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FRACTIONAL SOBOLEV REGULARITY FOR FULLY NONLINEAR ELLIPTIC EQUATIONS

EDGARD A. PIMENTEL, MAKSON S. SANTOS AND EDUARDO V. TEIXEIRA

Abstract: We prove higher-order fractional Sobolev regularity for fully nonlinear, uniformly elliptic equations in the presence of unbounded source terms. More precisely, we show the existence of a universal number $0 < \varepsilon < 1$, depending only on ellipticity constants and dimension, such that if u is a viscosity solution of $F(D^2u) = f(x) \in L^p$, then $u \in W^{1+\varepsilon,p}$, with appropriate estimates. Our strategy suggests a sort of fractional feature of fully nonlinear diffusion processes, as what we actually show is that $F(D^2u) \in L^p \implies (-\Delta)^\theta u \in L^p$, for a universal constant $\frac{1}{2} < \theta < 1$. We believe our techniques are flexible and can be adapted to various models and contexts.

Keywords: Fully nonlinear elliptic equations; regularity theory in fractional Sobolev spaces; $C^{1,\alpha}$-aperture functions.
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1. Introduction

In this paper we establish higher-order fractional regularity for the L^p-viscosity solutions of uniformly elliptic equations of the form

$$F(D^2u) = f(x) \qquad \text{in } B_1, \tag{1}$$

where $F\colon \mathcal{S}(d) \to \mathbb{R}$ is a (λ, Λ)-elliptic operator, and the source term $f \in L^p(B_1)$, for $p_0 < p \le \infty$; here, $p_0 \ge d/2$ is the Escauriaza exponent. We prove new interior estimates in Sobolev spaces $W^{1+\varepsilon,p}_{\text{loc}}(B_1)$, for a universal constant $0 < \varepsilon < 1$.

The regularity theory for fully nonlinear elliptic equations has played a prominent role in mathematical analysis since its launch in the early 1980's

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and, while substantial advances have been made by several authors through the years, fundamental questions remain open until today. The first development in the theory is the Krylov-Safonov Harnack inequality [10], which can be used to produced universal estimates in the C^α spaces, for a universal parameter $\alpha \in (0,1)$, for solutions of

$$F(D^2u) = 0 \quad \text{in} \quad B_1.$$

Through a linearization argument, $C^{1,\alpha}$-regularity results can be obtained; see, for instance, [3, 21]. Under convexity assumption upon the operator, the regularity of the solutions switches to the classical regime. That is, solutions to $F = 0$ become $C^{2,\alpha}$, with estimates; this is the subject of the so-called Evans-Krylov theory [9].

In [3], Caffarelli examines non-homogeneous fully nonlinear elliptic equations in the presence of variable coefficients. The findings reported in that paper cover regularity results in Hölder spaces – $C^{1,\alpha}$ and $C^{2,\alpha}$ estimates. Caffarelli also launches in that paper a regularity theory in Sobolev spaces, by establishing estimates in $W^{2,p}$, for $p > d$. An essential assumption in the $W^{2,p}$-regularity theory is a convexity-like condition on the operator. To be precise, in [3], it is assumed the homogeneous equation driven by the fixed-coefficients counterpart of the operator to have $C^{1,1}$-estimates. The requirement $p > d$ is weakened in [7], by means of an improved Harnack inequality proven in [8]. Caffarelli's estimates are then established in $W^{2,p}$ for $d/2 \leq p_0 < p$; the universal constant $p_0 = p_0(\lambda, \Lambda, d)$ is called *Escauriaza exponent* and plays a fundamental role in the theory of viscosity solutions of fully nonlinear equation. See [19] for sharp regularity estimates in such a regime.

Furthering the regularity theory in Sobolev spaces, local estimates in $W^{1,q}$ are the subject of [17]. In that paper, the author considers viscosity solutions to

$$F(D^2u, Du, u, x) = f \quad \text{in} \quad B_1$$

under usual structure conditions on F; see [5]. By supposing $f \in L^p(B_1)$, with $p_0 < p \leq d$, the author establish estimates in $W^{1,q}_{\text{loc}}(B_1)$ for every $q > 1$ satisfying

$$q < p^* := \frac{dp}{d-p} \quad \text{and} \quad d^* = \infty.$$

While striking, such estimates are restricted to the level of the gradient of the solutions. This raises a fundamental question: what's the *optimal* degree of differentiability of solutions under merely uniform ellipticity?

As concerns optimal regularity in Hölder spaces, this question was settled by the program carried out in [13, 14, 15, 16]. In those papers, the authors show that $C^{1,\alpha}$ estimates are indeed optimal. In Sobolev spaces, however, optimality remained largely open.

Our main result amounts to an integral estimate for fractional derivatives of order $1+\varepsilon$, where $\varepsilon \in (0,1)$ is a universal number, i.e. depends only on dimension and ellipticity constants (see the statement of Theorem 1 below). As a conclusion, one finds that ellipticity enforces a universal control on weak derivatives of order *strictly higher than 1*. Our main result reads as follows:

Theorem 1 (Fractional Sobolev Regularity). *Let $u \in C(B_1)$ be a viscosity solution to* (1). *Assume F is (λ,Λ)-elliptic and $d/2 \leq p_0 < p < \infty$. Let $\varepsilon \in (0,\alpha_0)$, where $\alpha_0 \in (0,1)$ is the exponent associated with the Hölder regularity for $F = 0$. Then $u \in W^{(1+\varepsilon),p}_{\mathrm{loc}}(B_1)$. In addition, there exists a positive constant $C = C(d,\lambda,\Lambda,\varepsilon)$ such that*

$$\|u\|_{W^{(1+\varepsilon),p}(B_{1/2})} \leq C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)}\right). \tag{2}$$

The proof of Theorem 1 draws inspiration from the arguments in [3], leading to $W^{2,p}$-estimates for convex operators. However, because we only impose uniform ellipticity, we cannot touch the graph of the solutions with paraboloids. The main key novelty of our approach is an alternative geometric construction that considers $C^{1,\alpha}$-cones of the form

$$\varphi(x) := \ell(x) \pm M|x-x_0|^{1+\alpha},$$

where $\ell\colon B_1 \to \mathbb{R}$ is an affine function and $M > 0$ is the opening of the cone. We are interested in the sets of points at which one can touch the graph of the solutions with cones of a certain opening. The idea is to show that information upon the measure of such sets can be translated into integral estimates of fractional order.

Indeed, an interesting corollary of our findings sheds light on an important issue underlying the regularizing effects in the fully nonlinear setting. Our reasoning suggests, at least heuristically, that while of second order, the diffusion process associated with F is no more efficient than α-stable Lévy process; see [18, 20] for further insights on diffusion efficiency versus regularity theory.

The remainder of this paper is organized as follows: Section 2.1 presents our main assumptions, whereas Section 2.2 gathers a few elementary notions and previous results. In Section 3, we resort to some geometric-measure techniques to produce a preliminary level of integrability for the aperture function. Section 4 refines this integrability using geometric and approximation methods. Sections 3 and 4 follow the reasonings of [4, Chapter 7] closely. The proof of Theorem 1 is the subject of Section 5, where we show viscosity solutions of (1) also verify a fractional diffusion equation.

2. Preliminaries

This section details the main hypotheses under which we work and presents preliminary facts and results used throughout the paper.

2.1. Main assumptions. In what follows, we put forward the main assumptions used in the article. We start with ellipticity condition on the operator F.

A 1 (Uniform ellipticity)**.** *The operator* $F\colon S(d) \to \mathbb{R}$ *is* (λ, Λ)*-uniformly elliptic. That is, for every* $M, N \in \mathcal{S}(d)$ *with* $N \geq 0$ *we have*

$$\lambda\|N\| \leq F(M) - F(M+N) \leq \Lambda\|N\|.$$

Ellipticity can also be phrased in terms of the extremal Pucci operators. For $M \in \mathcal{S}(d)$, we define the Pucci extremal operators $\mathcal{P}^{\pm}_{\lambda,\Lambda}$ as

$$\mathcal{P}^{+}_{\lambda,\Lambda}(M) := \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} \left(-\operatorname{Tr}(AM)\right)$$

and

$$\mathcal{P}^{-}_{\lambda,\Lambda}(M) := \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} \left(-\operatorname{Tr}(AM)\right),$$

where

$$\mathcal{A}_{\lambda,\Lambda} := \{A \in \mathcal{S}(d) : \lambda I \leq A \leq \Lambda I\}$$

is the class of (λ, Λ)-elliptic matrices. It is important to note that $\mathcal{P}^{+}_{\lambda,\Lambda}(M) = -\mathcal{P}^{-}_{\lambda,\Lambda}(-M)$. The condition in A1 is equivalent to

$$\mathcal{P}^{-}_{\lambda,\Lambda}(N) \leq F(M+N) - F(M) \leq \mathcal{P}^{+}_{\lambda,\Lambda}(N)$$

for any $M, N \in S(d)$. Our next assumption concerns the integrability condition imposed on the source term f; we denote with $p_0 \geq d/2$ the Escauriaza exponent.

A 2 (Integrability of the source term f)**.** *We suppose $f \in L^p(B_1)$, with $p \geq p_0$. In addition, there exists a constant $C > 0$ such that*

$$\|f\|_{L^p(B_1)} \leq C.$$

Remark 1 (Smallness regime)**.** *Our arguments depend on a smallness regime on the L^p-norm of the source term f; i.e., we require*

$$\|f\|_{L^p(B_1)} \leq \epsilon,$$

for some $0 < \epsilon \ll 1$, to be universally determined further. From usual scaling arguments, it shall be clear that such smallness requirement neither imposes further constraints on the data of the problem nor affects the resulting estimates.

2.2. Notations and preliminary results. For $r > 0$ and $x_0 \in \mathbb{R}^d$, $B_r(x_0)$ denotes the open ball of radius r centered at x_0. For simplicity, we denote $B_r(0)$ with B_r. Similarly, $Q_r(x_0)$ will denote the open cube with side r and center x_0, i.e.,

$$Q_r(x_0) := \left\{ x \in \mathbb{R}^d : |x - x_0|_\infty < \frac{r}{2} \right\},$$

where $|x|_\infty := \max\{|x_1|, \ldots |x_d|\}$. The space of real symmetric $d \times d$ matrices is denoted with $S(d)$. Lastly, we mention that some constants appearing in the paper depend only on the dimension d and the ellipticity λ and Λ; we refer to such a constant as *universal*.

For completeness, we next include the definition of viscosity solution; see [5, Definition 2.2].

Definition 1 (Viscosity solution)**.** *Let $p > p_0$. We say that $u \in C(B_1)$ is an L^p-viscosity sub-solution to*

$$F(D^2u, Du, u, x) = 0 \qquad in\ B_1 \tag{3}$$

if for every $x_0 \in B_1$ and every $\phi \in W^{2,p}_{\mathrm{loc}}(B_1)$ such that $u - \phi$ attains a local maximum at x_0, we have

$$\operatorname*{ess\,lim\,inf}_{x \to x_0} \left(F(D^2\phi(x), D\phi(x), u(x), x) \right) \leq 0.$$

We say that $u \in C(B_1)$ is an L^p-viscosity super-solution to (3) if for every $x_0 \in B_1$ and every $\phi \in W^{2,p}_{\mathrm{loc}}(B_1)$ such that $u - \phi$ attains a local minimum at x_0, we have

$$\operatorname*{ess\,lim\,sup}_{x \to x_0} \left(F(D^2\phi(x), D\phi(x), u(x), x) \right) \geq 0.$$

If $u \in C(B_1)$ is a viscosity sub and a super-solution to (3)*, we say that u is a viscosity solution to the equation.*

For $g \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, we define the maximal function of g, denoted with $m(g)$, as

$$m(g)(x) := \sup_{r>0} \frac{1}{|Q_r(x)|} \int_{Q_r(x)} |g(y)|\mathrm{d}y.$$

We recall that the maximal operator satisfies

$$|\{x \in \mathbb{R}^d : m(g)(x) \geq t\}| \leq \frac{C}{t} \|g\|_{L^1(\mathbb{R}^d)}, \quad \forall\, t > 0. \tag{4}$$

Two important structures in our analysis are the convex envelope of a function and the associated contact set.

Definition 2. *Let $\Omega \subset \mathbb{R}^d$ be an open set and $v \in C(\Omega)$. The convex envelope of v in Ω is defined by*

$$\Gamma(v)(x) := \sup_L \{L(x) \,|\, L \leq v \;\text{ in } \Omega,\; L \text{ is affine}\}.$$

The contact set of v is given by

$$\{x \in \Omega \mid v(x) = \Gamma(v)(x)\}.$$

Because our results rely solely on the ellipticity of the operator F, our strategy focuses on the $C^{1,\alpha}$-geometry of solutions. Consequently, we are interested in a contact set related to functions of class $C^{1,\alpha}$ for specific values of $\alpha \in (0,1)$. We proceed with the definition of $C^{1,\alpha}$-cone for $\alpha \in (0,1]$.

Definition 3 ($C^{1,\alpha}$-cone of opening M and vertex x_0)**.** *We say that ψ is a convex $C^{1,\alpha}$-cone of opening M and vertex x_0 if*

$$\psi(x) = L(x) + \frac{M}{2}|x - x_0|^{1+\alpha},$$

where M is a positive constant, and $L(x)$ is an affine function. Similarly, ψ is a concave $C^{1,\alpha}$-cone of opening M and vertex x_0 if

$$\psi(x) = L(x) - \frac{M}{2}|x - x_0|^{1+\alpha},$$

where M is a positive constant, and $L(x)$ is an affine function.

The sets collecting the points that can be touched by a $C^{1,\alpha}$-cone of certain opening play a pivotal role in our analysis since their measure yields information on the integrability of the solutions, as we shall explore later. Our next definition rigorously introduces those sets.

Definition 4. *Let $O \subset \Omega$ be an open subset, $0 < \tau_0 < \frac{\operatorname{diam}(O)}{5}$, and $M > 0$. We define*

$$\underline{G}_M(u, O) = \underline{G}_M(O)$$

as the set of all points $z \in O$ for which there exists a concave $C^{1,\alpha}$-cone ψ of opening M and vertex x_0 such that

(1) *$u(z) = \psi(z)$;*
(2) *$u(x) > \psi(x)$ for all $x \in B_{\tau_0}(z)$.*

We also define:

$$\overline{G}_M(u, O) = \overline{G}_M(O)$$

as the set of all points $z \in O$ for which there exists a convex $C^{1,\alpha}$-cone ψ of opening M and vertex x_0 such that

(1) *$u(z) = \psi(z)$;*
(2) *$u(x) < \psi(x)$ for all $x \in B_{\tau_0}(z)$.*

Finally

$$G_M(O) = \underline{G}_M(O) \cap \overline{G}_M(O).$$

Next, we note a monotonicity property related to the sets G_M. Let $M_1 > M_2$ and take $x_1 \in \underline{G}_{M_2}(u, O)$. By definition, there exists a concave $C^{1,\alpha}$-cone of the form

$$\psi(x) = L(x) - \frac{M_2}{2}|x - x_1|^{1+\alpha},$$

where L is an affine function, such that $\psi(x_1) = u(x_1)$ and $\psi(x) < u(x)$, for all $x \in B_{\frac{\operatorname{diam}(O)}{10}}(x_1)$. Hence,

$$u(x) > L(x) - \frac{M_2}{2}|x - x_1|^{1+\alpha} > L(x) - \frac{M_1}{2}|x - x_1|^{1+\alpha} = \tilde{\psi}(x),$$

for all $x \in B_{\frac{\operatorname{diam}(O)}{10}}(x_1)$. Notice that,

$$u(x_1) = \psi(x_1) = \tilde{\psi}(x_1).$$

We conclude that $x_1 \in \underline{G}_{M_1}(u, O)$; thus

$$\underline{G}_{M_2}(u, O) \subset \underline{G}_{M_1}(u, O).$$

Similarly, we have

$$\overline{G}_{M_2}(u, O) \subset \overline{G}_{M_1}(u, O).$$

Therefore,

$$G_{M_2}(u, O) \subset G_{M_1}(u, O).$$

The following definition accounts for a family of sets collecting points where a $C^{1,\alpha}$-cone cannot touch the graph of the solutions.

Definition 5. *Let $O \subset \Omega$ be an open subset, $0 < \tau_0 < \frac{\operatorname{diam}(O)}{5}$ and $M > 0$. We define*

$$\underline{A}_M(u,O) = \underline{A}_M(O) = O \backslash \underline{G}_M(u,O).$$

Similarly

$$\overline{A}_M(u,O) = \overline{A}_M(O) = O \backslash \overline{G}_M(u,O).$$

Finally

$$A_M(u,O) = A_M(O) = O \backslash G_M(u,O).$$

Now, we define the $C^{1,\alpha}$-aperture function. This structure is directly related to the integrability of the solutions to (1).

Definition 6 ($C^{1,\alpha}$-aperture function)**.** *Let $B \Subset B_1$. For $x \in B$ we define*

$$\theta(x) := \theta_{1+\alpha}(u,B)(x) = \inf\{M \mid x \in G_M(u,B)\} \in [0,\infty].$$

Our paper aims to understand how the aperture function encodes information on the integrability of fractional derivatives of solutions to (1).

In the sequel, we recall the primary ingredients of the theories unlocking this connection. We start with an auxiliary lemma.

Lemma 1. *For $\Omega \subset \mathbb{R}^d$, let $g : \Omega \to \mathbb{R}$ be a nonnegative and measurable function. Define $\mu_g : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ as*

$$\mu_g(t) := |\{x \in \Omega \mid g(x) > t\}|, \quad t > 0.$$

Let $\eta > 0$ and $M > 1$ be constants. Then for $0 < p < \infty$,

$$g \in L^p(\Omega) \Longleftrightarrow \sum_{k \geq 1} M^{pk} \mu_g(\eta M^k) = S < \infty$$

and

$$C^{-1} S \leq \|g\|^p_{L^p(\Omega)} \leq C(|\Omega| + S),$$

where $C > 0$ is a constant depending only on η, M and p.

The function μ_g, defined in Lemma 1, is known as the *distribution function of g*. Next, we recall a corollary of the Calderón-Zygmund decomposition. See [4, Lemma 4.2]. Let Q_1 be the unit cube and split it into 2^d cubes of half side. Then, split each of these 2^d cubes and iterate the process. The cubes obtained in this way are called dyadic cubes.

If Q is a dyadic cube different from Q_1, we say that $\tilde{Q}$ is the predecessor of Q if the latter is one of the 2^d cubes obtained by dividing $\tilde{Q}$.

Lemma 2 (Calderón-Zygmund decomposition)**.** *Let $A \subset B \subset Q_1$ be measurable sets and take $0 < \delta < 1$ such that*

(a) $|A| \leq \delta$*;*
(b) *if Q is a dyadic cube such that $|A \cap Q| > \delta |Q|$, then $\widetilde{Q} \subset B$, where $\widetilde{Q}$ is the predecessor of Q.*

Then

$$|A| \leq \delta |B|.$$

Now, we can state a connection between the distribution function μ_θ and the measure of the sets A_M. In fact,

$$\mu_\theta(t) \leq |A_t(u, B_{1/2})|;$$

therefore, we study the summability of

$$\sum_{k \leq 1} M^{pk} |A_{M^k}(u, B_{1/2})|$$

and the mechanism transmitting information from θ to u.

In what follows, we define the fractional Sobolev spaces. We refer to [6, Chapter 2] for further details.

Definition 7 (Fractional Sobolev spaces)**.** *Let $s \in (0,1)$. For any $p \in [1, +\infty)$ we define $W^{s,p}(\Omega)$ as*

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{n}{p} + s}} \in L^P(\Omega \times \Omega) \right\}.$$

We equip $W^{s,p}(\Omega)$ with a norm denoted with $\| \cdot \|_{W^{s,p}(\Omega)}$ and given by

$$\|u\|_{W^{s,p}(\Omega)} := \left(\int_\Omega |u|^p dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{d+sp}} \mathrm{d}x \mathrm{d}y \right)^{\frac{1}{p}}.$$

Moreover, we denote with $[u]_{W^{s,p}(\Omega)}$ the Gagliardo seminorm of u, defined as

$$[u]_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{d+sp}} \mathrm{d}x \mathrm{d}y \right)^{\frac{1}{p}}.$$

Sobolev spaces involving weak derivatives of higher fractional order are defined next.

Definition 8 (Higher fractional order Sobolev spaces)**.** *If $s > 1$, we write $s = m + \gamma$, where m is an integer and $\gamma \in (0, 1)$. We define*

$$W^{s,p}(\Omega) := \{u \in W^{m,p}(\Omega) \mid D^\alpha u \in W^{\gamma,p}(\Omega), \text{ for any } \alpha \text{ s.t. } |\alpha| = m\}.$$

In this case, $W^{s,p}(\Omega)$ is equipped with the following norm:

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|^p_{W^{m,p}(\Omega)} + \sum_{|\alpha|=m} \|D^\alpha u\|^p_{W^{\gamma,p}(\Omega)} \right)^{\frac{1}{p}}.$$

When $p = 2$, we write $W^{s,2}(\Omega) =: H^s(\Omega)$. The space $W^{s,p}_0(\Omega)$ consists of all functions $u \in W^{s,p}(\mathbb{R}^d)$ such that $u = 0$ in $\mathbb{R}^d \setminus \Omega$. In addition, $W^{-s,p}(\Omega)$ denotes the dual space of $W^{s,p}(\Omega)$.

Next, the facts we recall concern the Fourier transform and its relationship with the fractional Laplacian operator. For simplicity, we operate in the context of the Schwartz space, denoted with $\mathcal{S}$; we refer the reader to [6, Chapter 3]. Standard density arguments allow us to switch from $\mathcal{S}$ to $L^2(\mathbb{R}^d)$, as suitable. In what follows we recall the definition of the fractional Laplacian and the Fourier transform, to which we resort in our argument.

For $s \in (0, 1)$ and $v \in \mathcal{S}$, we define the fractional Laplacian operator as

$$(-\Delta)^s v(x) := -\frac{1}{2} C(d, s) \int_{\mathbb{R}^d} \frac{v(x + y) + v(x - y) - 2v(x)}{|y|^{d+2s}} \mathrm{d}y.$$

The Fourier transform of u is defined by

$$\mathcal{F}v(\zeta) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\zeta \cdot x} v(x) \mathrm{d}x.$$

The interaction between the Fourier transform and the fractional Laplacian operator is the subject of the upcoming proposition.

Proposition 1. *Let $s \in (0, 1)$ and let $(-\Delta)^s \colon \mathcal{S} \to L^2(\mathbb{R}^d)$ be the fractional Laplacian operator. Then, for any $u \in \mathcal{S}$,*

$$(-\Delta)^s u = \mathcal{F}^{-1}(|x|^{2s}(\mathcal{F}u)) \quad \text{for all} \quad x \in \mathbb{R}^d.$$

For a proof of this fact, we refer the reader to [6, Proposition 3.3]. We continue with the definition of a related functional space.

Definition 9. *For $s \in \mathbb{R}$ we define*

$$\bar{H}^s(\mathbb{R}^d) = \left\{ u \in L^2(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} (1 + |x|^{2s}) |\mathcal{F}u(x)|^2 \mathrm{d}x < \infty \right\}.$$

Proposition 2. *Let $s \in (0,1)$. The fractional Sobolev space $H^s(\mathbb{R}^d)$ coincides with $\bar{H}^s(\mathbb{R}^d)$. In particular, for any $u \in H^s(\mathbb{R}^d)$*

$$[u]^2_{H^s(\mathbb{R}^d)} = 2C(n,s)^{-1} \int_{\mathbb{R}^d} |x|^{2s} |\mathcal{F}u(x)|^2 \mathrm{d}x.$$

For the proof of Proposition 2, we mention [6, Proposition 3.4]. The next result concerns *local* regularity for the fractional Laplacian equation of order s. See [2, Theorem 1.4] for details.

Proposition 3. *Let $u \in W^{s,2}_0(\bar{\Omega})$ be the unique weak solution to*

$$\begin{cases} (-\Delta)^s u = f & in \quad \Omega \\ u = 0 & in \quad \mathbb{R}^d \setminus \Omega. \end{cases} \tag{5}$$

where $\Omega \subset \mathbb{R}^d$ is an arbitrary bounded open set and $s \in (0,1)$. If $f \in L^p(\Omega)$ with $1 < p < \infty$, then $u \in W^{2s,p}_{\mathrm{loc}}(\Omega)$. In addition, for every $\Omega_0 \Subset \Omega$ there exists $C > 0$ such that

$$\|u\|_{W^{2s,p}(\Omega_0)} \leq C \left(\|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)} \right),$$

where $C = C(d, \mathrm{diam}(\Omega), \mathrm{dist}(\Omega_0, \partial\Omega))$.

3. Geometric-Measure Tools

This section discusses an L^δ-estimate for the aperture function, introduced in Definition 6. This first level of integrability stems from the uniform ellipticity of F and the integrability of f; see A1 and A2. We refine such estimate further in the argument, where geometric techniques build upon the regularity available for $F = 0$ in $C^{1,\beta}$-spaces. To be more precise, following proposition holds true:

Proposition 4 (L^δ-estimate for the aperture function)**.** *Let $u \in C(B_1)$ be a normalized viscosity solution to* (1)*. Suppose A1 and A2 are in force. Then the $C^{1,\alpha}$-aperture function, $\theta_{1+\alpha} = \theta$, defined in Definition 6, is in $L^\delta(B_1)$, for some universal $0 < \delta \ll 1$. In additional, there exists a universal constant $C > 0$ such that*

$$\int_{B_{1/2}} |\theta(x)|^\delta \mathrm{d}x \leq C.$$

Proposition 4 is analogous to Lin's integral estimates, introduced in the linear setting in [11]. See [3, 4] for its fully nonlinear counterpart; for a more recent account of this result, see [1, 12]. The proof of Proposition

4 is standard and follows along the same lines as in [4, Chapter 7]. The important modification, from the technical viewpoint, concerns the use of $C^{1,\alpha}$-cones instead of paraboloids, as in [4]. Though we omit the proof, we recall some of its ingredients for further reference.

Lemma 3. *Let $u \in C(B_{6\sqrt{d}})$ be a viscosity solution to* (1) *in $B_{6\sqrt{d}}$. Suppose A1 and A2 are in force. Suppose further that Ω is a bounded domain such that $B_{6\sqrt{d}} \subset \Omega$. Then*

$$|\underline{G}_M(u,\Omega) \cap Q_1| \geq 1 - \sigma,$$

where $0 < \sigma < 1$ and $M > 1$ are universal constants.

For the proof of Lemma 3 we refer the reader to [4, Lemma 7.5]. We close this section with a decay rate for the measure of the sets $A_t(u,\Omega) \cap Q_1$. The conclusion of the next lemma is equivalent to the statement of Proposition 4. We state it next for convenience, as we resort to its formulation as a characterization of δ-integrability for θ.

Lemma 4. *Let $u \in C(B_{6\sqrt{d}})$ be a viscosity solution to* (1) *in $B_{6\sqrt{d}}$. Suppose A1 and A2 are in force and Ω is a bounded domain such that $B_{6\sqrt{d}} \subset \Omega$. Extend f by zero outside $B_{6\sqrt{d}}$. Then*

$$|A_t(u,\Omega) \cap Q_1| \leq c_2 t^{-\mu}, \quad \forall\, t > 0, \tag{6}$$

where c_2 and μ are positive universal constants.

The information in Proposition 4, or Lemma 4, is sub-optimal. This is because it does not consider the implications of ellipticity to the regularity of the solutions. In fact, by noticing that uniform ellipticity implies $C^{1,\alpha}$-estimates for the homogeneous problem governed by F, we manage to improve the decay rate in (6).

The next section produces an approximation lemma relating the solutions to (1) with viscosity solutions to $F = 0$. The regularity theory available for the latter refines the decay rate for $|A_{M^k}(u, B_{6\sqrt{d}}) \cap Q_1|$, ultimately improving the integrability of the aperture function θ.

4. Higher integrability of the aperture function

In what follows, we refine the decay rate of the measure of specific sets, leading to improved integrability of the aperture function. The core of this section is the following proposition.

Proposition 5 (Improved integrability of θ). *Let $u \in C(B_1)$ be a normalized viscosity solution to* (1). *Suppose A1-A2 are in force. Suppose further that $\|f\|_{L^p(B_1)} < \epsilon$, for some $\epsilon > 0$ to be determined. Then $\theta \in L^p(B_1)$ and there exists a universal constant $C > 0$ such that*

$$\|\theta\|_{L^p(B_{1/2})} \leq C.$$

In establishing Proposition 5, the key ingredient is an approximation lemma importing information from the solutions to $F = 0$.

Lemma 5 (Approximation Lemma). *Let $u \in C(B_{8\sqrt{d}})$ be a normalized viscosity solution to* (1) *in $B_{8\sqrt{d}}$. Suppose that A1-A2 are in force. Given $\delta > 0$, there exists $0 < \epsilon < \delta$ such that, if $\|f\|_{L^p(B_{8\sqrt{d}})} \leq \epsilon$, one can find a function $h \in C^{1,\alpha_0}_{\rm loc}(B_{7\sqrt{d}})$ satisfying*

$$\|u - h\|_{L^\infty_{\rm loc}(\overline{B}_{7\sqrt{d}})} \leq \delta,$$

where $\alpha_0 = \alpha_0(d, \lambda, \Lambda) \in (0, 1)$ accounts for the regularity available for $F = 0$. In addition, there exists a universal constant $C = C(d, \lambda, \Lambda, \alpha_0)$ such that

$$\|h\|_{C^{1,\alpha_0}(B_{13\sqrt{d}/2})} \leq C \|h\|_{L^\infty(B_{20\sqrt{d}/3})}.$$

Proof: We argue as in [19], by supposing, seeking a contradiction, that the statement of the proposition is false. In this case, there exists a number $\delta_0 > 0$, a sequence of (λ, Λ)-elliptic operators $(F_n)_{n\in\mathbb{N}}$ and sequences of functions $(u_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$, satisfying

$$F_n(D^2 u_n) = f_n \quad \text{in} \quad B_{8\sqrt{d}} \tag{7}$$

and

$$\|f_n\|_{L^p(B_{8\sqrt{d}})} \leq \frac{1}{n},$$

with

$$\|u_n - h\|_{L^\infty(B_{6\sqrt{d}})} > \delta_0, \tag{8}$$

for all $h \in C^{1,\alpha_0}_{\rm loc}(\overline{B}_{7\sqrt{d}})$.

From the regularity available for (7), the family $(u_n)_{n\in\mathbb{N}}$ is equibounded in some Hölder space C^β, for some $\beta \in (0, 1)$ unknown, though universal. Therefore, there exists a convergent subsequence, still denoted with $(u_n)_{n\in\mathbb{N}}$, and $u_\infty \in C^{\frac{\beta}{2}}_{\rm loc}(B_{7\sqrt{d}})$, such that $u_n \to u_\infty$ locally uniformly.

In addition, by already classical arguments, there exists a (λ, Λ)-elliptic operator F_∞ such that $F_n \to F_\infty$. At this point, standard stability results [5, Theorem 3.8] imply that u_∞ solves

$$F_\infty(D^2 u_\infty) = 0 \quad \text{in} \quad B_{7\sqrt{d}}.$$

Since F is a (λ, Λ)-uniformly elliptic operator, we have $u_\infty \in C^{1,\alpha_0}_{\text{loc}}(\overline{B}_{7\sqrt{d}})$. By taking $h \equiv u_\infty$, we get a contradiction with (8), and the proof is complete. ■

In the remainder of this section, we suppose the source term satisfies the smallness regime prescribed in Lemma 5. That is,

$$\|f\|_{L^p(B_{8\sqrt{d}})} \leq \epsilon,$$

for $0 < \epsilon \ll 1$ as in the statement of Lemma 5, though yet to be determined.

Lemma 6. *Let $u \in C(B_{8\sqrt{d}})$ be a normalized viscosity solution to (1) in $B_{8\sqrt{d}}$. Let $\Omega \subset \mathbb{R}$ be a bounded domain such that $B_{8\sqrt{d}} \subset \Omega$ and $h \in C^{1,\alpha_0}_{\text{loc}}(B_{7\sqrt{d}})$ be the function from Lemma 5. There exists a function $\tilde{h} \in C^2(\Omega)$ satisfying*

$$\|u - \tilde{h}\|_{L^\infty(\Omega)} \leq 1$$

and

$$\|\tilde{h}\|_{C^{1,\alpha_0}(B_{6\sqrt{d}})} \leq C\|h\|_{C^{1,\alpha_0}(B_{25\sqrt{d}/4})},$$

where $C > 0$ is a universal constant.

Proof: Set

$$\Omega_1 := \{x \in \mathbb{R}^d : \text{dist}(x, \Omega) < 1\}.$$

Extend u continuously to Ω_1. Also, extend h outside $\bar{B}_{13\sqrt{d}/2}$ continuously, such that $h = u$ in $\Omega_1 \setminus B_{20\sqrt{d}/3}$ and $\|u - h\|_{L^\infty(\Omega_1)} = \|u - h\|_{L^\infty\left(B_{13\sqrt{d}/2}\right)}$. Let $(\eta_n)_{n \in \mathbb{N}}$ be a sequence of standard mollifiers and define

$$h_n := h * \eta_n \quad \text{in} \quad \Omega_{1,n},$$

where

$$\Omega_{1,n} := \{x \in \Omega_1 : \text{dist}(x, \partial\Omega_1 < 1/n)\}.$$

It is clear that $\|h - h_n\|_{L^\infty_{\text{loc}}(\Omega_1)} \to 0$ as $n \to \infty$. In particular, for $\bar{n}$ sufficiently large we have

$$\|h - h_{\bar{n}}\|_{L^\infty(\Omega)} \leq \frac{1}{2}.$$

Therefore,

$$\|u - h_{\bar{n}}\|_{L^\infty(\Omega)} \leq \|u - h\|_{L^\infty(\Omega)} + \|h - h_{\bar{n}}\|_{L^\infty(\Omega)} \leq \delta + 1/2 \leq 1,$$

provided the approximation regime encoded by δ in Lemma 5 is such that $\delta \leq 1/2$. Define $B_{6\sqrt{d},\bar{n}}$ as

$$B_{6\sqrt{d},\bar{n}} := \left\{ x \in \mathbb{R}^d : \operatorname{dist}(x, B_{6\sqrt{d}}) < \frac{1}{\bar{n}} \right\},$$

and consider $h|_{B_{6\sqrt{d},\bar{n}}}$, which we still denote with h, for ease of notation. Thus for $x \in B_{6\sqrt{d}}$ we obtain

$$|h_{\bar{n}}(x)| \leq \int_{B_{6\sqrt{d},\bar{n}}} |h(y)\eta_n(x-y)|\mathrm{d}y \leq \sup_{B_{6\sqrt{d},\bar{n}}} |h(z)| \leq \|h\|_{C^{1,\alpha_0}(B_{25\sqrt{d}/4})};$$

hence, we have

$$\sup_{B_{6\sqrt{d}}} |h_{\bar{n}}(x)| \leq \|h\|_{C^{1,\alpha_0}(B_{25\sqrt{d}/4})}.$$

Because

$$Dh_n = Dh * \eta_n,$$

we similarly conclude that

$$\sup_{B_{6\sqrt{d}}} |Dh_{\bar{n}}(x)| \leq \|h\|_{C^{1,\alpha_0}(B_{25\sqrt{d}/4})}.$$

Finally notice that for $x, z \in B_{6\sqrt{d}}$

$$\begin{aligned} |Dh_n(x) - Dh_n(z)| &\leq \int_{B_{6\sqrt{d},\bar{n}}} |Dh(x-y) - Dh_n(z-y)|\eta_n(y)\mathrm{d}y \\ &\leq \sup_{y \in B_{6\sqrt{d},\bar{n}}} |Dh(x-y) - Dh_n(z-y)| \\ &\leq C|x-z|^{\alpha_0}. \end{aligned}$$

By combining the former inequalities and setting $\tilde{h} := h_{2\bar{n}}$ one completes the proof. ∎

In what follows, the former lemma builds upon measure-theoretic techniques to refine the L^δ-estimate available for θ.

Lemma 7. *Let $u \in C(B_{8\sqrt{d}})$ be a normalized viscosity solution to* (1) *in $B_{8\sqrt{d}}$. Suppose that A1-A2 are in force, with $\|f\|_{L^d(\Omega)} < \epsilon$. Suppose further that*

$$-|x|^{1+\alpha} \leq u(x) \leq |x|^{1+\alpha} \qquad in \qquad \Omega \setminus B_{6\sqrt{d}},$$

for some $\alpha \in (0,1)$, fixed. Then

$$|G_M(u,\Omega) \cap Q_1| \geq 1 - \rho, \tag{9}$$

for some $\rho \in (0,1)$, where $M > 1$ in a universal constant.

Proof: Fix $0 < \delta < 1$, yet to be determined. Let $\tilde{h}$ be the $C^2_{\mathrm{loc}}(\Omega)$ function whose existence follows from Lemma 6. Since

$$\|u - \tilde{h}\|_{L^\infty(\Omega)} \leq 1,$$

it follows that

$$-1 - |x|^{1+\alpha} \leq \tilde{h}(x) \leq 1 + |x|^{1+\alpha} \qquad \text{in} \qquad \Omega \backslash B_{6\sqrt{d}}.$$

Therefore, there exists $1 < N = N(d, C)$ such that

$$Q_1 \subset G_N(h, \Omega). \tag{10}$$

Define

$$w(x) = \frac{\min(1,\epsilon)}{2\delta}(u - \tilde{h})(x).$$

Notice that w solves

$$\tilde{F}(D^2 v, x) - \tilde{f}(x) = 0,$$

where

$$\tilde{F}(M, x) := \frac{\min(1,\epsilon)}{2\delta} F\left(\frac{2\delta}{\min(1,\epsilon)} M + D^2\tilde{h}\right),$$

and

$$\tilde{f}(x) := \frac{\min(1,\epsilon)}{(2\delta)} f(x).$$

As a consequence of the former inequality, we have

$$\|\tilde{f}\|_{L^d} \leq \frac{\min(1,\epsilon)}{(2\delta)} \|f\|_{L^d} \leq \epsilon.$$

Hence, w is entitled to the conclusions of Lemma 3 in Ω. Because of Lemma 4, we obtain

$$|A_t(w, \Omega) \cap Q_1| \leq t^{-\mu}, \quad \text{for all} \quad t > 0.$$

It follows that

$$|A_s(u - h, \Omega) \cap Q_1| \leq c s^{-\mu} \delta^{\mu} \quad \text{for all} \quad s > 0.$$

By choosing δ small enough we get

$$|G_N(u - h, \Omega) \cap Q_1| \geq 1 - \delta^{\mu} \geq 1 - \rho. \qquad \blacksquare$$

The proof of Lemma 7 sets the proximity-regime encoded by $\delta > 0$. As a by-product, it sets the smallness condition on the L^p-norm of the source term f, encoded by $\epsilon > 0$ in the statement of Lemma 5. In the remainder of this section, these constants remain fixed.

Lemma 8. *Let $u \in C(B_{8\sqrt{d}})$ be a normalized viscosity solution to* (1) *in $B_{8\sqrt{d}}$. Suppose that A1-A2 are in force, with $\|f\|_{L^d(\Omega)} < \epsilon$. If*

$$G_1(u,\Omega) \cap Q_3 \neq \emptyset, \tag{11}$$

then

$$|G_M(u,\Omega) \cap Q_1| \geq 1 - p,$$

with M and ρ as in Lemma 7.

Proof: Let $x_1 \in G_1(u,\Omega) \cap Q_3$. Hence, there exists an affine function $L(x)$, such that

$$-\frac{1}{2}|x - x_1|^{1+\alpha} \leq u(x) - L(x) \leq \frac{1}{2}|x - x_1|^{1+\alpha} \quad \text{in} \quad \Omega.$$

Define

$$v(x) = \frac{u(x) - L(x)}{c(d)},$$

where $c(d)$ is a constant depending only on d, large enough as to guarantee $|v(x)| \leq 1$ and

$$|v(x)| \leq |x|^{1+\alpha} \quad \text{in} \quad \Omega \backslash B_{6\sqrt{d}}.$$

In addition, v solves

$$\tilde{F}(D^2 u) - \tilde{f}(x) = 0,$$

where

$$\tilde{F}(M) := \frac{1}{c(d)} F(c(d)M),$$

and

$$\tilde{f}(x) := \frac{1}{c(d)} f(x).$$

Lemma 7 yields

$$|G_M(v,\Omega) \cap Q_1| \geq 1 - \rho,$$

therefore

$$|G_{c(d)M}(u,\Omega) \cap Q_1| \geq 1 - \rho$$

and the Lemma is proven. ■

The following result resorts once again to the Calderón-Zygmund decomposition.

Lemma 9. *Let $u \in C(B_{8\sqrt{d}})$ be a normalized viscosity solution to* (1) *in $B_{8\sqrt{d}}$. Suppose that A1-A2 are in force, with $\|f\|_{L^d(\Omega)} < \epsilon$. Extend f by zero outside $B_{8\sqrt{d}}$ and set*

$$A := A_{M^{k+1}}(u, B_{8\sqrt{d}}) \cap Q_1,$$

$$B := \left\{A_{M^k}(u, B_{8\sqrt{d}}) \cap Q_1\right\} \cup \left\{x \in Q_1 : m(f^d)(x) \geq c_3^d M^{kd}\right\},$$

for $k \in \mathbb{N}$. Then

$$|A| \leq \rho |B|,$$

where $M > 1$ is a universal constant and $c_3 > 0$ depends only on d, λ, Λ and ρ.

Proof: We start by noticing that $|u| \leq 1 \leq |x|^{1+\alpha}$ in $B_{8\sqrt{d}} \backslash B_{6\sqrt{d}}$. Hence Lemma 7 applied with $\Omega = B_{8\sqrt{d}}$, implies

$$|G_{M^{k+1}}(u, B_{8\sqrt{d}}) \cap Q_1| \geq |G_{M^k}(u, B_{8\sqrt{d}}) \cap Q_1| \geq 1 - \rho.$$

It leads to $|A| \leq \rho$.

The remainder of the proof relies on the Calderón-Zygmund decomposition, as stated in Lemma 2. Hence, we need to show that if $Q = Q_{1/2^i}(x_0)$ is a dyadic cube Q_1 such that

$$|A_{M^{k+1}}(u, B_{8\sqrt{d}}) \cap Q| = |A \cap Q| > \rho |Q|, \tag{12}$$

we have $\tilde{Q} \subset B$. We suppose otherwise and produce a contradiction. Suppose that $\tilde{Q} \not\subset B$ and let x_1 be such that

$$x_1 \in \tilde{Q} \cap G_{M^k}(u, B_{8\sqrt{d}}) \tag{13}$$

and

$$m(f^d)(x_1) \leq \left(c_3 M^k\right)^d. \tag{14}$$

Consider as before the transformation

$$x = x_0 + \frac{1}{2^i} y, \quad x \in B_{8\sqrt{d}}, \tag{15}$$

and define

$$v(y) = \frac{2^{2i}}{M^k} u\left(x_0 + \frac{1}{2^i} y\right).$$

Finally, let $\tilde{\Omega}$ be the image of $B_{8\sqrt{d}}$ under the transformation (15). We need to verify that v satisfies the hypothesis of Lemma 8. Note that v solves

$$\tilde{F}(D^2u) - \tilde{f}(x) = 0 \;\; in \;\; \tilde{\Omega},$$

where

$$\tilde{F}(N) := \frac{1}{M^k} F\left(M^k N\right)$$

and

$$\tilde{f}(x) := \frac{1}{M^k} f\left(x_0 + \frac{1}{2^i}x\right).$$

Since $B_{8\sqrt{d}} \subset \tilde{\Omega}$, the function v satisfies the equation in $B_{8\sqrt{d}}$, in the viscosity sense. Furthermore $|x_1 - x_0|_\infty \leq 3/2^{i+1}$ implies that $B_{8\sqrt{d}/2^i}(x_0) \subset Q_{19\sqrt{d}/2^i}(x_1)$. Hence

$$\begin{aligned}
\|\tilde{f}\|^d_{B_{8\sqrt{d}}} &= \frac{2^{id}}{M^{kd}} \int_{B_{8\sqrt{d}/2^i}(x_0)} |f(x)|^d \mathrm{d}x \\
&\leq \frac{c(d)}{M^{kd}} \frac{1}{|Q_{19\sqrt{d}/2^i}|} \int_{Q_{19\sqrt{d}/2^i}(x_1)} |f(x)|^d \mathrm{d}x \\
&\leq c(d) c_3^d \\
&\leq \epsilon,
\end{aligned}$$

for c_3 small enough.

Now, by (13) there exists a convex and a concave $C^{1,\alpha}$-cones of opening M^k, ψ_1 and ψ_2 respectively, such that ψ_1 touches u from above at x_1 and ψ_2 touches u from below at x_1. Define

$$\tilde{\psi_1}(y) := \psi_1\left(x_0 + \frac{1}{2^i}y\right)$$

and

$$\tilde{\psi_2}(y) := \psi_2\left(x_0 + \frac{1}{2^i}y\right).$$

It is easy to see that $\tilde{\psi_1}$ (resp. $\tilde{\psi_2}$) touches v from above (respectively from below) in a point y_1 such that $x_1 = x_0 + \frac{1}{2^i}y_1$. Therefore $G_1(v, \tilde{\Omega}) \neq \emptyset$. By Lemma 8 we obtain

$$|G_M(v, \tilde{\Omega}) \cap Q_1| \geq 1 - \rho = (1-\rho)|Q_1|.$$

Hence

$$|G_{M^{k+1}}(u, B_{8\sqrt{d}}) \cap Q| \geq (1-\rho)|Q|,$$

which implies

$$|A_{M^{k+1}}(u, B_{8\sqrt{d}}) \cap Q| \leq \rho|Q|.$$

This fact produces a contradiction with (12) and finishes the proof. ■

Next, we detail the proof of Proposition 5.

Proof of Proposition 5: Let M be as in Lemma 9 and take ρ such that

$$\rho M^p = \frac{1}{2}.$$

For $k \geq 0$, define

$$\alpha_k := |A_{M^k}(u, B_{8\sqrt{d}}) \cap Q_1|$$

and

$$\beta_k := \left| \left\{ x \in Q_1 \mid m(f^d)(x) \geq \left(c_3 M^k\right)^d \right\} \right|.$$

By Lemma 9 we obtain $\alpha_{k+1} \leq \rho(\alpha_k + \beta_k)$. Hence

$$\alpha_k \leq \rho^k + \sum_{i=0}^{k-1} \rho^{k-i}\beta_i. \tag{16}$$

Since $f^d \in L^{p/d}(B_{8\sqrt{d}})$, we have that $m(f^d) \in L^{p/d}(B_{8\sqrt{d}})$ and

$$\|m(f^d)\|_{L^{p/d}(B_{8\sqrt{d}})} \leq c\|f\|^d_{L^p(B_{8\sqrt{d}})} \leq C.$$

Therefore, by Lemma 1 we obtain

$$\sum_{k\geq 0} \left(M^d\right)^{\frac{pk}{d}} |x \in Q_1 \mid m(f^d)(x) \geq c_3^d M^{dk}| \leq C.$$

The former inequality implies

$$\sum_{k\geq 0} M^{pk}\beta_k \leq C. \tag{17}$$

Since $B_{1/2} \subset Q_1$, the distribution function of θ is bounded from above as follows:

$$\mu_\theta(t) \leq |A_t(u, B_{1/2})| \leq |A_t(u, B_{8\sqrt{d}}) \cap Q_1|.$$

Hence

$$
\begin{aligned}
\sum_{k\geq 1} M^{pk}\alpha_k &\leq \sum_{k\geq 1}(\rho M^p)^k + \sum_{k\geq 1}\sum_{i=0}^{k-1}\rho^{k-i}M^{pk}\beta_i \\
&= \sum_{k\geq 1}2^{-k} + \sum_{k\geq 1}\sum_{i=0}^{k-1}\rho^{k-i}M^{p(k-i)}M^{pi}\beta_i \\
&= \sum_{k\geq 1}2^{-k} + \sum_{k\geq 1}\sum_{i=0}^{k-1}2^{-(k-i)}M^{pi}\beta_i \\
&= \sum_{k\geq 1}2^{-k} + \left(\sum_{i\geq 0}M^{pi}\beta_i\right)\left(\sum_{j\geq 1}2^{-j}\right) \\
&\leq C.
\end{aligned}
$$

Applying Lemma 1 once again we conclude that $\|\theta\|_{L^p(B_{1/2})} \leq C$ and the proof is complete. ■

5. Fractional diffusion and a proof of Theorem 1

In this section, we detail the proof of Theorem 1.

Proof of Theorem 1: We start by noticing that if the graph of u is touched at a point x_0 by a $C^{1,\alpha}$-cone, we may assume x_0 is the vertex of the cone. Let ψ be a $C^{1,\alpha}$-cone of opening $\pm M$ and vertex x_0. We have:

$$
\begin{aligned}
\Delta_h^{1+\alpha}\psi(x_0) &:= \tfrac{\psi(x_0+h)+\psi(x_0-h)-2\psi(x_0)}{|h|^{1+\alpha}} \\
&= \pm M.
\end{aligned}
$$

Also, notice that touching u strictly in $B_{\frac{1}{10}}(x_0)$ from above at x_0 by a convex $C^{1,\alpha}$-cone ψ of opening M and vertex x_0 gives for all $0<h<\frac{1}{10}$

$$
\begin{aligned}
\Delta_h^{1+\alpha}u(x_0) &:= \tfrac{u(x_0+h)+u(x_0-h)-2u(x_0)}{|h|^{1+\alpha}} \\
&< \tfrac{\psi(x_0+h)+\psi(x_0-h)-2\psi(x_0)}{|h|^{1+\alpha}} \\
&\leq \theta(u,B_{1/2})(x_0).
\end{aligned}
$$

Similarly, touching u strictly in $B_{\frac{1}{10}}(x_0)$ from below at x_0 by a concave $C^{1,\alpha}$-cone ψ of opening M and vertex x_0 gives, for all $0<h<\frac{1}{10}$:

$$
-\theta(u,B_{1/2})(x_0) < \Delta_h^{1+\alpha}u(x_0).
$$

Hence, by hypothesis,

$$\|\Delta_h^{1+\alpha} u\|_{L^p(B_{1/2})} \le C,$$

uniformly for all $0 < h < \frac{1}{10}$, for $0 < \alpha < \alpha_0$. At this point, we set $\varphi := u\chi_{B_{1/2}}$ in $\mathbb{R}^d$. Then we have that $\varphi \in L^\infty(\mathbb{R}^d)$. Next, for

$$1 < \sigma < 1 + \alpha,$$

we define the singular integral operator:

$$I_{\sigma/2}(v)(x_0) := \int_{\mathbb{R}^d} \frac{v(x_0 + y) + v(x_0 - y) - 2v(y)}{|y|^{d+\sigma}} \mathrm{d}y.$$

Notice that, up to constants, $I_{\sigma/2}(v) \sim \Delta^{\sigma/2}(v)$. For $x_0 \in B_{1/2}$ we estimate

$$\begin{aligned}
I_{\sigma/2}(\varphi)(x_0) &= \int_{\mathbb{R}^d} \frac{\varphi(x_0 + y) + \varphi(x_0 - y) - 2\varphi(y)}{|y|^{d+\sigma}} \mathrm{d}y \\
&= \int_{B_{1/10}} \frac{u(x_0 + y) + u(x_0 - y) - 2u(y)}{|y|^{d+\sigma}} \mathrm{d}y \\
&\quad + \int_{B_1 \setminus B_{1/10}} \frac{u(x_0 + y) + u(x_0 - y) - 2u(y)}{|y|^{d+\sigma}} \mathrm{d}y \\
&\le \theta(u, B_{1/2})(x_0) \int_{B_{1/10}} \frac{dy}{|y|^{d-\mu}} + C\|u\|_{L^\infty(B_1)} \\
&= \frac{C}{\mu 10^d} \cdot \big(\theta(u, B_{1/2}) + \|u\|_{L^\infty(B_1)}\big)
\end{aligned}$$

where $\mu := 1 + \alpha - \sigma$, and C is a universal constant. Hence, we have proven that

$$(-\Delta)^{\sigma/2}\varphi \in L^p(B_{1/2}).$$

By setting $g := (-\Delta)^{\sigma/2}\varphi$ in $B_{1/2}$ we conclude that φ satisfies

$$\begin{cases} (-\Delta)^{\sigma/2}\varphi = g & \text{in} \quad B_{1/2} \\ \varphi = 0 & \text{in} \quad \mathbb{R}^d \setminus B_{1/2}. \end{cases}$$

Now, we aim at showing that $\varphi \in W^{\sigma/2,2}(\mathbb{R}^d)$. Extend g by zero outside $B_{1/2}$. It is clear that $g \in L^2(\mathbb{R}^d)$. Hence $\mathcal{F}(g) \in L^2(\mathbb{R}^d)$. In addition, since $\varphi \in L^2(\mathbb{R}^d)$, $\mathcal{F}(\varphi) \in L^2(\mathbb{R}^d)$ as well. By Proposition 1 (applied to functions in $L^2(\mathbb{R}^d)$)

$$\mathcal{F}(g)(x) = (|x|)^\sigma \mathcal{F}(\varphi).$$

Furthermore, we have that

$$(1+|x|^\sigma)\mathcal{F}(\varphi) = \mathcal{F}(\varphi) + \mathcal{F}(g).$$

It follows that $(1+|x|^\sigma)\mathcal{F}(\varphi) \in L^2(\mathbb{R}^d)$. In particular,

$$\int_{\mathbb{R}^d} (1+|x|^\sigma)|\mathcal{F}(\varphi)(x)|^2 dx < \infty.$$

Hence, by Proposition 2, we conclude that $\varphi \in W^{\sigma/2,2}(\mathbb{R}^d)$. Finally, by Theorem 5, we obtain that $\varphi \in W^{\sigma,p}_{\text{loc}}(B_{1/2})$. Therefore

$$u \in W^{\sigma,p}_{\text{loc}}(B_{1/2}),$$

with universal estimates, which ends the proof. ■

References

[1] Scott N. Armstrong, Luis Silvestre, Charles K. Smart. Partial regularity of solutions of fully nonlinear, uniformly elliptic equations. *Comm. Pure Appl. Math.*, 65 (2012), no. 8, 1169–1184, 2015.

[2] Umberto Biccari, Mahamadi Warma, and Enrique Zuazua. Addendum: Local elliptic regularity for the Dirichlet fractional Laplacian [MR3641649]. *Adv. Nonlinear Stud.*, 17(4):837–839, 2017.

[3] Luis Caffarelli. Interior a priori estimates for solutions of fully nonlinear equations. *Ann. of Math. (2)*, 130(1):189–213, 1989.

[4] Luis Caffarelli and Xavier Cabré. *Fully nonlinear elliptic equations*, volume 43 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1995.

[5] Luis Caffarelli, Michael G. Crandall, Maciej Kocan, and Andrzej Święch. On viscosity solutions of fully nonlinear equations with measurable ingredients. *Comm. Pure Appl. Math.*, 49(4):365–397, 1996.

[6] Eleonora Di Nezza, Giampiero Palatucci and Enrico Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.

[7] Luis Escauriaza. $W^{2,n}$ a priori estimates for solutions to fully nonlinear equations. *Indiana Univ. Math. J.*, 42(2), 413–423, 1993.

[8] Eugene Fabes and Daniel Stroock. The L^p-integrability of Green's functions and fundamental solutions for elliptic and parabolic equations. *Duke Math. J.*, 51(4), 997–1016, 1984.

[9] Nikolai V. Krylov. Boundedly inhomogeneous elliptic and parabolic equations. *Izv. Akad. Nauk SSSR Ser. Mat.*, 46(3), 487–523, 670, 1982.

[10] Nikolai V. Krylov and Mikhail Safonov. A property of the solutions of parabolic equations with measurable coefficients. *Izv. Akad. Nauk SSSR Ser. Mat.*, 44(1), 161–175, 239, 1980.

[11] Fang-Hua Lin. Second derivative L^p-estimates for elliptic equations of nondivergent type. *Proc. Amer. Math. Soc.*, 96 (1986), no. 3, 447–451.

[12] Connor Mooney. A proof of the Krylov-Safonov theorem without localization. *Comm. Partial Differential Equations*, 44 (2019), no. 8, 681–690.

[13] Nikolai S. Nadirashvili and Serge Vlăduţ. Nonclassical solutions of fully nonlinear elliptic equations. *Geom. Funct. Anal.* 17 (2007), no. 4, 1283–1296.

[14] Nikolai S. Nadirashvili and Serge Vlăduţ. Singular viscosity solutions to fully nonlinear elliptic equations. *J. Math. Pures Appl.* (9) 89 (2008), no. 2, 107–113.
[15] Nikolai S. Nadirashvili and Serge Vlăduţ. Octonions and singular solutions of Hessian elliptic equations. *Geom. Funct. Anal.* 21 (2011), no. 2, 483–498.
[16] Nikolai S. Nadirashvili and Serge Vlăduţ. Singular solutions of Hessian fully nonlinear elliptic equations. *Adv. Math.* 228 (2011), no. 3, 1718–1741.
[17] Andrzej Święch. $W^{1,p}$–interior estimates for solutions of fully nonlinear, uniformly elliptic equations. *Adv. Differential Equations* 2 (1997), no. 6, 1005–1027
[18] Eduardo V. Teixeira. Regularity for quasilinear equations on degenerate singular sets. *Math. Ann.*, 358(1-2):241–256, 2014.
[19] Eduardo V. Teixeira. Universal moduli of continuity for solutions to fully nonlinear elliptic equations. *Arch. Ration. Mech. Anal.*, 211(3):911–927, 2014.
[20] Eduardo V. Teixeira Regularity theory for nonlinear diffusion processes, *Notices Amer. Math. Soc.* 67 (2020), no. 4, 475–483.
[21] Neil S. Trudinger. Hölder gradient estimates for fully nonlinear elliptic equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 108(1-2), 57–65, 1988.

Edgard A. Pimentel
University of Coimbra, CMUC, Department of Mathematics, 3001-501 Coimbra, Portugal and Pontifical Catholic University of Rio de Janeiro – PUC-Rio 22451-900, Gávea, Rio de Janeiro-RJ, Brazil

E-mail address: `edgard.pimentel@mat.uc.pt`
URL: `http://www.mat.uc.pt/∼edgard.pimentel`

Makson S. Santos
Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo – USP, 13566-590, Centro, São Carlos - SP, Brazil

E-mail address: `makson.santos@icmc.usp.br`
URL: `https://sites.google.com/view/makson-santos`

Eduardo V. Teixeira
University of Central Florida, 4393 Andromeda Loop N, Orlando, FL 32816, USA

E-mail address: `eduardo.teixeira@ucf.edu`
URL: `https://sciences.ucf.edu/math/eteixeira/`