in x \oplus x$. ■

5.3.5. Theorem. *$\mathcal{U}_r$ is admissible on L, that is, a uniformity.*

Proof: Let $x \in L$. By (A1), $x = \bigvee\{y \mid \exists a \in N, a * y \le x\}$. Then, by 5.3.2,

$$x \le \bigvee\{y \mid \exists a \in N, E_r(a) \circ (y \oplus y) \subseteq x \oplus x\} = \bigvee\{y \mid y \lhd_{\mathcal{U}_r} x\} \le x. \quad \blacksquare$$

5.4. Corollary. *It follows, in particular, that a G-quantale L is (completely) regular.*

5.5. Uniformities induced by metrics. Let S be a completely ordered semigroup with an operation + and least element 0. An S-metric (further just metric) on a frame L is a mapping

$$d\colon L \times L \longrightarrow S$$

together with an $N \subseteq S$ such that

(M1) if $y \neq 0$ then $d(x,z) \le d(x,y) + d(y,z)$,
(M2) $d(\bigvee_i a_i, \bigvee_j b_j) = \bigvee_{ij} d(a_i, b_j)$,
(M3) for every $a \in N$ there is a $b \in N$ such that $b + b \le a$,
(M4) for each $a \in N$, $C(a) = \{u \mid d(u,u) \le a\}$ is a cover, and
(M5) for every $a \in L$, $a = \bigvee\{b \mid b \lhd_{\mathcal{C}} a\}$ where $\mathcal{C} = \{C(a) \mid a \in N\}$.

It is easy to check that the system of covers $\{C(a) \mid a \in N\}$ constitutes a cover uniformity in the sense of 5.2. We speak of an *S-metric uniformity.*

5.5.1. Examples. 1. For a classical metric space (X, ρ) consider on the frame $\Omega(X)$ the $\mathbb{R}$-metric $d(U,V) = \operatorname{diam}(U \cup V)$. Then the $\mathbb{R}$-metric uniformity is the classical metric uniformity.

2. More generally, consider a metric diameter δ on a frame L (see e.g. [11, Chapter XI]). Then we have an $\mathbb{R}$-metric $d(U,V) = \delta(U \cup V)$ and the resulting uniformity is the standard point-free metric uniformity.

5.5.2. The metric nature of the $*$-uniformity on a G-quantale. If $(L, *)$ is a G-quantale then $d_r(x,y) = x * y^{-1}$ is easily seen to be an L-metric on L. We will show that the resulting metric uniformity coincides with the uniformity $\mathcal{U}_r$ above.

5.5.3. Lemma. *Let $(L, *)$ be a G-quantale and $a, b \in N$. Then*

$$C(a)x \le a * x$$

and if $b * b * b^{-1} \le a$ then

$$b * x \le C(a)x.$$

Proof: I. Let $y \in C(a)$, that is, $y * y^{-1} \le a$, and $y \wedge x \ne 0$. Then $y^{-1} * x \in N$ and $y \le y * y^{-1} * x \le a * x$.

II. We can assume that $b * x \ne 0$. We have $\bigvee\{u \mid u * u^{-1} \le b\} = 1$ and hence

$$x = \bigvee\{x \wedge u \mid u * u^{-1} \le b\} = \bigvee\{y \mid y * y^{-1} \le b, 0 \ne y \le x\}$$

and

$$b * x = \bigvee\{b * y \mid y * y^{-1} \le b, 0 \ne y \le x\}.$$

We have for the y on the right hand side

$$d(b * y, b * y) = b * y * y^{-1} * b^{-1} \le b * b * b^{-1} \le a \text{ and } 0 \ne y \le (b * y) \wedge x$$

so that $b * y \le C(a)x$ and, finally,

$$b * x = \bigvee\{b * y \mid y * y^{-1} \le b, y \le x\} \le C(a)x. \quad \blacksquare$$

5.5.4. Corollary. *The metric uniformity of d_r and the right group uniformity of a localic group coincide; similarly for the metric $d_l(x, y) = x^{-1} * y$ and the left group uniformity.*

6. Homomorphisms and uniformity

6.1. Uniformness of homomorphisms. Recall that a frame homomorphism $h: L \longrightarrow M$ is *uniform* with respect to uniformities $\mathcal{E}$ on L and $\mathcal{F}$ on M if

$$\forall E \in \mathcal{E},\ (h \oplus h)(E) \in \mathcal{F}$$

(in terms of bases, $\forall E \in \mathcal{E}\ \exists F \in \mathcal{F}, (h \oplus h)(E) \supseteq F$).

In the language of cover uniformities we have equivalently (in terms of bases) $h: (L, \mathcal{A}) \longrightarrow (M, \mathcal{B})$ uniform if

$$\forall A \in \mathcal{A}\ \exists B \in \mathcal{B} \quad \text{such that} \quad h[A] \ge B$$

(the last $\ge$ in the sense of refinement as in 5.2).

It is known that localic group homomorphism are uniform [13]. In this short section we will discuss this phenomenon from the point of view of the G-quantale structure.

6.2. For any localic group homomorphism $h\colon L \longrightarrow M$, the identity $\mu_M h = (h \oplus h)\mu_L$ means precisely that

$$\text{(H)} \qquad \bigvee\{h(a) \oplus h(b) \mid a * b \le c\} = \bigvee\{u \oplus v \mid u * v \le h(c)\},$$

since $\mu(c) = \bigvee\{a \oplus b \mid a * b \le c\}$.

6.2.1. Observations. 1. It follows easily from the distributivity law (Q2) on $*$ that $\bigcup\{a \oplus b \mid a * b \le c\}$ is saturated. Hence

$$\bigvee\{a \oplus b \mid a * b \le c\} = \bigcup\{a \oplus b \mid a * b \le c\}$$

and (H) reduces to

$$\text{(H')} \qquad \bigcup\{h(a) \oplus h(b) \mid a * b \le c\} = \bigcup\{u \oplus v \mid u * v \le h(c)\}.$$

2. From the properties of the involution, it also follows easily that $\bigcup\{a \oplus b \mid a * b^{-1} \le c\}$ is saturated. Hence

$$\bigvee\{a \oplus b \mid a * b^{-1} \le c\} = \bigcup\{a \oplus b \mid a * b^{-1} \le c\}.$$

In particular, $E_r(a) = \bigcup\{x \oplus y \mid x * y^{-1} \le a\}$.

6.2.2. Proposition. *Let $h\colon L \longrightarrow M$ be a frame homomorphism between G-quantales $(L, *, (-)^{-1}, N_L)$ and $(M, *, (-)^{-1}, N_M)$. If h satisfies (H), $h[N_L] \subseteq N_M$ and $h(x^{-1}) = h(x)^{-1}$ for every $x \in L$, then*

$$h\colon (L, \mathcal{U}_r^L) \longrightarrow (M, \mathcal{U}_r^M)$$

is a uniform homomorphism.

Proof: Let $a \in N_L$ and $E_r^L(a) \in \mathcal{U}_r^L$. Since $h(a) \in N_M$, it suffices to show that $(h \oplus h)(E_r^L(a)) \supseteq E_r^M(h(a))$. By 6.2.1, this amounts to

$$\bigcup\{h(x) \oplus h(y) \mid x * y^{-1} \le a\} \supseteq \bigcup\{u \oplus v \mid u * v^{-1} \le h(a)\}.$$

Let $u * v^{-1} \le h(a)$ and denote v^{-1} by w. Then $v = w^{-1}$. By (H), $(u, w) \le (h(x), h(y))$ for some x, y such that $x * y \le a$. Hence $(u, v) = (u, w^{-1}) \le (h(x), h(y)^{-1}) = (h(x), h(y^{-1}))$ and $x * (y^{-1})^{-1} = x * y \le a$. ■

6.3. Observation. It is easy to see that the other inclusion in the proof above also holds and we have indeed $(h \oplus h)(E_r(a)) = E_r(h(a))$ for every $a \in N_L$.

6.4. Note. The inclusion $\supseteq$ in (H') is clearly equivalent to

$$\text{(H1)} \qquad u * v \le h(c) \ \Rightarrow \ \exists a, b \in L: \ u \le h(a), \ v \le h(b), \ a * b \le c$$

while the converse inclusion is equivalent to the quantale homomorphism condition $h(a) * h(b) \le h(a * b)$.

Hence, for quantale homomorphisms, (H) is equivalent to (H1); it seems that to obtain the uniform property for the homomorphisms between general G-quantales, the condition (H1) in some form will have to be assumed.

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