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ON PRESHEAF SUBMONADS OF QUANTALE-ENRICHED CATEGORIES

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ABSTRACT: This paper focus on the presheaf monad and its submonads on the realm of V-categories, for a quantale V. First we present two characterisations of presheaf submonads, both using V-distributors: one based on admissible classes of V-distributors, and other using Beck-Chevalley conditions on V-distributors. Then we focus on the study of the corresponding Eilenberg-Moore categories of algebras, having as main examples the formal ball monad and the so-called Lawvere monad.

KEYWORDS: Quantale, V-category, distributor, lax idempotent monad, Presheaf monad, Ball monad, Lawvere monad.
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Introduction

Having as guideline Lawvere's point of view that it is worth to regard metric spaces as categories enriched in the extended real half-line $[0,\infty]_+$ (see [17]), we regard both the formal ball monad and the monad that identifies Cauchy complete spaces as its algebras – which we call here the *Lawvere monad* – as submonads of the presheaf monad on the category **Met** of $[0,\infty]_+$-enriched categories. This leads us to the study of general presheaf submonads on the category of V-enriched categories, for a given quantale V.

Here we expand on known general characterisations of presheaf submonads and their algebras, and introduce a new ingredient – conditions of Beck-Chevalley type – which allows us to identify properties of functors and natural transformations, and, most importantly, contribute to a new facet of the behaviour of presheaf submonads.

In order to do that, after introducing the basic concepts needed to the study of V-categories in Section 1, Section 2 presents the presheaf monad and a characterisation of its submonads using admissible classes of V-distributors

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which is based on [2]. Next we introduce the already mentioned Beck-Chevalley conditions (BC*) which resemble those discussed in [5], with V-distributors playing the role of V-relations. In particular we show that lax idempotency of a monad $\mathbb{T}$ on V-**Cat** can be identified via a BC* condition, and that the presheaf monad satisfies fully BC*. This leads to the use of BC* to present a new characterisation of presheaf submonads in Section 4.

The remaining sections are devoted to the study of the Eilenberg-Moore category induced by presheaf submonads. In Section 5, based on [2], we detail the relationship between the algebras, (weighted) cocompleteness, and injectivity. Next we focus on the algebras and their morphisms, first for the formal ball monad, and later for a general presheaf submonad. We end by presenting the relevant example of the presheaf submonad whose algebras are the so-called Lawvere complete V-categories [3], which, when $V=[0,\infty]_+$, are exactly the Cauchy complete (generalised) metric spaces, while their morphisms are the V-functors which preserve the limits for Cauchy sequences.

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1. Preliminaries

Our work focus on V-categories (or V-enriched categories, cf. [7, 17, 15]) in the special case of V being a quantale.

Throughout V is a *commutative and unital quantale*; that is, V is a complete lattice endowed with a symmetric tensor product $\otimes$, with unit $k\neq\bot$, commuting with joins, so that it has a right adjoint hom; this means that, for $u,v,w\in V$,

$$u\otimes v\le w\;\Leftrightarrow\; v\le \hom(u,w).$$

As a category, V is a complete and cocomplete (thin) symmetric monoidal closed category.

Definition 1.1. A *V-category* is a pair (X,a) where X is a set and $a\colon X\times X\to V$ is a map such that:

(R) for each $x\in X$, $k\le a(x,x)$;

(T) for each $x,x',x''\in X$, $a(x,x')\otimes a(x',x'')\le a(x,x'')$.

If (X,a), (Y,b) are V-categories, a *V-functor* $f\colon(X,a)\to(Y,b)$ is a map $f\colon X\to Y$ such that

(C) for each $x,x'\in X$, $a(x,x')\le b(f(x),f(x'))$.

The category of V-categories and V-functors will be denoted by V-**Cat**. Sometimes we will use the notation $X(x,y) = a(x,y)$ for a V-category (X,a) and $x,y \in X$.

We point out that V has itself a V-categorical structure, given by the right adjoint to $\otimes$, hom; indeed, $u \otimes k \le u \;\Rightarrow\; k \le \hom(u,u)$, and $u \otimes \hom(u,u') \otimes \hom(u',u'') \le u' \otimes \hom(u',u'') \le u''$ gives that $\hom(u,u') \otimes \hom(u',u'') \le \hom(u,u'')$. Moreover, for every V-category (X,a), one can define its *opposite V-category* $X^{\mathrm{op}} = (X,a^\circ)$, with $a^\circ(x,x') = a(x',x)$ for all $x,x' \in X$.

Examples 1.2. (1) For $V = \mathbf{2} = (\{0<1\},\wedge,1)$, a $\mathbf{2}$-category is an *ordered set* (not necessarily antisymmetric) and a $\mathbf{2}$-functor is a *monotone map*. We denote $\mathbf{2}$-**Cat** by **Ord**.

(2) The lattice $V = [0,\infty]$ ordered by the "greater or equal" relation $\ge$ (so that $r \wedge s = \max\{r,s\}$, and the supremum of $S \subseteq [0,\infty]$ is given by $\inf S$) with tensor $\otimes = +$ will be denoted by $[0,\infty]_+$. A $[0,\infty]_+$-category is a *(generalised) metric space* and a $[0,\infty]_+$-functor is a *non-expansive map* (see [17]). We denote $[0,\infty]_+$-**Cat** by **Met**. We note that

$$\hom(u,v) = v \ominus u := \max\{v-u,0\},$$

for all $u,v \in [0,\infty]$.

If instead of $+$ one considers the tensor product $\wedge$, then $[0,\infty]_\wedge$-**Cat** is the category **UMet** of *ultrametric spaces* and *non-expansive maps*.

(3) The complete lattice $[0,1]$ with the usual "less or equal" relation $\le$ is isomorphic to $[0,\infty]$ via the map $[0,1] \to [0,\infty]$, $u \mapsto -\ln(u)$ where $-\ln(0) = \infty$. Under this isomorphism, the operation $+$ on $[0,\infty]$ corresponds to the multiplication $*$ on $[0,1]$. Denoting this quantale by $[0,1]_*$, one has $[0,1]_*$-**Cat** isomorphic to the category **Met** $= [0,\infty]_+$-**Cat** of (generalised) metric spaces and non-expansive maps.

Since $[0,1]$ is a frame, so that finite meets commute with infinite joins, we can also consider it as a quantale with $\otimes = \wedge$. The category $[0,1]_\wedge$-**Cat** is isomorphic to the category **UMet**.

Another interesting tensor product in $[0,1]$ is given by the *Lukasiewicz tensor* $\odot$ where $u \odot v = \max(0, u+v-1)$; here $\hom(u,v) = \min(1, 1-u+v)$. Then $[0,1]_\odot$-**Cat** is the category of *bounded-by-1 (generalised) metric spaces* and *non-expansive maps*.

(4) We consider now the set

$$\Delta = \{\varphi\colon [0,\infty] \to [0,1] \mid \text{for all } \alpha \in [0,\infty]\colon\ \varphi(\alpha) = \bigvee_{\beta<\alpha} \varphi(\beta)\},$$

of *distribution functions*. With the pointwise order, it is a complete lattice. For $\varphi, \psi \in \Delta$ and $\alpha \in [0,\infty]$, define $\varphi \otimes \psi \in \Delta$ by

$$(\varphi \otimes \psi)(\alpha) = \bigvee_{\beta+\gamma\leq\alpha} \varphi(\beta) * \psi(\gamma).$$

Then $\otimes : \Delta \times \Delta \to \Delta$ is associative and commutative, and

$$\kappa : [0,\infty] \to [0,1],\ \alpha \mapsto \begin{cases} 0 & \text{if } \alpha = 0, \\ 1 & \text{else} \end{cases}$$

is a unit for $\otimes$. Finally, $\psi \otimes - : \Delta \to \Delta$ preserves suprema since, for all $u \in [0,1]$, $u * -\colon [0,1] \to [0,1]$ preserves suprema. A Δ-category is a *(generalised) probabilistic metric space* and a Δ-functor is a *probabilistic non-expansive map* (see [13] and references there).

We will also make use of two additional categories we describe next, the category V-**Rel**, of sets and V-relations, and the category V-**Dist**, of V-categories and V-distributors.

Objects of V-**Rel** are sets, while morphisms are V-relations, i.e., if X and Y are sets, a *V-relation* $r\colon X \to\!\!\!\!\!\!\mapsto Y$ is a map $r\colon X \times Y \to V$. Composition of V-relations is given by *relational composition*, so that the composite of $r\colon X \to\!\!\!\!\!\!\mapsto Y$ and $s\colon Y \to\!\!\!\!\!\!\mapsto Z$ is given by

$$(s \cdot r)(x,z) = \bigvee_{y\in Y} r(x,y) \otimes s(y,z),$$

for every $x \in X$, $z \in Z$. Identities in V-**Cat** are simply identity relations, with $1_X(x,x') = k$ if $x = x'$ and $1_X(x,x') = \bot$ otherwise. The category V-**Rel** has an involution $(\ )^\circ$, assigning to each V-relation $r\colon X \to\!\!\!\!\!\!\mapsto Y$ the V-relation $r^\circ\colon Y \to\!\!\!\!\!\!\mapsto X$ defined by $r^\circ(y,x) = r(x,y)$, for every $x \in X$, $y \in Y$.

Since every map $f\colon X \to Y$ can be thought as a V-relation through its graph $f_\circ\colon X \times Y \to V$, with $f_\circ(x,y) = k$ if $f(x) = y$ and $f_\circ(x,y) = \bot$ otherwise, there is an injective on objects and faithful functor $\mathbf{Set} \to V$-**Rel**. When no confusion may arise, we use also f to denote the V-relation $f_\circ$.

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Hence, any such pair induces a (equivalence class of) Cauchy sequence(s) $(x_n)_n$, and a representative for

$$\begin{tikzcd} X \arrow[r, "(1_X)_*", "\circ" marking] \arrow[d, "\varphi"', "\circ" marking] & X \\ E \arrow[ur, "{[\varphi,(1_X)_*]}"', "\circ" marking] \arrow[u, phantom, ""] & \end{tikzcd} \quad (\le)$$

is nothing but a limit point for $(x_n)_n$. Conversely, it is easily checked that every Cauchy sequence $(x_n)_n$ in X gives rise to a pair of adjoint distributors

$$\varphi = \lim_n X(-, x_n) \text{ and } \psi = \lim_n X(x_n, -).$$

We point out that the $\mathbb{L}$-embeddings, i.e. the fully faithful and fully dense V-functors $f\colon X \to Y$ do not coincide with the $\mathbb{L}$-dense ones (so that f_* is a right adjoint). For instance, assuming for simplicity that V is integral, a V-functor $y\colon E \to X$ ($y \in X$) is fully dense if and only if $y \simeq x$ for all $x \in X$, while it is an $\mathbb{L}$-embedding if and only if $y \le x$ for all $x \in X$. Indeed, $y\colon E \to X$ is $\mathbb{L}$-dense if, and only if,

– there is a distributor $\varphi\colon X {\to\!\!\!\!\circ\,} E$, i.e.

$$(\forall x, x' \in X) \;\; X(x, x') \otimes \varphi(x') \le \varphi(x), \tag{7.i}$$

such that

– $k \ge \varphi \cdot y_*$, which is trivially true, and $a \le y_* \cdot \varphi$, i.e.

$$(\forall x, x' \in X) \;\; X(x, x') \le \varphi(x) \otimes X(y, x'). \tag{7.ii}$$

Since (7.i) follows from (7.ii),

$$y \text{ is } \mathbb{L}\text{-dense} \;\Leftrightarrow\; (\forall x, x' \in X) \;\; X(x, x') \le \varphi(x) \otimes X(y, x').$$

In particular, when $x = x'$, this gives $k \le \varphi(x) \otimes X(y, x)$, and so we can conclude that, for all $x \in X$, $y \le x$ and $\varphi(x) = k$. The converse is also true; that is

$$y \text{ is } \mathbb{L}\text{-dense} \;\Leftrightarrow\; (\forall x \in X) \;\; y \le x.$$

Still, it was shown in [14] that injectivity with respect to fully dense and fully faithful V-functors (called L-dense in [14]) characterizes also the $\mathbb{L}$-algebras.

The category V-**Rel** is a 2-category, when equipped with the 2-cells given by the pointwise order; that is, for $r, r'\colon X {\to\!\!\!\!\!\shortmid\,} Y$, one defines $r \le r'$ if, for all $x \in X$, $y \in Y$, $r(x, y) \le r'(x, y)$. This gives us the possibility of studying adjointness between V-relations. We note in particular that, if $f\colon X \to Y$ is a map, then $f_\circ \cdot f^\circ \le 1_Y$ and $1_X \le f^\circ \cdot f_\circ$, so that $f_\circ \dashv f^\circ$.

Objects of V-**Dist** are V-categories, while morphisms are V-distributors (also called V-bimodules, or V-profunctors); i.e., if (X, a) and (Y, b) are V-categories, a *V-distributor* – or, simply, a *distributor* – $\varphi\colon (X, a) {\to\!\!\!\!\circ\,} (Y, b)$ is a V-relation $\varphi\colon X {\to\!\!\!\!\!\shortmid\,} Y$ such that $\varphi \cdot a \le \varphi$ and $b \cdot \varphi \le \varphi$ (in fact $\varphi \cdot a = \varphi$ and $b \cdot \varphi = \varphi$ since the other inequalities follow from (R)). Composition of distributors is again given by relational composition, while the identities are given by the V-categorical structures, i.e. $1_{(X,a)} = a$. Moreover, V-**Dist** inherits the 2-categorical structure from V-**Rel**.

Each V-functor $f\colon (X, a) \to (Y, b)$ induces two distributors, $f_*\colon (X, a) {\to\!\!\!\!\circ\,} (Y, b)$ and $f^*\colon (Y, b) {\to\!\!\!\!\circ\,} (X, a)$, defined by $f_*(x, y) = Y(f(x), y)$ and $f^*(y, x) = Y(y, f(x))$, that is, $f_* = b \cdot f_\circ$ and $f^* = f^\circ \cdot b$. These assignments are functorial, as we explain below.

First we define 2-cells in V-**Cat**: for $f, f'\colon (X, a) \to (Y, b)$ V-functors, $f \le f'$ when $f^* \le (f')^*$ as distributors, so that

$$f \le f' \;\;\Leftrightarrow\;\; \forall x \in X,\, y \in Y,\; Y(y, f(x)) \le Y(y, f'(x)).$$

V-**Cat** is then a 2-category, and we can define two 2-functors

$$\begin{array}{rcl} (\;)_*\colon V\text{-}\mathbf{Cat}^{\mathrm{co}} & \longrightarrow & V\text{-}\mathbf{Dist} \\ X & \longmapsto & X \\ f & \longmapsto & f_* \end{array} \quad \text{and} \quad \begin{array}{rcl} (\;)^*\colon V\text{-}\mathbf{Cat}^{\mathrm{op}} & \longrightarrow & V\text{-}\mathbf{Dist} \\ X & \longmapsto & X \\ f & \longmapsto & f^* \end{array}$$

Note that, for any V-functor $f\colon (X, a) \to (Y, b)$,

$$f_* \cdot f^* = b \cdot f_\circ \cdot f^\circ \cdot b \le b \cdot b = b \text{ and } f^* \cdot f_* = f^\circ \cdot b \cdot b \cdot f_\circ \ge f^\circ \cdot f_\circ \cdot a \ge a;$$

hence every V-functor induces a pair of adjoint distributors, $f_* \dashv f^*$. A V-functor $f\colon X \to Y$ is said to be *fully faithful* if $f^* \cdot f_* = a$, i.e. $X(x, x') = Y(f(x), f(x'))$ for all $x, x' \in X$, while it is *fully dense* if $f_* \cdot f^* = b$, i.e. $Y(y, y') = \bigvee_{x \in X} Y(y, f(x)) \otimes Y(f(x), y')$, for all $y, y' \in Y$. A fully faithful V-functor $f\colon X \to Y$ does not need to be an injective map; it is so in case

X and Y are separated V-categories (as defined below).

Remark 1.3. In V-**Cat** adjointness between V-functors

$$\begin{tikzcd} Y \arrow[r, bend left, "g"] \arrow[r, phantom, "\top"] & X \arrow[l, bend left, "f"] \end{tikzcd}$$

can be equivalently expressed as:

$$f \dashv g \;\Leftrightarrow\; f_* = g^* \;\Leftrightarrow\; g^* \dashv f^* \;\Leftrightarrow\; (\forall x \in X)\,(\forall y \in Y)\, X(x, g(y)) = Y(f(x), y).$$

In fact the latter condition encodes also V-functoriality of f and g; that is, if $f\colon X \to Y$ and $g\colon Y \to X$ are maps satisfying the condition

$$(\forall x \in X)\ (\forall y \in Y)\ \ X(x, g(y)) = Y(f(x), y),$$

then f and g are V-functors, with $f \dashv g$.

Furthermore, it is easy to check that, given V-categories X and Y, a map $f\colon X \to Y$ is a V-functor whenever f_* is a distributor (or whenever f^* is a distributor).

The order defined on V-**Cat** is in general not antisymmetric. For V-functors $f, g\colon X \to Y$, one says that $f \simeq g$ if $f \le g$ and $g \le f$ (or, equivalently, $f^* = g^*$). For elements x, y of a V-category X, one says that $x \le y$ if, considering the V-functors $x, y\colon E = (\{*\}, k) \to X$ (where $k(*,*) = k$) defined by $x(*) = x$ and $y(*) = y$, one has $x \le y$; or, equivalently, $X(x, y) \ge k$. Then, for any V-functors $f, g\colon X \to Y$, $f \le g$ if, and only if, $f(x) \le g(x)$ for every $x \in X$.

Definition 1.4. A V-category Y is said to be *separated* if, for $f, g\colon X \to Y$, $f = g$ whenever $f \simeq g$; equivalently, if, for all $x, y \in Y$, $x \simeq y$ implies $x = y$.

The tensor product $\otimes$ on V induces a tensor product on V-**Cat**, with $(X, a) \otimes (Y, b) = (X \times Y, a \otimes b) = X \otimes Y$, where $(X \otimes Y)((x, y), (x', y')) = X(x, x') \otimes Y(y, y')$. The V-category E is a $\otimes$-neutral element. With this tensor product, V-**Cat** becomes a monoidal closed category. Indeed, for each V-category X, the functor $X \otimes (\ )\colon V\text{-}\mathbf{Cat} \to V\text{-}\mathbf{Cat}$ has a right adjoint $(\ )^X$ defined by $Y^X = (V\text{-}\mathbf{Cat}(X, Y), [\![\ ,\ ]\!])$, with $[\![f, g]\!] = \bigwedge_{x \in X} Y(f(x), g(x))$ (see [7, 17, 15] for details).

It is interesting to note the following well-known result (see, for instance, [3, Theorem 2.5]).

which is the condition corresponding to condition (2) of Corollary 6.7.

Finally, as for the formal ball monad, Theorem 5.1 gives the following characterisation of $\mathbb{T}$-algebra morphisms.

Corollary 7.3. *Let $(X, \alpha), (Y, \beta)$ be $\mathbb{T}$-algebras. Then a V-functor $f : X \to Y$ is a $\mathbb{T}$-algebra morphism if and only if*

$$(\forall \varphi \in TX)\ \ \beta(\varphi \cdot f^*) \ge f(\alpha(\varphi)).$$

Example 7.4. The Lawvere monad. Among the examples presented in [2] there is a special submonad of $\mathbb{P}$ which is inspired by the crucial remark of Lawvere in [17] that Cauchy completeness for metric spaces is a kind of cocompleteness for V-categories. Indeed, the submonad $\mathbb{L}$ of $\mathbb{P}$ induced by

$$\Phi = \{\varphi\colon X {\circ\!\!\to} Y\ ;\ \varphi \text{ is a right adjoint } V\text{-distributor}\}$$

has as $\mathbb{L}$-algebras the *Lawvere complete V-categories*. These were studied also in [3], and in [14] under the name L-complete V-categories. When $V = [0, \infty]_+$, using the usual order in $[0, \infty]$, for distributors $\varphi\colon X {\circ\!\!\to} E$, $\psi\colon E {\circ\!\!\to} X$ to be adjoint

$$\begin{tikzcd} X \arrow[r, bend left, "\varphi"] \arrow[r, phantom, "\top"] & E \arrow[l, bend left, "\psi"] \end{tikzcd}$$

means that

$$\begin{aligned} (\forall x, x' \in X)\ \ X(x, x') &\le \varphi(x) + \psi(x'), \\ 0 &\ge \inf_{x \in X} (\psi(x) + \varphi(x)). \end{aligned}$$

This means in particular that

$$(\forall n \in \mathbb{N})\ (\exists x_n \in X)\ \ \psi(x_n) + \varphi(x_n) \le \frac{1}{n},$$

and, moreover,

$$X(x_n, x_m) \le \varphi(x_n) + \psi(x_m) \le \frac{1}{n} + \frac{1}{m}.$$

This defines a *Cauchy sequence* $(x_n)_n$, so that

$$(\forall \varepsilon > 0)\,(\exists p \in \mathbb{N})\,(\forall n, m \in \mathbb{N})\, n \ge p \wedge m \ge p \;\Rightarrow\; X(x_n, x_m) + X(x_m, x_n) < \varepsilon.$$

7. On algebras for submonads of $\mathbb{P}$ and their morphisms

In the following $\mathbb{T} = (T, \mu, \eta)$ is a submonad of the presheaf monad $\mathbb{P} = (P, m, y)$ in V-$\mathbf{Cat}_{\mathrm{sep}}$ For simplicity we will assume that the injective and fully faithful components of the monad morphism $\sigma : T \to P$ are inclusions. Theorem 5.1 gives immediately that:

Proposition 7.1. *Let (X, a) be a V-category and $\alpha : TX \to X$ be a V-functor. The following are equivalent:*

(1) (X, α) is a $\mathbb{T}$-algebra;
(2) $\forall\, x \in X : \alpha(x^*) = x$.

We would like to identify the $\mathbb{T}$-algebras directly, as we did for $\mathbb{B}_\bullet$ or $\mathbb{B}$ in Proposition 6.4. First of all, we point out that a $\mathbb{T}$-algebra structure $\alpha\colon TX \to X$ must satisfy, for every $\varphi \in TX$ and $x \in X$,

$$X(\alpha(\varphi), x) = TX(\varphi, x^*),$$

and so, in particular,

$$\alpha(\varphi) \le x \;\Leftrightarrow\; \varphi \le x^*;$$

hence α must assign to each $\varphi \in TX$ an $x_\varphi \in X$ so that

$$x_\varphi = \min\{x \in X\,;\, \varphi \le x^*\}.$$

Moreover, for such map $\alpha\colon TX \to X$, α is a V-functor if, and only if,

$$\begin{aligned} &(\forall \varphi, \rho \in TX)\;\; TX(\varphi, \rho) \le X(x_\varphi, x_\rho) = TX(X(-, x_\varphi), X(-, x_\rho))\\ \Leftrightarrow\; &(\forall \varphi, \rho \in TX)\;\; TX(\varphi, \rho) \le \bigwedge_{x \in X} \hom(X(x, x_\varphi), X(x, x_\rho))\\ \Leftrightarrow\; &(\forall x \in X)\; (\forall \varphi, \rho \in TX)\;\; X(x, x_\varphi) \otimes TX(\varphi, \rho) \le X(x, x_\rho). \end{aligned}$$

Proposition 7.2. *A V-category X is a $\mathbb{T}$-algebra if, and only if:*

(1) for all $\varphi \in TX$ there exists $\min\{x \in X\,;\, \varphi \le x^*\}$;
(2) for all $\varphi, \rho \in TX$ and for all $x \in X$, $X(x, x_\varphi) \otimes TX(\varphi, \rho) \le X(x, x_\rho)$.

We remark that condition (2) can be equivalently stated as:

(2') for each $\rho \in TX$, the distributor $\rho_1 = \displaystyle\bigvee_{\varphi \in TX} X(-, x_\varphi) \otimes TX(\varphi, \rho)$ satisfies $x_{\rho_1} = x_\rho$,

Theorem 1.5. *For V-categories X and Y, and a V-relation $\varphi\colon X \to\!\!\!\!\!\!\mapsto Y$, the following conditions are equivalent:*

(i) $\varphi\colon X \,{-\!\!\circ\!\!\to}\, Y$ *is a distributor;*
(ii) $\varphi\colon X^{\mathrm{op}} \otimes Y \to (V, \hom)$ *is a V-functor.*

In particular, the V-categorical structure a of (X, a) is a V-distributor $a\colon (X, a) \,{-\!\!\circ\!\!\to}\, (X, a)$, and therefore a V-functor $a\colon (X, a)^{\mathrm{op}} \otimes (X, a) \to (V, \hom)$, which induces, via the closed monoidal structure of V-$\mathbf{Cat}$, the *Yoneda V-functor* $y_X\colon (X, a) \to (V, \hom)^{(X,a)^{\mathrm{op}}}$. Thanks to the theorem above, $V^{X^{\mathrm{op}}}$ can be equivalently described as

$$PX := \{\varphi\colon X \,{-\!\!\circ\!\!\to}\, E \mid \varphi \ V\text{-distributor}\}.$$

Then the structure $\widetilde{a}$ on PX is given by

$$\widetilde{a}(\varphi, \psi) = [\![\varphi, \psi]\!] = \bigwedge_{x \in X} \hom(\varphi(x), \psi(x)),$$

for every $\varphi, \psi\colon X \,{-\!\!\circ\!\!\to}\, E$, where by $\varphi(x)$ we mean $\varphi(x, *)$, or, equivalently, we consider the associated V-functor $\varphi\colon X \to V$. The Yoneda functor $y_X\colon X \to PX$ assigns to each $x \in X$ the distributor $x^*\colon X \,{-\!\!\circ\!\!\to}\, E$, where we identify again $x \in X$ with the V-functor $x\colon E \to X$ assigning x to the (unique) element of E. Then, for every $\varphi \in PX$ and $x \in X$, we have that

$$[\![y_X(x), \varphi]\!] = \varphi(x),$$

as expected. In particular y_X is a fully faithful V-functor, being injective on objects (i.e. an injective map) when X is a separated V-category. We point out that $(V, \hom)$ is separated, and so is PX for every V-category X.

For more information on V-$\mathbf{Cat}$ we refer to [12, Appendix].

2. The presheaf monad and its submonads

The assignment $X \mapsto PX$ defines a functor $P\colon V\text{-}\mathbf{Cat} \to V\text{-}\mathbf{Cat}$: for each V-functor $f\colon X \to Y$, $Pf\colon PX \to PY$ assigns to each distributor $X \xrightarrow{\varphi} E$ the distributor $Y \xrightarrow{f^*} X \xrightarrow{\varphi} E$. It is easily checked that the Yoneda functors $(y_X\colon X \to PX)_X$ define a natural transformation $y\colon 1 \to P$. Moreover, since, for every V-functor f, the adjunction $f_* \dashv f^*$ yields an adjunction $Pf = (\;)\cdot f^* \dashv (\;)\cdot f_* =: Qf$, Py_X has a right adjoint, which we denote by $m_X\colon PPX \to PX$. It is straightforward to check that $\mathbb{P} = (P, m, y)$ is a

2-monad on V-**Cat** – the so-called *presheaf monad* –, which, by construction of $\mathfrak{m}_X$ as the right adjoint to $P\mathfrak{y}_X$, is lax idempotent (see [11] for details).

Next we present a characterisation of the submonads of $\mathbb{P}$ which is partially in [2]. We recall that, given two monads $\mathbb{T} = (T, \mu, \eta)$, $\mathbb{T}' = (T', \mu', \eta')$ on a category **C**, a monad morphism $\sigma\colon \mathbb{T} \to \mathbb{T}'$ is a natural transformation $\sigma\colon T \to T'$ such that

$$\begin{tikzcd} 1 \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & T \arrow[d, "\sigma"] \\ & T' \end{tikzcd} \qquad \begin{tikzcd} TT \arrow[r, "\sigma_T"] \arrow[d, "\mu"'] & T'T \arrow[r, "T'\sigma"] & T'T' \arrow[d, "\mu'"] \\ T \arrow[rr, "\sigma"'] & & T' \end{tikzcd} \qquad (2.i)$$

By *submonad of* $\mathbb{P}$ we mean a 2-monad $\mathbb{T} = (T, \mu, \eta)$ on V-**Cat** with a monad morphism $\sigma : \mathbb{T} \to \mathbb{P}$ such that σ_X is an embedding (i.e. both fully faithful and injective on objects) for every V-category X.

Definition 2.1. Given a class Φ of V-distributors, for every V-category X let

$$\Phi X = \{\varphi\colon X \,{\circ\!\!\!\longrightarrow}\, E \mid \varphi \in \Phi\}$$

have the V-category structure inherited from the one of PX. We say that Φ is *admissible* if, for every V-functor $f\colon X \to Y$ and V-distributors $\varphi\colon Z \,{\circ\!\!\!\longrightarrow}\, Y$ and $\psi\colon X \,{\circ\!\!\!\longrightarrow}\, Z$ in Φ,

(1) $f^* \in \Phi$;
(2) $\psi \cdot f^* \in \Phi$ and $f^* \cdot \varphi \in \Phi$;
(3) $\varphi \in \Phi \;\Leftrightarrow\; (\forall y \in Y)\; y^* \cdot \varphi \in \Phi$;
(4) for every V-distributor $\gamma\colon PX \,{\circ\!\!\!\longrightarrow}\, E$, if the restriction of γ to ΦX belongs to Φ, then $\gamma \cdot (\mathfrak{y}_X)_* \in \Phi$.

Lemma 2.2. *Every admissible class Φ of V-distributors induces a submonad $\Phi = (\Phi, \mathfrak{m}^\Phi, \mathfrak{y}^\Phi)$ of $\mathbb{P}$.*

Proof: For each V-category X, equip ΦX with the initial structure induced by the inclusion $\sigma_X\colon \Phi X \to PX$, that is, for every $\varphi, \psi \in \Phi X$, $\Phi X(\varphi, \psi) = PX(\varphi, \psi)$. For each V-functor $f\colon X \to Y$ and $\varphi \in \Phi X$, by condition (2), $\varphi \cdot f^* \in \Phi$, and so Pf (co)restricts to $\Phi f\colon \Phi X \to \Phi Y$.
Condition (1) guarantees that $\mathfrak{y}_X\colon X \to PX$ corestricts to $\mathfrak{y}^\Phi_X\colon X \to \Phi X$.
Finally, condition (4) guarantees that $\mathfrak{m}_X\colon PPX \to PX$ also (co)restricts to $\mathfrak{m}^\Phi_X : \Phi\Phi X \to \Phi X$: with $\gamma\colon \Phi X \,{\circ\!\!\!\longrightarrow}\, E$, also $\widetilde{\gamma} := \gamma \cdot (\sigma_X)^*\colon PX \,{\circ\!\!\!\longrightarrow}\, E$

and therefore, because V is cancellative, $Y(z', z) = k$; analogously one proves that $Y(z, z') = k$, and so $z = z'$ because Y is separated.

To prove the converse, for each $y \in Y$ we denote by $\overline{y}$ the only $z \in X$ satisfying (6.ii), and define

$$Bh_\sharp(y, r) = (\overline{y}, Y(\overline{y}, y) \otimes r).$$

When $x \in X$, it is immediate that $\overline{x} = x$, and so $Bh_\sharp \cdot Bh = 1_{BX}$. Using Remark 1.3, to prove that $Bh_\sharp$ is a V-functor and $Bh \dashv Bh_\sharp$ it is enough to show that

$$BX((x, r), Bh_\sharp(y, s)) = BY(Bh(x, r), (y, s)),$$

for every $x \in X$, $y \in Y$, $r, s \in V$. By definition of $Bh_\sharp$ this means

$$BX((x, r), (\overline{y}, Y(\overline{y}, y) \otimes s)) = BY((x, r), (y, s)),$$

that is,

$$\hom(r, Y(x, \overline{y}) \otimes Y(\overline{y}, y) \otimes s) = \hom(r, Y(x, y) \otimes s),$$

which follows directly from (6.ii). ∎

Corollary 6.17. *In* **Met***, if $X \subseteq [0, \infty]$, then its inclusion $h\colon X \to [0, \infty]$ is a B-embedding if, and only if, X is a closed interval.*

Proof: If $X = [x_0, x_1]$, with $x_0, x_1 \in [0, \infty]$, $x_0 \le x_1$, then it is easy to check that, defining $\overline{y} = x_0$ if $y \le x_0$, $\overline{y} = y$ if $y \in X$, and $\overline{y} = x_1$ if $y \ge x_1$, for every $y \in [0, \infty]$, condition (6.ii) is fulfilled.

We divide the proof of the converse in two cases:
(1) If X is not an interval, i.e. if there exists $x, x' \in X$, $y \in [0, \infty] \setminus X$ with $x < y < x'$, then either $\overline{y} < y$, and then

$$0 = y \ominus x' \neq (y \ominus x') + (y \ominus \overline{y}) = y - \overline{y},$$

or $\overline{y} > y$, and then

$$y - x = y \ominus x \neq (\overline{y} \ominus x) + (y \ominus \overline{y}) = \overline{y} - x.$$

(2) If $X = [x_0, x_1[$ and $y > x_1$, then there exists $x \in X$ with $\overline{y} < x < y$, and so

$$y - x = y \ominus x \neq (\overline{y} \ominus x) + (y \ominus \overline{y}) = y - \overline{y}.$$

An analogous argument works for $X =]x_0, x_1]$. ∎

Thanks to Theorem 5.5 $\mathbb{B}$-algebras are characterized via an injectivity property with respect to special embeddings. We end this section studying in more detail these embeddings. Since we are working in V-$\mathbf{Cat}_{\rm sep}$, a B-embedding $h\colon X\to Y$, being fully faithful, is injective on objects. Therefore, for simplicity, we may think of it as an inclusion. With $Bh_\sharp\colon BY\to BX$ the right adjoint and left inverse of $Bh\colon BX\to BY$, we denote $Bh_\sharp(y,r)$ by (y_r,r_y).

Lemma 6.15. *Let $h\colon X\to Y$ be a B-embedding. Then:*

(1) $(\forall y\in Y)\ (\forall x\in X)\ (\forall r\in V)\ BY((x,r),(y,r))=BY((x,r),(y_r,r_y))$;
(2) $(\forall\, y\in Y)\colon\, k_y=Y(y_k,y)$;
(3) $(\forall\, y\in Y)\ (\forall x\in X)\colon\ \ Y(x,y)=Y(x,y_k)\otimes Y(y_k,y)$.

Proof: (1) From $Bh_\sharp\cdot Bh=1_{BX}$ and $Bh\cdot Bh_\sharp\le 1_{BY}$ one gets, for any $(y,r)\in BY$, $(y,r)\le(y_r,r_y)$, i.e. $BY((y,r),(y_r,r_y))=\hom(r_y,Y(y_r,y)\otimes r)=k$. Therefore, for all $x\in X$, $y\in Y$, $r\in V$,

$$\begin{aligned}BY((x,r),(y,r))&\le BX((x,r),(y_r,r_y))=BY((x,r),(y_r,r_y))\\&=BY((x,r),(y_r,r_y))\otimes BY((y_r,r_y),(y,r))\\&\le BY((x,r),(y,r)),\end{aligned}$$

that is

$$BY((x,r),(y,r))=BY((x,r),(y_r,r_y)).$$

(2) Let $y\in Y$. Then

$$Y(y_k,y)=BY((y_k,k),(y,k))=BY((y_k,k),(y_k,k_y))=k_y.$$

(3) Let $y\in Y$ and $x\in X$. Then

$$\begin{aligned}Y(x,y)&=BY((x,k),(y,k))=BY((x,k),(y_k,k_y))\\&=Y(x,y_k)\otimes k_y=Y(x,y_k)\otimes Y(y_k,y).\end{aligned}$$ ∎

Proposition 6.16. *Let X and Y be V-categories. A V-functor $h\colon X\to Y$ is a B-embedding if and only if h is fully faithful and*

$$(\forall y\in Y)\ (\exists! z\in X)\ (\forall x\in X)\quad Y(x,y)=Y(x,z)\otimes Y(z,y). \tag{6.ii}$$

Proof: If h is a B-embedding, then it is fully faithful by Proposition 5.3 and, for each $y\in Y$, $z=y_k\in X$ fulfils the required condition. To show that such z is unique, assume that $z,z'\in X$ verify the equality of condition (6.ii). Then

$$Y(z,y)=Y(z,z')\otimes Y(z',y)\le Y(z',y)=Y(z',z)\otimes Y(z,y)\le Y(z,y),$$

belongs to Φ by (2), and then, since γ is the restriction of $\widetilde{\gamma}$ to ΦX, by (4) $\mu_X(\widetilde{\gamma})=\gamma\cdot(\sigma_X)^*\cdot(y_X)_*=\gamma\cdot(\sigma_X)^*\cdot(\sigma_X)_*\cdot(y_X^\Phi)_*=\gamma\cdot(y_X^\Phi)_*\in\Phi$.

By construction, $(\sigma_X)_X$ is a natural transformation, each σ_X is an embedding, and σ makes diagrams (2.i) commute. ∎

Theorem 2.3. *For a 2-monad $\mathbb{T}=(T,\mu,\eta)$ on V-$\mathbf{Cat}$, the following assertions are equivalent:*

(i) *$\mathbb{T}$ is isomorphic to Φ, for some admissible class of V-distributors Φ.*
(ii) *$\mathbb{T}$ is a submonad of $\mathbb{P}$.*

Proof: (i) $\Rightarrow$ (ii) follows from the lemma above.

(ii) $\Rightarrow$ (i): Let $\sigma\colon\mathbb{T}\to\mathbb{P}$ be a monad morphism, with σ_X an embedding for every V-category X, which, for simplicity, we assume to be an inclusion. First we show that

$$\Phi=\{\varphi\colon X{\longrightarrow\!\!\!\!\!\!\circ\;\;}Y\mid\forall y\in Y\ y^*\cdot\varphi\in TX\} \tag{2.ii}$$

is admissible. In the sequel $f\colon X\to Y$ is a V-functor.

(1) For each $x\in X$, $x^*\cdot f^*=f(x)^*\in TY$, and so $f^*\in\Phi$.

(2) If $\psi\colon X{\longrightarrow\!\!\!\!\!\!\circ\;\;}Z$ is a V-distributor in Φ, and $z\in Z$, since $z^*\cdot\psi\in TX$, $Tf(z^*\cdot\psi)=z^*\cdot\psi\cdot f^*\in TY$, and therefore $\psi\cdot f^*\in\Phi$ by definition of Φ. Now, if $\varphi\colon Z{\longrightarrow\!\!\!\!\!\!\circ\;\;}Y\in\Phi$, then, for each $x\in X$, $x^*\cdot f^*\cdot\varphi=f(x)^*\cdot\varphi\in TZ$ because $\varphi\in\Phi$, and so $f^*\cdot\varphi\in\Phi$.

(3) follows from the definition of Φ.

(4) If the restriction of $\gamma\colon PX{\longrightarrow\!\!\!\!\!\!\circ\;\;}E$ to TX, i.e., $\gamma\cdot(\sigma_X)_*$, belongs to Φ, then $\mu_X(\gamma\cdot(\sigma_X)_*)=\gamma\cdot(\sigma_X)_*\cdot(\eta_X)_*=\gamma\cdot(y_X)_*$ belongs to TX. ∎

We point out that, with $\mathbb{P}$, also $\mathbb{T}$ is lax idempotent. This assertion is shown at the end of next section, making use of the Beck-Chevalley conditions we study next. (We note that the arguments of [6, Prop. 16.2], which states conditions under which a submonad of a lax idempotent monad is still lax idempotent, cannot be used directly here.)

3. The presheaf monad and Beck-Chevalley conditions

In this section our aim is to show that $\mathbb{P}$ verifies some interesting conditions of Beck-Chevalley type, that resemble the BC conditions studied in [5]. We recall from [5] that a commutative square in **Set**

$$\begin{tikzcd} W \arrow[r, "l"] \arrow[d, "g"'] & Z \arrow[d, "h"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is said to be a *BC-square* if the following diagram commutes in **Rel**

$$\begin{tikzcd} W \arrow[r, "l_\circ", "\shortmid"{marking}] & Z \\ X \arrow[u, "g^\circ", "\shortmid"{marking}] \arrow[r, "f_\circ"', "\shortmid"{marking}] & Y, \arrow[u, "h^\circ"', "\shortmid"{marking}] \end{tikzcd}$$

where, given a map $t\colon A \to B$, $t_\circ\colon A \to\!\!\!\!\!\shortmid\;\, B$ denotes the relation defined by t and $t^\circ\colon B \to\!\!\!\!\!\shortmid\;\, A$ its opposite. Since $t_\circ \dashv t^\circ$ in **Rel**, this is in fact a kind of Beck-Chevalley condition. A **Set**-endofunctor T is said to satisfy BC if it preserves BC-squares, while a natural transformation $\alpha\colon T \to T'$ between two **Set**-endofunctors satisfies BC if, for each map $f\colon X \to Y$, its naturality square

$$\begin{tikzcd} TX \arrow[r, "\alpha_X"] \arrow[d, "Tf"'] & T'X \arrow[d, "T'f"] \\ TY \arrow[r, "\alpha_Y"'] & T'Y \end{tikzcd}$$

is a BC-square.

In our situation, for endofunctors and natural transformations in V-**Cat**, the role of **Rel** is played by V-**Dist**.

Definition 3.1. A commutative square in V-**Cat**

$$\begin{tikzcd} (W,d) \arrow[r, "l"] \arrow[d, "g"'] & (Z,c) \arrow[d, "h"] \\ (X,a) \arrow[r, "f"'] & (Y,b) \end{tikzcd}$$

Proposition 6.13. *Let V be an integral quantale. The following assertions are equivalent:*

(i) *BV is separated;*
(ii) *V is cancellative;*
(iii) *If X is separated then BX is separated.*

Proof: (i) $\Rightarrow$ (ii): Let $r, s \in V$, $r \neq \bot$ and $r = s \otimes r$. Note that

$$BV((k,r),(s,r)) = \hom(r, \hom(k,s) \otimes r) = \hom(r, s \otimes r) = \hom(r,r) = k$$

and

$$\begin{aligned} BV((s,r),(k,r)) &= \hom(r, \hom(s,k) \otimes r) = \hom(r, \hom(s,k) \otimes s \otimes r) \\ &= \hom(s \otimes r, s \otimes r) = k. \end{aligned}$$

Therefore, since BV is separated, $(s,r) = (k,r)$ and it follows that $s = k$.

(ii) $\Rightarrow$ (iii): If $(x,r) \simeq (y,s)$ in BX, then

$$BX((x,r),(y,s)) = k \Leftrightarrow r \leq X(x,y) \otimes s, \text{ and}$$
$$BX((y,s),(x,r)) = k \Leftrightarrow s \leq X(y,x) \otimes r.$$

Therefore $r \leq s$ and $s \leq r$, that is $r = s$. Moreover, since $r \leq X(x,y) \otimes r \leq r$ we have that $X(x,y) = k$. Analogously, $X(y,x) = k$ and we conclude that $x = y$.

(iii) $\Rightarrow$ (i): Since V is separated it follows immediately from (iii) that BV is separated. ∎

We can now show that $\mathbb{B}$ is a submonad of $\mathbb{P}$ in the adequate setting. *From now on we will be working with a cancellative and integral quantale V, and $\mathbb{B}$ will be the (co)restriction of the formal ball monad to V-$\mathbf{Cat}_{\mathrm{sep}}$.*

Proposition 6.14. *Let V be a cancellative and integral quantale. Then $\mathbb{B}$ is a submonad of $\mathbb{P}$ in V-$\mathbf{Cat}_{\mathrm{sep}}$.*

Proof: Thanks to Proposition 6.3, all that remains is to show that σ_X is injective on objects, for any V-category X. Let $\sigma(x,r) = \sigma(y,s)$, or, equivalently, $X(-,x) \otimes r = X(-,y) \otimes s$. Then, in particular,

$$r = X(x,x) \otimes r = X(x,y) \otimes s \leq s = X(y,y) \otimes s = X(y,x) \otimes r \leq r.$$

Therefore $r = s$ and $X(y,x) = X(x,y) = k$. We conclude that $(x,r) = (y,s)$. ∎

(2) X is a $\mathbb{B}$-algebra if, and only if, $X =]a,b]$ or $X = [a,b]$ for some $a,b \in [0,\infty]$.

Let X be a $\mathbb{B}_\bullet$-algebra. From Proposition 6.4 one has

$$(\forall x \in X, r \in [0,\infty])(\exists x \oplus r \in X)(\forall y \in X)\, y \ominus (x \oplus r) = (y \ominus x) \ominus r = y \ominus (x + r).$$

This implies that, if $y \in X$, then $y > x \otimes r \iff y > x + r$. Therefore, if $x + r \in X$, then $x \oplus r = x + r$, and, moreover, X is an interval: given $x, y, z \in [0,\infty]$ with $x < y < z$ and $x, z \in X$, then, with $r = y - x \in [0,\infty]$, $x + r = y$ must belong to X:

$$\begin{aligned} z \ominus (x \oplus r) = z - (x + r) = z - y > 0 \;\Rightarrow\; & z \ominus (x \oplus r) = z - (x \oplus r) = z - y \\ \Leftrightarrow\; & y = x \oplus r \in X. \end{aligned}$$

In addition, X must have bottom element (that is a maximum with respect to the classical order of the real half-line): for any $x \in X$ and $b = \sup X$, $x \oplus (b - x) = \sup\{z \in X\,;\, z \leq b\} = b \in X$. For $r = \infty$ and any $x \in X$, $x \oplus \infty$ must be the top element of X, so $X = [a,b]$ for $a, b \in [0,\infty]$.

Conversely, if $X =]a,b]$, for $x \in X$ and $r \in [0,\infty[$, define $x \oplus r = x + r$ if $x + r \in X$ and $x \oplus r = b$ elsewhere. It is easy to check that condition (ii) of Proposition 6.4 is satisfied for $r \neq \infty$.

Analogously, if $X = [a,b]$, for $x \in X$ and $r \in [0,\infty]$, we define $x \oplus r$ as before in case $r \neq \infty$ and $x \oplus \infty = a$.

As we will see, (co)restricting $\mathbb{B}$ to V-$\mathbf{Cat}_{\mathrm{sep}}$ will allows us to obtain some interesting results. Unfortunately X being separated does not entail BX being so. Because of this we will need to restrict our attention to the *cancellative* quantales which we define and characterize next.

Definition 6.11. A quantale V is said to be *cancellative* if

$$\forall r, s \in V,\, r \neq \bot :\; r = s \otimes r \;\Rightarrow\; s = k. \tag{6.i}$$

Remark 6.12. We point out that this notion of cancellative quantale does not coincide with the notion of cancellable ccd quantale introduced in [4]. On the one hand cancellative quantales are quite special, since, for instance, when V is a locale, and so with $\otimes = \wedge$ is a quantale, V is not cancellative since condition (6.i) would mean, for $r \neq \bot$, $r = s \wedge r \;\Rightarrow\; s = \top$. On the other hand, $[0,1]_\odot$, that is $[0,1]$ with the usual order and having as tensor product the Łukasiewicz sum, is cancellative but not cancellable. In addition we remark that every *value quantale* [16] is cancellative.

is said to be a *BC*-square* if the following diagram commutes in V-$\mathbf{Dist}$

$$\begin{tikzcd} (W,d) \arrow[r, "l_*"] & (Z,c) \\ (X,a) \arrow[u, "g^*"] \arrow[r, "f_*"'] & (Y,b) \arrow[u, "h^*"'] \end{tikzcd} \tag{3.i}$$

(or, equivalently, $h^* \cdot f_* \leq l_* \cdot g^*$).

Remarks 3.2. (1) For a V-functor $f\colon (X,a) \to (Y,b)$, to be fully faithful is equivalent to

$$\begin{tikzcd} (X,a) \arrow[r, "1"] \arrow[d, "1"'] & (X,a) \arrow[d, "f"] \\ (X,a) \arrow[r, "f"'] & (Y,b) \end{tikzcd}$$

being a BC*-square (exactly in parallel with the characterisation of monomorphisms via BC-squares).

(2) We point out that, contrarily to the case of BC-squares, in BC*-squares the horizontal and the vertical arrows play different roles; that is, the fact that diagram (3.i) is a BC*-square is not equivalent to

$$\begin{tikzcd} (W,d) \arrow[r, "g"] \arrow[d, "l"'] & (X,a) \arrow[d, "f"] \\ (Z,c) \arrow[r, "h"'] & (Y,b) \end{tikzcd}$$

being a BC*-square; it is indeed equivalent to its *dual*

$$\begin{tikzcd} (W,d^\circ) \arrow[r, "g"] \arrow[d, "l"'] & (X,a^\circ) \arrow[d, "f"] \\ (Z,c^\circ) \arrow[r, "h"'] & (Y,b^\circ) \end{tikzcd}$$

being a BC*-square.

Definitions 3.3. (1) A *functor* $T\colon V$-$\mathbf{Cat} \to V$-$\mathbf{Cat}$ *satisfies BC** if it preserves BC*-squares.

(2) Given two endofunctors T, T' on V-**Cat**, a *natural transformation* $\alpha\colon T \to T'$ *satisfies BC** if the naturality diagram

$$\begin{tikzcd} TX \arrow[r, "\alpha_X"] \arrow[d, "Tf"'] & T'X \arrow[d, "T'f"] \\ TY \arrow[r, "\alpha_Y"'] & T'Y \end{tikzcd}$$

is a BC*-square for every morphism f in V-**Cat**.

(3) *A 2-monad* $\mathbb{T} = (T, \mu, \eta)$ on V-**Cat** is said to satisfy *fully BC** if T, μ, and η satisfy BC*.

Remark 3.4. In the case of **Set** and **Rel**, since the condition of being a BC-square is equivalent, under the Axiom of Choice (AC), to being a weak pullback, a **Set**-monad $\mathbb{T}$ *satisfies fully BC* if, and only if, it is *weakly cartesian* (again, under (AC)). This, together with the fact that there are relevant **Set**-monads – like for instance the ultrafilter monad – whose functor and multiplication satisfy BC but the unit does not, led the authors of [5] to name such monads as *BC-monads*. This is the reason why we use *fully BC** instead of BC* to identify these V-**Cat**-monads.

As a side remark we recall that, still in the **Set**-context, a partial BC-condition was studied by Manes in [18]: for a **Set**-monad $\mathbb{T} = (T, \mu, \eta)$ to be *taut* requires that T, μ, η satisfy BC for commutative squares where f is monic.

Our first use of BC* is the following characterisation of lax idempotency for a 2-monad $\mathbb{T}$ on V-**Cat**.

Proposition 3.5. *Let* $\mathbb{T} = (T, \mu, \eta)$ *be a 2-monad on* V-**Cat**.

(1) The following assertions are equivalent:

(i) $\mathbb{T}$ *is lax idempotent.*

(ii) *For each* V*-category* X*, the diagram*

$$\begin{tikzcd} TX \arrow[r, "T\eta_X"] \arrow[d, "\eta_{TX}"'] & TTX \arrow[d, "\mu_X"] \\ TTX \arrow[r, "\mu_X"'] & TX \end{tikzcd} \tag{3.ii}$$

is a BC-square.*

(2) If $\mathbb{T}$ *is lax idempotent, then* μ *satisfies BC*.*

Corollary 6.7. *If* $BX \xrightarrow{-\oplus-} X$ *is a* $\mathbb{B}$*-algebra structure, then, for* $x \in X$*,* $r, s \in V \setminus \{\bot\}$*:*

(1) $x \oplus k = x$*;*

(2) $x \oplus (r \otimes s) = (x \oplus r) \oplus s$*;*

(3) $X(x, x \oplus r) \geq r$*.*

Lemma 6.8. *Let* X *and* Y *be* V*-categories equipped with* $\mathbb{B}$*-algebra structures* $BX \xrightarrow{-\oplus-} X$ *and* $BY \xrightarrow{-\oplus-} Y$ *. Then a map* $f : X \to Y$ *is a* V*-functor if and only if*

$$f \text{ is monotone and } f(x) \oplus r \leq f(x \oplus r),$$

for all $(x, r) \in BX$*.*

Proof: Assume that f is a V-functor. Then it is, in particular, monotone, and, from Theorem 5.1 we know that $f(x) \oplus r \leq f(x \oplus r)$.

Conversely, assume that f is monotone and that $f(x) \oplus r \leq f(x \oplus r)$, for all $(x, r) \in BX$. Let $x, x' \in X$. Then $x \oplus X(x, x') \leq x'$ since $(x \oplus -) \dashv X(x, -)$ by Corollary 6.5, and then

$$\begin{aligned} f(x) \oplus X(x, x') &\leq f(x \oplus X(x, x')) && \text{(by hypothesis)} \\ &\leq f(x') && \text{(by monotonicity of } f\text{)}. \end{aligned}$$

Now, using the adjunction $f(x) \oplus - \dashv Y(f(x), -))$, we conclude that

$$X(x, x') \leq Y(f(x), f(x')).$$

∎

The following results are now immediate:

Corollary 6.9. *(1) Let* $(X, \oplus), (Y, \oplus)$ *be* $\mathbb{B}$*-algebras. Then a map* $f\colon X \to Y$ *is a* $\mathbb{B}$*-algebra morphism if and only if, for all* $(x, r) \in BX$*,*

$$f \text{ is monotone and } f(x \oplus r) = f(x) \oplus r.$$

(2) Let $(X, \oplus), (Y, \oplus)$ *be* $\mathbb{B}$*-algebras. Then a* V*-functor* $f\colon X \to Y$ *is a* $\mathbb{B}$*-algebra morphism if and only if, for all* $(x, r) \in BX$*,*

$$f(x \oplus r) \leq f(x) \oplus r.$$

Example 6.10. If $X \subseteq [0, \infty]$, with the V-category structure inherited from hom, then

(1) X is a $\mathbb{B}_\bullet$-algebra if, and only if, $X = [a, b]$ for some $a, b \in [0, \infty]$.

The V-categories X satisfying (iii), and therefore satisfying the above (equivalent) conditions, are called *tensored*. This notion was originally introduced in the article [1] by Borceux and Kelly for general V-categories (for our special V-categories we suggest to consult [19]).

Note that, thanks to condition (ii), we get the following characterisation of tensored categories.

Corollary 6.5. *A V-category X is tensored if, and only if, for every $x \in X$,*

$$\begin{tikzcd} X \arrow[r, bend left, "X(x,-)"] \arrow[r, phantom, "\top"] & V \arrow[l, bend left, "x\oplus -"] \end{tikzcd}$$

*is an adjunction in V-**Cat**.*

We now shift our attention to the formal ball monad $\mathbb{B}$. The characterisation of $\mathbb{B}_\bullet$-algebras given by the Proposition 6.4 may be adapted to obtain a characterisation of $\mathbb{B}$-algebras. Indeed, the only difference is that a $\mathbb{B}$-algebra structure $BX \to X$ does not include the existence of $x \oplus \bot$ for $x \in X$, which, when it exists, is the top element with respect to the order in X. Moreover, the characterisation of $\mathbb{B}$-algebras given in [10, Proposition 3.4] can readily be generalised to V-**Cat** as follows.

Proposition 6.6. *For a V-functor $\alpha\colon BX \to X$ the following conditions are equivalent.*

(i) *α is a $\mathbb{B}$-algebra structure.*
(ii) *For every $x \in X$, $r, s \in V \setminus \{\bot\}$, $\alpha(x,k) = x$ and $\alpha(x, r\otimes s) = \alpha(\alpha(x,r),s)$.*
(iii) *For every $x \in X$, $r \in V \setminus \{\bot\}$, $\alpha(x,k) = x$ and $X(x,\alpha(x,r)) \geq r$.*
(iv) *For every $x \in X$, $\alpha(x,k) = x$.*

Proof: By definition of $\mathbb{B}$-algebra, (i) $\Leftrightarrow$ (ii), while (i) $\Leftrightarrow$ (iv) follows from Theorem 5.1, since $\mathbb{B}$ is lax-idempotent. (iii) $\Rightarrow$ (iv) is obvious, and so it remains to prove that, if α is a $\mathbb{B}$-algebra structure, then $X(x,\alpha(x,r)) \geq r$, for $r \neq \bot$. But

$$X(x,\alpha(x,r)) \geq r \;\Leftrightarrow\; k \leq \hom(r, X(x,\alpha(x,r)) = X(\alpha(x,r),\alpha(x,r)),$$

because $\alpha(x,-) \dashv X(x,-)$ by Corollary 6.5. ∎

Since we know that, if X has a $\mathbb{B}$-algebra structure α, then $\alpha(x,r) = x \oplus r$, we may state the conditions above as follows.

Proof: (1) (i) $\Rightarrow$ (ii): The monad $\mathbb{T}$ is lax idempotent if, and only if, for every V-category X, $T\eta_X \dashv \mu_X$, or, equivalently, $\mu_X \dashv \eta_{TX}$. These two conditions are equivalent to $(T\eta_X)_* = (\mu_X)^*$ and $(\mu_X)_* = (\eta_{TX})^*$. Hence $(\mu_X)^*(\mu_X)_* = (T\eta_X)_*(\eta_{TX})^*$ as claimed.

(ii) $\Rightarrow$ (i): From $(\mu_X)^*(\mu_X)_* = (T\eta_X)_*(\eta_{TX})^*$ it follows that

$$(\mu_X)_* = (\mu_X)_*(\mu_X)^*(\mu_X)_* = (\mu_X\cdot T\eta_X)_*(\eta_{TX})^* = (\eta_{TX})^*,$$

that is, $\mu_X \dashv \eta_{TX}$.

(2) BC* for μ follows directly from lax idempotency of $\mathbb{T}$, since

$$\begin{tikzcd} TTX \arrow[r, "(\mu_X)_*", "\circ" marking] & TX & & TTX \arrow[r, "(\eta_{TX})^*", "\circ" marking] & TX \\ TTY \arrow[u, "(TTf)^*", "\circ" marking] \arrow[r, "(\mu_Y)_*"', "\circ" marking] & TY \arrow[u, "(Tf)^*"', "\circ" marking] & = & TTY \arrow[u, "(TTf)^*", "\circ" marking] \arrow[r, "(\eta_{TY})^*"', "\circ" marking] & TY \arrow[u, "(Tf)^*"', "\circ" marking] \end{tikzcd}$$

and the latter diagram commutes trivially. ∎

Remark 3.6. Thanks to Remarks 3.2 we know that, if we invert the role of η_{TX} and $T\eta_X$ in (3.ii), we get a characterisation of oplax idempotent 2-monad: $\mathbb{T}$ is oplax idempotent if, and only if, the diagram

$$\begin{tikzcd} TX \arrow[r, "\eta_{TX}"] \arrow[d, "T\eta_X"'] & TTX \arrow[d, "\mu_X"] \\ TTX \arrow[r, "\mu_X"'] & TX \end{tikzcd}$$

is a BC*-square.

Theorem 3.7. *The presheaf monad $\mathbb{P} = (P, \mathfrak{m}, \mathfrak{y})$ satisfies fully BC*.*

Proof: (1) *P satisfies BC**: Given a BC*-square

$$\begin{tikzcd} (W,d) \arrow[r, "l"] \arrow[d, "g"'] & (Z,c) \arrow[d, "h"] \\ (X,a) \arrow[r, "f"'] & (Y,b) \end{tikzcd}$$

in V-**Cat**, we want to show that

$$\begin{tikzcd} PW \arrow[r, "(Pl)_*", "\circ"{marking}] & PZ \\ PX \arrow[u, "(Pg)^*", "\circ"{marking}] \arrow[r, "(Pf)_*"', "\circ"{marking}] \arrow[ur, phantom, "\geq"] & PY. \arrow[u, "(Ph)^*"', "\circ"{marking}] \end{tikzcd} \tag{3.iii}$$

For each $\varphi \in PX$ and $\psi \in PZ$, we have

$$\begin{aligned}
(Ph)^*(Pf)_*(\varphi,\psi) &= (Ph)^\circ \cdot \widetilde{b} \cdot Pf(\varphi,\psi) \\
&= \widetilde{b}(Pf(\varphi), Ph(\psi)) \\
&= \bigwedge_{y\in Y} \hom(\varphi\cdot f^*(y), \psi\cdot h^*(y)) \\
&\leq \bigwedge_{x\in X} \hom(\varphi\cdot f^*\cdot f_*(x), \psi\cdot h^*\cdot f_*(x)) \\
&\leq \bigwedge_{x\in X} \hom(\varphi(x), \psi\cdot l_*\cdot g^*(x)) && (\varphi\leq\varphi\cdot f^*\cdot f_*,\ \text{(3.iii) is BC*}) \\
&= \widetilde{a}(\varphi, \psi\cdot l_*\cdot g^*) \\
&\leq \widetilde{a}(\varphi, \psi\cdot l_*\cdot g^*)\otimes\widetilde{c}(\psi\cdot l_*\cdot l^*, \psi) && (\text{because } \psi\cdot l_*\cdot l^*\leq\psi) \\
&= \widetilde{a}(\varphi, Pg(\psi\cdot l_*))\otimes\widetilde{c}(Pl(\psi\cdot l_*), \psi) \\
&\leq \bigvee_{\gamma\in PW} \widetilde{a}(\varphi, Pg(\gamma))\otimes\widetilde{c}(Pl(\gamma),\psi) \\
&= (Pl)_*(Pg)^*(\varphi,\psi).
\end{aligned}$$

(2) μ *satisfies BC**: For each V-functor $f\colon X\to Y$, from the naturality of $\mathfrak{y}$ it follows that the following diagram

$$\begin{tikzcd} PPX \arrow[r, "(\mathfrak{y}_{PX})^*", "\circ"{marking}] & PX \\ PPY \arrow[u, "(PPf)^*", "\circ"{marking}] \arrow[r, "(\mathfrak{y}_{PY})^*"', "\circ"{marking}] & PY \arrow[u, "(Pf)^*"', "\circ"{marking}] \end{tikzcd}$$

commutes. Lax idempotency of $\mathbb{P}$ means in particular that $\mathfrak{m}_X \dashv \mathfrak{y}_{PX}$, or, equivalently, $(\mathfrak{m}_X)_* = (\mathfrak{y}_{PX})^*$, and therefore the commutativity of this diagram shows BC* for $\mathfrak{m}$.

more precisely, for each $y\in X$, $\sigma_X(x,r)(y) = X(y,x)\otimes r$.

Moreover, σ_X is fully faithful: for each $(x,r),(y,s)\in B_\bullet X$,

$$\begin{aligned}
B_\bullet X((x,r),(y,s)) = \hom(r, X(x,y)\otimes s) &\geq \hom(X(x,x)\otimes r, X(x,y)\otimes s) \\
&\geq \bigwedge_{z\in X} \hom(X(z,x)\otimes r, X(z,y)\otimes s) = PX(\sigma(x,r),\sigma(y,s)).
\end{aligned}$$

$\blacksquare$

It is clear that $\sigma\colon \mathbb{B}_\bullet\to\mathbb{P}$ is not pointwise monic; indeed, if $r=\bot$, then $\sigma_X(x,\bot)\colon X{\multimap}E$ is the distributor that is constantly $\bot$, for any $x\in X$. Still it is interesting to identify the $\mathbb{B}_\bullet$-algebras via the existence of special weighted colimits.

Proposition 6.4. *For a V-category X, the following conditions are equivalent:*

(i) *X has a $\mathbb{B}_\bullet$-algebra structure $\alpha\colon B_\bullet X\to X$;*
(ii) $(\forall x\in X,\ r\in V)\,(\exists x{\oplus}r\in X)\,(\forall y\in X)\ X(x{\oplus}r, y) = \hom(r, X(x,y))$;
(iii) *for all $(x,r)\in B_\bullet X$, every diagram of the sort*

$$\begin{tikzcd} X \arrow[r, "(1_X)_*", "\circ"{marking}] \arrow[d, "\sigma_X(x,r)"', "\circ"{marking}] \arrow[dr, phantom, "\leq"] & X \\ E \arrow[ur, "{[\sigma_X(x,r),(1_X)_*]}"', "\circ"{marking}] & \end{tikzcd}$$

has a (weighted) colimit.

Proof: (i) $\Rightarrow$ (ii): The adjunction $\alpha\dashv\eta_X$ gives, via Remark 1.3,

$$X(\alpha(x,r),y) = B_\bullet X((x,r),(y,k)) = \hom(r, X(x,y)).$$

For $x\oplus r := \alpha(x,r)$, condition (ii) follows.

(ii) $\Rightarrow$ (iii): The calculus of the distributor $[\sigma_X(x,r),(1_X)_*]$ shows that it is represented by $x\oplus r$:

$$[\sigma_X(x,r),(1_X)_*](*,y) = \hom(r, X(x,y)).$$

(iii) $\Rightarrow$ (i) For each $(x,r)\in B_\bullet X$, let $x\oplus r$ represent $[\sigma_X(x,r),(1_X)_*]$. In case $r=k$, we choose $x\oplus k = x$ to represent the corresponding distributor (any $x'\simeq x$ would fit here but x is the right choice for our purpose). Then $\alpha\colon B_\bullet X\to X$ defined by $\alpha(x,r) = x\oplus r$ is, by construction, left adjoint to η_X, and $\alpha\cdot\eta_X = 1_X$. $\blacksquare$

- a functor $B_\bullet\colon V\text{-}\mathbf{Cat}\to V\text{-}\mathbf{Cat}$ which maps each V-category X to $B_\bullet X$ with underlying set $X\times V$ and
$$B_\bullet X((x,r),(y,s))=\hom(r,X(x,y)\otimes s)$$
and every V-functor $f\colon X\to Y$ to the V-functor $B_\bullet f\colon B_\bullet X\to B_\bullet Y$ with $B_\bullet f(x,r)=(f(x),r)$;
- natural transformations $\eta\colon 1\to B_\bullet$ and $\mu\colon B_\bullet B_\bullet\to B_\bullet$ with $\eta_X(x)=(x,k)$ and $\mu_X((x,r),s)=(x,r\otimes s)$, for every V-category X, $x\in X$, $r,s\in V$.

The *formal ball monad* $\mathbb{B}$ is the submonad of $\mathbb{B}_\bullet$ obtained when we only consider balls with radius different from $\bot$.

Remark 6.2. Note that $\mathbb{B}_\bullet X$ is not separated if X has more than one element (for any $x,y\in X$, $(x,\bot)\simeq(y,\bot)$), while, as shown in 6.13, for X separated, separation of $\mathbb{B}X$ depends on an extra property of the quantale V.

Using Corollaries 4.2 and 3.8, it is easy to check that

Proposition 6.3. *There is a pointwise fully faithful monad morphism $\sigma\colon\mathbb{B}_\bullet\to\mathbb{P}$. In particular, $\mathbb{B}_\bullet$ is lax-idempotent.*

Proof: First of all let us check that η satisfies BC*, i.e., for any V-functor $f\colon X\to Y$,
$$\begin{tikzcd} X \arrow[r, "(\eta_X)_*", "\circ"{marking}] & B_\bullet X \\ Y \arrow[u, "f^*", "\circ"{marking}] \arrow[r, "(\eta_Y)_*"', "\circ"{marking}] \arrow[ur, phantom, "\geq"] & B_\bullet Y \arrow[u, "(B_\bullet f)^*"', "\circ"{marking}] \end{tikzcd}$$
For $y\in Y$, $(x,r)\in B_\bullet X$,
$$\begin{aligned}((B_\bullet f)^*(\eta_Y)_*)(y,(x,r))&=B_\bullet Y((y,k),(f(x),r))=Y(y,f(x))\otimes r\\&\le\bigvee_{z\in X}Y(y,f(z))\otimes X(z,x)\otimes r\\&=\bigvee_{z\in X}Y(y,f(z))\otimes B_\bullet X((z,k),(x,r))\\&=((\eta_X)_*f^*)(y,(x,r)).\end{aligned}$$
Then, by Corollary 4.2, for each V-category X, σ_X is defined as in the proof of Theorem 4.1, i.e. for each $(x,r)\in B_\bullet X$,
$$\sigma_X(x,r)=B_\bullet X((-,k),(x,r))\colon X\to V;$$

(3) *y satisfies BC**: Once again, for each V-functor $f\colon(X,a)\to(Y,b)$, we want to show that the diagram
$$\begin{tikzcd} X \arrow[r, "(y_X)_*", "\circ"{marking}] & PX \\ Y \arrow[u, "f^*", "\circ"{marking}] \arrow[r, "(y_Y)_*"', "\circ"{marking}] & PY \arrow[u, "(Pf)^*"', "\circ"{marking}] \end{tikzcd}$$
commutes. Let $y\in Y$ and $\varphi\colon X{\circ\!\!\longrightarrow}E$ belong to PX. Then
$$\begin{aligned}((Pf)^*(y_Y)_*)(y,\varphi)&=((Pf)^\circ\cdot\widetilde{b}\cdot y_Y)(y,\varphi)=\widetilde{b}(y_Y(y),Pf(\varphi))=Pf(\varphi)(y)\\&=\bigvee_{x\in X}b(y,f(x))\otimes\varphi(x)=\bigvee_{x\in X}b(y,f(x))\otimes\widetilde{a}(y_X(x),\varphi)\\&=(\widetilde{a}\cdot y_X\cdot f^\circ\cdot b)(y,\varphi)=(y_X)_*\cdot f^*(y,\varphi),\end{aligned}$$
as claimed. ∎

Corollary 3.8. *Let $\mathbb{T}=(T,\mu,\eta)$ on V-$\mathbf{Cat}$ be a 2-monad on V-$\mathbf{Cat}$, and $\sigma\colon\mathbb{T}\to\mathbb{P}$ be a monad morphism, pointwise fully faithful. Then $\mathbb{T}$ is lax idempotent.*

Proof: We know that $\mathbb{P}$ is lax idempotent, and so, for every V-category X, $(m_X)_*=(y_{PX})^*$. Consider diagram (2.i). The commutativity of the diagram on the right gives that $(\mu_X)_*=(\sigma_X)^*(\sigma_X)_*(\mu_X)_*=(\sigma_X)^*(m_X)_*(P\sigma_X)_*(\sigma_{TX})_*$; using the equality above, and preservation of fully faithful V-functors by $\mathbb{P}$ – which follows from BC* – we obtain:
$$\begin{aligned}(\mu_X)_*&=(\sigma_X)^*(y_{PX})^*(P\sigma_X)_*(\sigma_{TX})_*=(\sigma_X)^*(\eta_{PX})^*(\sigma_{PX})^*(P\sigma_X)_*(\sigma_{TX})_*=\\&=(\eta_{TX})^*\cdot(\sigma_{TX})^*(P\sigma_X)^*(P\sigma_X)_*(\sigma_{TX})_*=(\eta_{TX})^*.\end{aligned}$$
∎

4. Presheaf submonads and Beck-Chevalley conditions

In this section, for a general 2-monad $\mathbb{T}=(T,\mu,\eta)$ on V-$\mathbf{Cat}$, we relate its BC* properties with the existence of a (sub)monad morphism $\mathbb{T}\to\mathbb{P}$. We remark that a necessary condition for $\mathbb{T}$ to be a submonad of $\mathbb{P}$ is that TX is separated for every V-category X, since PX is separated and separated V-categories are stable under monomorphisms.

Theorem 4.1. *For a 2-monad $\mathbb{T}=(T,\mu,\eta)$ on V-$\mathbf{Cat}$ with TX separated for every V-category X, the following assertions are equivalent:*

(i) $\mathbb{T}$ *is a submonad of* $\mathbb{P}$.
(ii) $\mathbb{T}$ *is lax idempotent and satisfies BC*, and both* η_X *and* $Q\eta_X \cdot y_{TX}$ *are fully faithful, for each* V*-category* X.
(iii) $\mathbb{T}$ *is lax idempotent,* μ *and* η *satisfy BC*, and both* η_X *and* $Q\eta_X \cdot y_{TX}$ *are fully faithful, for each* V*-category* X.
(iv) $\mathbb{T}$ *is lax idempotent,* η *satisfies BC*, and both* η_X *and* $Q\eta_X \cdot y_{TX}$ *are fully faithful, for each* V*-category* X.

Proof: (i) $\Rightarrow$ (ii): By (i) there exists a monad morphism $\sigma\colon \mathbb{T} \to \mathbb{P}$ with σ_X an embedding for every V-category X. By Corollary 3.8, with $\mathbb{P}$, also $\mathbb{T}$ is lax idempotent. Moreover, from $\sigma_X \cdot \eta_X = y_X$, with y_X, also η_X is fully faithful. (In fact this is valid for any monad with a monad morphism into $\mathbb{P}$.)

To show that $\mathbb{T}$ satisfies BC* we use the characterisation of Theorem 2.3; that is, we know that there is an admissible class Φ of distributors so that $\mathbb{T} = \Phi$. Then BC* for T follows directly from the fact that Φf is a (co)restriction of Pf, for every V-functor f.

BC* for η follows from BC* for y and full faithfulness of σ since, for any commutative diagram in V-**Cat**

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{1}"] & \cdot \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\boxed{2}"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$

with $\boxed{1}\boxed{2}$ satisfying BC*, and f and g fully faithful, also $\boxed{1}$ satisfies BC*.

Thanks to Proposition 3.5, BC* for μ follows directly from lax idempotency of $\mathbb{T}$.

The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious.

(iv) $\Rightarrow$ (i): For each V-category (X, a), we denote by $\widehat{a}$ the V-category structure on TX, and define the V-functor

$$(\,TX \xrightarrow{\sigma_X} PX\,) = (\,TX \xrightarrow{y_{TX}} PTX \xrightarrow{Q\eta_X} PX\,);$$

that is, $\sigma_X(\mathfrak{x}) = (\,X \xrightarrow{\eta_X} TX \xrightarrow{\widehat{a}} TX \xrightarrow{\mathfrak{x}^\circ} E\,) = \widehat{a}(\eta_X(\ ), \mathfrak{x})$. As a composite of fully faithful V-functors, σ_X is fully faithful; moreover, it is an embedding because, by hypothesis, TX and PX are separated V-categories.

$$[\varphi, a](*, x) = \bigwedge_{x' \in X} \hom(\varphi(x', *), a(x', x)) = TX(\varphi, x^*).$$

Therefore, if $\alpha\colon TX \to X$ is a $\mathbb{T}$-algebra structure, then

$$[\varphi, a](*, x) = TX(\varphi, x^*) = X(\alpha(\varphi), x),$$

that is, $[\varphi, a] = \alpha(\varphi)_*$; this means that α assigns to each distributor $\varphi\colon X \to\!\!\!\!\circ\, E$ the representative of $[\varphi, (1_X)_*]$.

Hence, we may describe the category of $\mathbb{T}$-algebras as follows.

Theorem 5.9. *(1) A map* $\alpha\colon TX \to X$ *is a* $\mathbb{T}$*-algebra structure if, and only if, for each distributor* $\varphi\colon X \to\!\!\!\!\circ\, E$ *in* TX, $\alpha(\varphi)_* = [\varphi, (1_X)_*]$.
(2) If X *and* Y *are* $\mathbb{T}$*-algebras, then a* V*-functor* $f\colon X \to Y$ *is a* $\mathbb{T}$*-homomorphism if, and only if,* f *preserves* φ*-weighted colimits for any* $\varphi \in TX$*, i.e., if* $x \in X$ *represents* $[\varphi, (1_X)_*]$*, then* $f(x)$ *represents* $[\varphi \cdot f^*, (1_Y)_*]$.

6. On algebras for submonads of $\mathbb{P}$: the special case of the formal ball monad

From now on we will study more in detail $(V\text{-}\mathbf{Cat})^{\mathbb{T}}$ for special submonads $\mathbb{T}$ of $\mathbb{P}$. In our first example, the formal ball monad $\mathbb{B}$, we will need to consider the (co)restriction of $\mathbb{B}$ and $\mathbb{P}$ to V-$\mathbf{Cat}_{\mathrm{sep}}$. We point out that the characterisations of $\mathbb{T}$-algebras of Theorem 5.5 remain valid for these (co)restrictions.

The space of formal balls is an important tool in the study of (quasi-)metric spaces. Given a metric space (X, d) its *space of formal balls* is simply the collection of all pairs (x, r), where $x \in X$ and $r \in [0, \infty[$. This space can itself be equipped with a (quasi-)metric. Moreover this construction can naturally be made into a monad on the category of (quasi-)metric spaces (cf. [10, 16] and references there).

This monad can readily be generalised to V-categories, using a V-categorical structure in place of the (quasi-)metric. We will start by considering an extended version of the formal ball monad, the *extended formal ball monad* $\mathbb{B}_\bullet$, which we define below.

Definitions 6.1. The *extended formal ball monad* $\mathbb{B}_\bullet = (B_\bullet, \eta, \mu)$ is given by the following:

(3) Given a class Φ of V-distributors, a V-category Z is called Φ-*cocomplete* if it is φ-cocomplete for every $\varphi\in\Phi$. When $\Phi = V$-$\mathbf{Dist}$, then Z is said to be *cocomplete*.

The proof of the following result can be found in [11, 2].

Theorem 5.5. *Given a submonad $\mathbb{T}$ of $\mathbb{P}$, for a V-category X the following assertions are equivalent:*

(i) *X is a $\mathbb{T}$-algebra.*
(ii) *X is injective with respect to T-embeddings.*
(iii) *X is $\Phi(\mathbb{T})$-cocomplete.*

$\Phi(\mathbb{T})$-cocompleteness of a V-category X is guaranteed by the existence of some special weighted colimits, as we explain next. (Here we present very briefly the properties needed. For more information on this topic see [19].)

Lemma 5.6. *For a distributor $\varphi\colon X\to Y$ and a V-functor $f\colon X\to Z$, the following assertions are equivalent:*

(i) *there exists the φ-colimit of f;*
(ii) *there exists the $(\varphi\cdot f^*)$-colimit of 1_Z;*
(iii) *for each $y\in Y$, there exists the $(y^*\cdot\varphi)$-colimit of f.*

Proof: (i) $\Leftrightarrow$ (ii): It is straightforward to check that

$$[\varphi,f_*]=[\varphi\cdot f^*,(1_Z)_*].$$

(i) $\Leftrightarrow$ (iii): Since $[\varphi,f_*]$ is defined pointwise, it is easily checked that, if g represents $[\varphi,f_*]$, then, for each $y\in Y$, the V-functor $E\xrightarrow{y}Y\xrightarrow{g}Z$ represents $[y^*\cdot\varphi,f_*]$.

Conversely, if, for each $y\colon E\to Y$, $g_y\colon E\to Z$ represents $[y^*\cdot\varphi,f_*]$, then the map $g\colon Y\to Z$ defined by $g(y)=g_y(*)$ is such that $g_*=[\varphi,f_*]$; hence, as stated in Remark 1.3, g is automatically a V-functor. ■

Corollary 5.7. *Given a submonad $\mathbb{T}$ of $\mathbb{P}$, a V-category X is a $\mathbb{T}$-algebra if, and only if, $[\varphi,(1_X)_*]$ has a colimit for every $\varphi\in TX$.*

Remark 5.8. Given $\varphi\colon X\,{\circ\!\!\!\rightarrow}\,E$ in TX, in the diagram

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\leq"] & X \\
Y \arrow[ur, "{[\varphi,a]}"'] &
\end{tikzcd}$$

To show that $\sigma=(\sigma_X)_X\colon T\to P$ *is a natural transformation*, that is, for each V-functor $f\colon X\to Y$, the outer diagram

$$\begin{tikzcd}
TX \arrow[r, "\mathpzc{y}_{TX}"] \arrow[d, "Tf"'] \arrow[dr, phantom, "\boxed{1}"] & PTX \arrow[r, "Q\eta_X"] \arrow[d, "PTf"] \arrow[dr, phantom, "\boxed{2}"] & PX \arrow[d, "Pf"] \\
TY \arrow[r, "\mathpzc{y}_{TY}"'] & PTY \arrow[r, "Q\eta_Y"'] & PY
\end{tikzcd}$$

commutes, we only need to observe that $\boxed{1}$ is commutative and BC* for η implies that $\boxed{2}$ is commutative.

It remains to show σ *is a monad morphism*: for each V-category (X,a) and $x\in X$,

$$(\sigma_X\cdot\eta_X)(x)=\widehat{a}(\eta_X(\ ),\eta_X(x))=a(-,x)=x^*=\mathpzc{y}_X(x),$$

and so $\sigma\cdot\eta=\mathpzc{y}$. To check that, for every V-category (X,a), the following diagram commutes

$$\begin{tikzcd}
TTX \arrow[r, "\sigma_{TX}"] \arrow[d, "\mu"'] & PTX \arrow[r, "P\sigma_X"] & PPX \arrow[d, "\mathpzc{m}_X"] \\
TX \arrow[rr, "\sigma_X"'] & & PX,
\end{tikzcd}$$

let $\mathfrak{X}\in TTX$. We have

$$\mathpzc{m}_X\cdot P\sigma_X\cdot\sigma_{TX}(\mathfrak{X})=$$

$$=(\ X\xrightarrow{\mathpzc{y}_X}PX\xrightarrow{\widetilde{a}}PX\xrightarrow{\sigma_X^\circ}TX\xrightarrow{\eta_{TX}}TTX\xrightarrow{\widehat{\widehat{a}}}TTX\xrightarrow{\mathfrak{X}^\circ}E\ )$$

$$=(\ X\xrightarrow{\eta_X}TX\xrightarrow{\widehat{a}}TX\xrightarrow{\eta_{TX}}TTX\xrightarrow{\widehat{\widehat{a}}}TTX\xrightarrow{\mathfrak{X}^\circ}E\ ),$$

since $\sigma_X^\circ\cdot\widetilde{a}\cdot\mathpzc{y}_X(x,\mathfrak{x})=\widetilde{a}(\mathpzc{y}_X(x),\sigma_X(\mathfrak{x}))=\sigma_X(\mathfrak{x})(x)=\widehat{a}\cdot\eta_X(x,\mathfrak{x})$, and

$$\sigma_X\cdot\mu_X(\mathfrak{x})=(\ X\xrightarrow{\eta_X}TX\xrightarrow{\widehat{a}}TX\xrightarrow{\mu_X^\circ}TTX\xrightarrow{\mathfrak{X}^\circ}E\ ).$$

Hence the commutativity of the diagram follows from the equality $\widehat{\widehat{a}}\cdot\eta_{TX}\cdot\widehat{a}\cdot\eta_X=\mu_X^\circ\cdot\widehat{a}\cdot\eta_X$ we show next. Indeed,

$$\begin{aligned}\widehat{\widehat{a}}\cdot\eta_{TX}\cdot\widehat{a}\cdot\eta_X&=(\eta_{TX})_*(\eta_X)_*=(\eta_{TX}\cdot\eta_X)_*=(T\eta_X\cdot\eta_X)_*=(T\eta_X)_*(\eta_X)_*\\&=\mu_X^*(\eta_X)_*=\mu_X^\circ\cdot\widehat{a}\cdot\eta_X.\end{aligned}$$ ■

The proof of the theorem allows us to conclude immediately the following result.

Corollary 4.2. *Given a 2-monad $\mathbb{T} = (T, \mu, \eta)$ on V-**Cat** such that η satisfies BC*, there is a monad morphism $\mathbb{T} \to \mathbb{P}$ if, and only if, η is pointwise fully faithful.*

5. On algebras for submonads of $\mathbb{P}$: a survey

In the remainder of this paper we will study, given a submonad $\mathbb{T}$ of $\mathbb{P}$, the category $(V\text{-}\mathbf{Cat})^{\mathbb{T}}$ of (Eilenberg-Moore) $\mathbb{T}$-algebras. Here we collect some known results which will be useful in the following sections. We will denote by $\Phi(\mathbb{T})$ the admissible class of distributors that induces the monad $\mathbb{T}$ (defined in (2.ii)).

The following result, which is valid for any lax idempotent monad $\mathbb{T}$, asserts that, for any V-category, to be a $\mathbb{T}$-algebra is a property (see, for instance, [9] and [6]).

Theorem 5.1. *Let $\mathbb{T}$ be lax idempotent monad on V-**Cat**.*

(1) For a V-category X, the following assertions are equivalent:
- (i) *$\alpha\colon TX \to X$ is a $\mathbb{T}$-algebra structure on X;*
- (ii) *there is a V-functor $\alpha\colon TX \to X$ with $\alpha \dashv \eta_X$ and $\alpha \cdot \eta_X = 1_X$;*
- (iii) *there is a V-functor $\alpha\colon TX \to X$ such that $\alpha \cdot \eta_X = 1_X$;*
- (iv) *$\alpha\colon TX \to X$ is a split epimorphism in V-**Cat**.*

(2) If (X, α) and (Y, β) are $\mathbb{T}$-algebra structures, then every V-functor $f\colon X \to Y$ satisfies $\beta \cdot Tf \le f \cdot \alpha$.

Next we formulate characterisations of $\mathbb{T}$-algebras that can be found in [11, 2], using *injectivity* with respect to certain *embeddings*, and using the existence of certain *weighted colimits*, notions that we recall very briefly in the sequel.

Definition 5.2. [8] A V-functor $f\colon X \to Y$ is a *T-embedding* if Tf is a left adjoint right inverse; that is, there exists a V-functor $Tf_\sharp$ such that $Tf \dashv Tf_\sharp$ and $Tf_\sharp \cdot Tf = 1_{TX}$.

For each submonad $\mathbb{T}$ of $\mathbb{P}$, the class $\Phi(\mathbb{T})$ allows us to identify easily the T-embeddings.

Proposition 5.3. *For a V-functor $h\colon X \to Y$, the following assertions are equivalent:*

- (i) *h is a T-embedding;*
- (ii) *h is fully faithful and h_* belongs to $\Phi(\mathbb{T})$.*

In particular, P-embeddings are exactly the fully faithful V-functors.

Proof: (ii) $\Rightarrow$ (i): Let h be fully faithful with $h_* \in \Phi(\mathbb{T})$. As in the case of the presheaf monad, $\Phi h : \Phi X \to \Phi Y$ has always a right adjoint whenever $h_* \in \Phi(\mathbb{T})$, $\Phi^{\dashv} h := (-) \cdot h_*\colon \Phi Y \to \Phi X$; that is, for each distributor $\psi : Y \,{\circ\!\!\to}\, E$ in ΦY, $\Phi^{\dashv} h(\psi) = \psi \cdot h_*$, which is well defined because by hypothesis $h_* \in \Phi(\mathbb{T})$. If h is fully faithful, that is, if $h^* \cdot h_* = (1_X)^*$, then $(\Phi^{\dashv} h \cdot \Phi h)(\varphi) = \varphi \cdot h^* \cdot h_* = \varphi$.

(i) $\Rightarrow$ (ii): If $\Phi^{\dashv} h$ is well-defined, then $y^* \cdot h_*$ belongs to $\Phi(\mathbb{T})$ for every $y \in Y$, hence $h_* \in \Phi(\mathbb{T})$, by 2.1(3), and so $h_* \in \Phi(\mathbb{T})$. Moreover, if $\Phi^{\dashv} h \cdot \Phi h = 1_{\Phi X}$, then in particular $x^* \cdot h^* \cdot h_* = x^*$, for every $x \in X$, which is easily seen to be equivalent to $h^* \cdot h_* = (1_X)^*$. ■

In V-**Dist**, given a V-distributor $\varphi\colon (X, a) \,{\circ\!\!\to}\, (Y, b)$, the functor $(\ ) \cdot \varphi$ preserves suprema, and therefore it has a right adjoint $[\varphi, -]$ (since the hom-sets in V-**Dist** are complete ordered sets):

$$\begin{tikzcd} \mathbf{Dist}(X,Z) \arrow[r, bend left, "{[\varphi,-]}"] \arrow[r, phantom, "\top"] & \mathbf{Dist}(Y,Z). \arrow[l, bend left, "(\ )\cdot\varphi"] \end{tikzcd}$$

For each distributor $\psi\colon X \,{\circ\!\!\to}\, Z$,

$$\begin{tikzcd} X \arrow[r, "\psi", "\circ"{marking}] \arrow[d, "\varphi"', "\circ"{marking}] & Z \\ Y \arrow[ur, "{[\varphi,\psi]}"', "\circ"{marking}] \arrow[u, phantom, shift right=3ex, "\le"] \end{tikzcd}$$

$[\varphi, \psi]\colon Y \,{\circ\!\!\to}\, Z$ is defined by

$$[\varphi, \psi](y, z) = \bigwedge_{x \in X} \hom(\varphi(x, y), \psi(x, z)).$$

Definitions 5.4. (1) Given a V-functor $f\colon X \to Z$ and a distributor (here called *weight*) $\varphi\colon X \,{\circ\!\!\to}\, Y$, a *$\varphi$-weighted colimit* of f (or simply a *φ-colimit* of f), whenever it exists, is a V-functor $g\colon Y \to Z$ such that $g_* = [\varphi, f_*]$. One says then that *g represents* $[\varphi, f_*]$.

(2) A V-category Z is called *φ-cocomplete* if it has a colimit for each weighted diagram with weight $\varphi\colon (X, a) \,{\circ\!\!\to}\, (Y, b)$; i.e. for each V-functor $f\colon X \to Z$, the φ-colimit of f exists.