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SKEW-SYMMETRIC MATRICES RELATED TO THE VECTOR CROSS PRODUCT IN $\mathbb{C}^7$

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Abstract: Skew-symmetric matrices of order 7 defined through the 2-fold vector cross product in $\mathbb{C}^7$, and other related matrices, are presented. More concretely, matrix properties, namely invertibility, nullspace, powers and index, are studied. Furthermore, results on vector cross product equations, vector cross product differential equations and vector cross product difference equations are established. In particular, known results in $\mathbb{R}^3$, $\mathbb{R}^7$ and $\mathbb{C}^3$ are extended to $\mathbb{C}^7$.

Keywords: 2-fold vector cross product, hermitian inner product, skew-symmetric matrix, generalized inverses, (vector cross product, vector cross product differential, vector cross product difference) equations.
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1. Introduction

Assuming the usual definition, as explained by Elduque in the elementary account [14] on vector cross products and their connections with the exceptional basic classical simple Lie superalgebras, r-fold vector cross products exist only for d-dimensional vector spaces with: $r = 1$ and d even; $r = 2$ and $d = 3$ or 7; $r = 3$ and $d = 8$; and $r = d - 1$ for an arbitrary d. The first proof of this classical result, and an extension of it, goes back to the work [8], where Brown and Gray presented an algebraic proof. An algebraic-topologic proof of the same result for real euclidean spaces was given by Eckmann, who in [13] assumed continuity – a weaker condition – instead of multilinearity. Based on the results in [13], a variation of the latter proof was given by Whitehead in [26]. In addition, in [15], citing the articles [13] and [26], Gray established results about vector cross products on manifolds. An elementary proof of the classical result, although only valid over a field of characteristic 0, was given by Rost in [23]. Later on, Meyberg simplified this proof in [22].

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The mentioned classical result can be seen as a consequence of another classical result on the classification of Hurwitz algebras (that is, unital composition algebras, [3], [17]). The real and complex cases are due to Hurwitz, who presented the classification in [18]. Jacobson established the classification, in [20], over a field F of characteristic different from 2. More concretely, the generalized Hurwitz Theorem asserts that, over F, if $\mathcal{A}$ is a finite dimensional composition algebra with identity, then its dimension is equal to 1, 2, 4 or 8. Furthermore, as Jacobson was interested in the study of the automorphisms of Hurwitz algebras, he proved that $\mathcal{A}$ is isomorphic either to the base field, a separable quadratic extension of the base field (a quadratic commutative and associative separable algebra), a generalized quaternion algebra (four dimensional algebra that is associative but not commutative) or a generalized octonion algebra (or Cayley algebra, eight dimensional algebra that is alternative but not associative), [20].

Throughout the years, the interest in 2-fold vector cross products has remained alive. In [19], Ikramov studies the complex vector cross product in $\mathbb{C}^3$. Costa, Facas Vicente, Beites, Martins, Serôdio and Tadeu, in [11], use the vector cross product in $\mathbb{R}^7$ to study the orthogonal projection of a point onto a line. In [10], Catarino and Vitória express the distance between two skew lines in $\mathbb{R}^7$ in terms of the double vector cross product. Antić and Vrancken, in [1], recall the best possible almost complex structure on $S^6(1)$ that is induced by the vector cross product in $\mathbb{R}^7$. In [4], some vector cross product differential and difference equations are studied by Beites, Nicolás, Saraiva and Vitória. A generalization of the standard definition of 2-fold vector cross product is proposed in [21] by Lewintan. In [6], Beites, Nicolás and Vitória pursue an arithmetic for closed balls in $\mathbb{R}^n$ which includes operations involving the 2-fold vector cross product. Using this product in $\mathbb{R}^3$, Beites and Catarino establish Gelin-Cesàro's identity for Leonardo quaternions in [2].

The structure of the present work, divided into three main sections, is as follows. In section 2, where some background is presented, known definitions, results and notations related to the 2-fold vector cross product, to the 7-dimensional complex vector space $\mathbb{C}^7$, to generalized inverses and to differential and difference equations are recalled. In section 3, properties of matrices related to the 2-fold vector cross product in $\mathbb{C}^7$, namely on invertibility, nullspace, powers and index, are established. Partially following the ideas of Agudo for $\mathbb{R}^3$ in [12], where he uses the term "vector division", vector cross product equations in $\mathbb{C}^7$ are considered in section 4. Moreover, in $\mathbb{C}^7$, vector cross product differential equations and vector cross product difference equations are studied. Several results presented in the works [4] – of Beites, Nicolás, Saraiva and Vitória –, [5] – due to Beites, Nicolás and Vitória –, [16] – whose authors are Gross, Trenkler and Troschke –, [24] – of Trenkler –, and [25] – by Trenkler and Trenkler – are extended.

2. Preliminaries

Let V be a d-dimensional vector space over a field F of characteristic different from 2, endowed with a nondegenerate symmetric bilinear form $(\cdot,\cdot)$. A bilinear map $\times : V^2 \to V$ is a 2-*fold vector cross product in* V if, for any $u, v \in V$:

(1) $(u \times v, u) = (u \times v, v) = 0$,

(2) $(u \times v, u \times v) = \begin{vmatrix} (u,u) & (u,v) \\ (v,u) & (v,v) \end{vmatrix}$.

Recall that 1. implies the skew-symmetry of the trilinear map $(\cdot \times \cdot, \cdot)$, and so the anticommutativity of $\times$, [14]. In the present article, the 2-fold vector cross product in the 7-dimensional complex vector space $\mathbb{C}^7$, denoted by $\times$, is considered.

Equip the 7-dimensional complex vector space $\mathbb{C}^7$ with the *standard Hermitian inner product* $\langle \cdot,\cdot \rangle : \mathbb{C}^7 \times \mathbb{C}^7 \to \mathbb{C}$ defined by

$$\langle x, y \rangle = \sum_{t=1}^{7} x_t \overline{y_t},$$

for all $x = \begin{bmatrix} x_1 & \ldots & x_7 \end{bmatrix}^T$, $y = \begin{bmatrix} y_1 & \ldots & y_7 \end{bmatrix}^T \in \mathbb{C}^7$. It satisfies, respectively, linearity in the first coordinate, positive definiteness and hermitian (or conjugate) symmetry:

$$\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle, \tag{2.1}$$

$$\langle x, y \rangle = \overline{\langle y, x \rangle}, \tag{2.2}$$

$$\langle x, x \rangle \geq 0 \text{ and } \langle x, x \rangle = 0 \Leftrightarrow x = 0. \tag{2.3}$$

Recall that (2.2) implies that $\langle x, x \rangle \in \mathbb{R}$. Recall also that (2.1) and (2.2) imply conjugate linearity in the second coordinate, that is,

$$\langle x, \alpha y + \beta z \rangle = \overline{\alpha} \langle x, y \rangle + \overline{\beta} \langle x, z \rangle. \tag{2.4}$$

When considering the 2-fold vector cross product in the 7-dimensional complex vector space $\mathbb{C}^7$, observe that the nondegenerate symmetric bilinear form $(\cdot,\cdot)$ referred in the first definition is defined by

$$(x, y) = \langle x, \overline{y} \rangle,$$

for all $x = \begin{bmatrix} x_1 & \dots & x_7 \end{bmatrix}^T$, $y = \begin{bmatrix} y_1 & \dots & y_7 \end{bmatrix}^T \in \mathbb{C}^7$.

Throughout the work, $\mathbb{C}^{m\times n}$ denotes the set of all $m \times n$ complex matrices. When $n = 1$, $\mathbb{C}^{m\times 1}$ is identified with $\mathbb{C}^m$. When $m = n = 1$, $\mathbb{C}^{1\times 1}$ is identified with $\mathbb{C}$.

Let $B \in \mathbb{C}^{m\times n}$. A matrix $B^{(1)} \in \mathbb{C}^{n\times m}$ is a *generalized inverse* of B if $BB^{(1)}B = B$. See [7] for more details on generalized inverses, also known as (1)-*inverses* or *g-inverses*, where the subsequent result appears.

Theorem 2.1 ([7]). *Let $B \in \mathbb{C}^{m\times n}$, $b \in \mathbb{C}^m$. Then, the equation $Bx = b$ is consistent if and only if, for some $B^{(1)}$, $BB^{(1)}b = b$.*

Let $A \in \mathbb{C}^{n\times n}$.

The *index* $\operatorname{Ind}(A)$ of A is the smallest $l \in \mathbb{N}_0$ such that $R(A^l) = R(A^{l+1})$ or, equivalently, $N(A^l) = N(A^{l+1})$, where C and N stand for the column space (or range) and the nullspace, [9]. Alternatively, but equivalently, it can also be defined as the smallest $l \in \mathbb{N}_0$ such that $\mathbb{C}^n = R(A^l) \oplus N(A^l)$.

Let $\operatorname{Ind}(A) = l$. The *Drazin inverse* of A is the unique matrix $A^D \in \mathbb{C}^{n\times n}$ which satisfies

$$AA^D = A^D A,\ A^D AA^D = A^D,\ A^{l+1}A^D = A^l.$$

When $\operatorname{Ind}(A) \in \{0, 1\}$, A^D is sometimes called the *group-inverse* of A and the last equality assumes the form $AA^D A = A$. There are several methods for computing A^D, as described in [9] and references therein, some of which require all eigenvalues to be well determined.

Let $A, B \in \mathbb{C}^{n\times n}$ and $t_0 \in \mathbb{R}$. Let $f = f(t)$ be a $\mathbb{C}^n$-valued function of the real variable t. Throughout the work, $x = x(t)$ stands for an unknown $\mathbb{C}^n$-valued function of the real variable t and $\dot{x} = \dfrac{dx}{dt}$ denotes the corresponding derivative vector of x.

A vector $x_0 \in \mathbb{C}^n$ is a *consistent initial vector* for the differential equation

$$A\dot{x} + Bx = f \tag{2.5}$$

if the initial value problem

$$A\dot{x} + Bx = f,\ x(t_0) = x_0, \tag{2.6}$$

possesses at least one solution. In this case, $x(t_0) = x_0$ is said to be a *consistent initial condition.* Further, (2.5) is called *tractable* if (2.6) has a unique solution for each consistent initial vector x_0, [9].

Theorem 2.2. [9] *Let $A, B \in \mathbb{C}^{n\times n}$. The homogeneous differential equation $A\dot{x} + Bx = 0$ is tractable if and only if $(\lambda A + B)^{-1}$ exists for some $\lambda \in \mathbb{C}$.*

Let $A, B \in \mathbb{C}^{n\times n}$. Let $f^{(k)} = f^{(k)}(t) \in \mathbb{C}^n$ be the k-th term of a sequence of vectors, $k = 0, 1, 2, ...$ Throughout the work, $x^{(k)} = x^{(k)}(t) \in \mathbb{C}^n$ stands for the k-th term of an unknown sequence of vectors, $k = 0, 1, 2, \ldots$ We assume that $x^{(0)} = x_0$ is given.

A vector $x_0 \in \mathbb{C}^n$ is a *consistent initial vector* for the difference equation

$$Ax^{(k+1)} = Bx^{(k)} + f^{(k)} \tag{2.7}$$

if the initial value problem

$$Ax^{(k+1)} = Bx^{(k)} + f^{(k)},\ k = 1, 2, \ldots,\ x^{(0)} = x_0, \tag{2.8}$$

has a solution for $x^{(k)}$. In this case, $x^{(0)} = x_0$ is said to be a *consistent initial condition.* Furthermore, (2.7) is called *tractable* if (2.8) has a unique solution for each consistent initial vector x_0, [9].

Theorem 2.3. [9] *Let $A, B \in \mathbb{C}^{n\times n}$. The homogeneous difference equation $Ax^{(k+1)} = Bx^{(k)}$ is tractable if and only if $(\lambda A + B)^{-1}$ exists for some $\lambda \in \mathbb{C}$.*

3. Properties

Let $a = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \end{bmatrix}^T \in \mathbb{C}^7$. Consider the linear mapping

$$\begin{array}{rccl} a_\times : & \mathbb{C}^7 & \to & \mathbb{C}^7 \\ & x & \mapsto & a_\times(x) = a \times x. \end{array}$$

For each $a \in \mathbb{C}^7$, there exists a unique matrix $S_a \in \mathbb{C}^{7\times 7}$ such that

$$a \times x = S_a x, \tag{3.1}$$

where

$$S_a = \begin{bmatrix} 0 & -a_3 & a_2 & -a_5 & a_4 & -a_7 & a_6 \\ a_3 & 0 & -a_1 & -a_6 & a_7 & a_4 & -a_5 \\ -a_2 & a_1 & 0 & a_7 & a_6 & -a_5 & -a_4 \\ a_5 & a_6 & -a_7 & 0 & -a_1 & -a_2 & a_3 \\ -a_4 & -a_7 & -a_6 & a_1 & 0 & a_3 & a_2 \\ a_7 & -a_4 & a_5 & a_2 & -a_3 & 0 & -a_1 \\ -a_6 & a_5 & a_4 & -a_3 & -a_2 & a_1 & 0 \end{bmatrix}. \tag{3.2}$$

In the following result, some properties related to the matrices defined in (3.1)-(3.2) are established.

Proposition 3.1. *Let $a, b, c \in \mathbb{C}^7$. Let $\alpha, \beta \in \mathbb{C}$. Then:*

(1) $S_{\alpha a+\beta b}c = \alpha S_a c + \beta S_b c$;
(2) $S_{\overline{a}} = \overline{S_a}$;
(3) $S_a = -S_a^T$;
(4) $S_a^* = -S_{\overline{a}}$,
where $\cdot^*$ stands for the conjugate transpose of a matrix;
(5) $S_a b = -S_b a$;
(6) $S_a a = 0$;
(7) $$S_a\overline{a} = 2\mathbf{i}\begin{bmatrix} \mathrm{Im}(a_2\overline{a_3}) + \mathrm{Im}(a_4\overline{a_5}) + \mathrm{Im}(a_6\overline{a_7}) \\ -\mathrm{Im}(a_1\overline{a_3}) + \mathrm{Im}(a_4\overline{a_6}) - \mathrm{Im}(a_5\overline{a_7}) \\ \mathrm{Im}(a_1\overline{a_2}) - \mathrm{Im}(a_4\overline{a_7}) - \mathrm{Im}(a_5\overline{a_6}) \\ -\mathrm{Im}(a_1\overline{a_5}) - \mathrm{Im}(a_2\overline{a_6}) + \mathrm{Im}(a_3\overline{a_7}) \\ \mathrm{Im}(a_1\overline{a_4}) + \mathrm{Im}(a_2\overline{a_7}) + \mathrm{Im}(a_3\overline{a_6}) \\ -\mathrm{Im}(a_1\overline{a_7}) + \mathrm{Im}(a_2\overline{a_4}) - \mathrm{Im}(a_3\overline{a_5}) \\ \mathrm{Im}(a_1\overline{a_6}) - \mathrm{Im}(a_2\overline{a_5}) - \mathrm{Im}(a_3\overline{a_4}) \end{bmatrix};$$
(8) $S_{\overline{a}}\overline{b} = \overline{S_a b}$;
(9) S_a *is singular;*
(10) $S_a^2 = aa^T - \langle a, \overline{a}\rangle I_7$;
(11) $S_a^3 = -\langle a, \overline{a}\rangle S_a$;
(12) *the eigenvalues of S_a are 0, $\sqrt{|\langle a, \overline{a}\rangle|}\mathrm{e}^{\boldsymbol{i}\frac{\theta}{2}}$ and $\sqrt{|\langle a, \overline{a}\rangle|}\mathrm{e}^{\boldsymbol{i}(\frac{\theta}{2}+\pi)}$, with θ an argument of $-\langle a, \overline{a}\rangle$;*
(13) *the nullspace of S_a, where $a \neq 0$, is $N(S_a) = \{\alpha a : \alpha \in \mathbb{C}\}$.*

Proof: Properties 1. and 5. are direct consequences of, respectively, the bilinearity and the anticommutativity of $\times$ in $\mathbb{C}^7$.

From (3.2) it is straightforward to prove 2. and 3.

Concerning 4., invoking 2. and 3. leads to $S_a^* = (\overline{S_a})^T = (S_{\overline{a}})^T = -S_{\overline{a}}$.

Taking $b = a$ in 5. leads to 6.

By property 5., $S_a\overline{a} + S_{\overline{a}}a = 0$ which, by 2., is equivalent to $S_a\overline{a} + \overline{S_a}a = 0 \Leftrightarrow S_a\overline{a} + \overline{S_a\overline{a}} = 0$. The last equality means that each entry of $S_a\overline{a}$ is either zero or a purely imaginary complex number. Concretely, from (3.2), $S_a\overline{a}$ is the matrix

$$\begin{bmatrix} a_2\overline{a_3} - a_3\overline{a_2} + a_4\overline{a_5} - a_5\overline{a_4} + a_6\overline{a_7} - a_7\overline{a_6} \\ -a_1\overline{a_3} + a_3\overline{a_1} + a_4\overline{a_6} - a_6\overline{a_4} - a_5\overline{a_7} + a_7\overline{a_5} \\ a_1\overline{a_2} - a_2\overline{a_1} - a_4\overline{a_7} + a_7\overline{a_4} - a_5\overline{a_6} + a_6\overline{a_5} \\ -a_1\overline{a_5} + a_5\overline{a_1} - a_2\overline{a_6} + a_6\overline{a_2} + a_3\overline{a_7} - a_7\overline{a_3} \\ a_1\overline{a_4} - a_4\overline{a_1} + a_2\overline{a_7} - a_7\overline{a_2} + a_3\overline{a_6} - a_6\overline{a_3} \\ -a_1\overline{a_7} + a_7\overline{a_1} + a_2\overline{a_4} - a_4\overline{a_2} - a_3\overline{a_5} + a_5\overline{a_3} \\ a_1\overline{a_6} - a_6\overline{a_1} - a_2\overline{a_5} + a_5\overline{a_2} - a_3\overline{a_4} + a_4\overline{a_3} \end{bmatrix}$$

$$= \begin{bmatrix} 2\mathbf{i}\mathrm{Im}(a_2\overline{a_3}) + 2\mathbf{i}\mathrm{Im}(a_4\overline{a_5}) + 2\mathbf{i}\mathrm{Im}(a_6\overline{a_7}) \\ -2\mathbf{i}\mathrm{Im}(a_1\overline{a_3}) + 2\mathbf{i}\mathrm{Im}(a_4\overline{a_6}) - 2\mathbf{i}\mathrm{Im}(a_5\overline{a_7}) \\ 2\mathbf{i}\mathrm{Im}(a_1\overline{a_2}) - 2\mathbf{i}\mathrm{Im}(a_4\overline{a_7}) - 2\mathbf{i}\mathrm{Im}(a_5\overline{a_6}) \\ -2\mathbf{i}\mathrm{Im}(a_1\overline{a_5}) - 2\mathbf{i}\mathrm{Im}(a_2\overline{a_6}) + 2\mathbf{i}\mathrm{Im}(a_3\overline{a_7}) \\ 2\mathbf{i}\mathrm{Im}(a_1\overline{a_4}) + 2\mathbf{i}\mathrm{Im}(a_2\overline{a_7}) + 2\mathbf{i}\mathrm{Im}(a_3\overline{a_6}) \\ -2\mathbf{i}\mathrm{Im}(a_1\overline{a_7}) + 2\mathbf{i}\mathrm{Im}(a_2\overline{a_4}) - 2\mathbf{i}\mathrm{Im}(a_3\overline{a_5}) \\ 2\mathbf{i}\mathrm{Im}(a_1\overline{a_6}) - 2\mathbf{i}\mathrm{Im}(a_2\overline{a_5}) - 2\mathbf{i}\mathrm{Im}(a_3\overline{a_4}) \end{bmatrix},$$

from where 7. follows.

Applying 2. allows to arrive at 8. since $S_{\overline{a}}\overline{b} = \overline{S_a}\ \overline{b}$.

As far as 9., on the one hand, if $a = 0$ then $S_a = 0$, a singular matrix. On the other hand, if $a \neq 0$ then, from 6., $S_a a = 0$. If S_a were invertible then $a = 0$, a contradiction.

As $S_a^2 = [s_{ij}]_{7\times 7}$ with

$$s_{ij} = \begin{cases} -\sum\limits_{\substack{t=1, \\ t\neq i}}^{7} a_t^2 & \text{if } \ i = j \\ a_i a_j & \text{if } \ i \neq j \end{cases},$$

$aa^T = [d_{ij}]_{7\times 7}$ with

$$d_{ij} = \begin{cases} a_i^2 & \text{if } \ i = j \\ a_i a_j & \text{if } \ i \neq j \end{cases},$$

and

$$\langle a, \overline{a}\rangle = \sum_{t=1}^{7} a_t^2,$$

then 10. follows.

Taking into account 10., $S_a^3 = aa^T S_a - \langle a, \overline{a}\rangle S_a$. By 3. and 6., $aa^T S_a = a(S_a^T a)^T = -a(S_a a)^T = 0$. Hence, 11. follows.

Regarding 12., the characteristic equation of S_a is

$$\begin{aligned}\det(S_a - \lambda I_7) = 0 &\Leftrightarrow -\lambda(\lambda^2 + \langle a, \overline{a}\rangle)^3 = 0\\ &\Leftrightarrow \lambda = 0 \vee \lambda^2 = -\langle a, \overline{a}\rangle\\ &\Leftrightarrow \lambda = 0 \vee \lambda = \sqrt{|\langle a, \overline{a}\rangle|}\mathrm{e}^{\mathbf{i}\frac{\theta}{2}} \vee \lambda = \sqrt{|\langle a, \overline{a}\rangle|}\mathrm{e}^{\mathbf{i}(\frac{\theta}{2}+\pi)},\end{aligned}$$

with θ an argument of $-\langle a, \overline{a}\rangle$.

Let $a \in \mathbb{C}^7\backslash\{0\}$. The inclusion $\supseteq$ in 13. follows from property 6. since, for all $\gamma \in \mathbb{C}$, $S_a(\gamma a) = \gamma S_a a = 0$. By the proof of 12., the eigenvalue 0 has algebraic multiplicity 1. As $0 \neq a \in N(S_a)$, the geometric multiplicity of 0 is 1. Hence, $\dim N(S_a) = \dim \{\alpha a : \alpha \in \mathbb{C}\} = 1$, and 13. is obtained. ■

The subsequent results concern powers and traces of the matrices defined in (3.1)-(3.2).

Lemma 3.2. *Let $a \in \mathbb{C}^7$ such that $\langle a, \overline{a}\rangle \neq 0$. For $m \in \mathbb{N}$,*

$$S_a^{2m} = (-1)^{m+1}\langle a, \overline{a}\rangle^{m-1}aa^T + (-1)^m\langle a, \overline{a}\rangle^m I_7 \tag{3.3}$$

and

$$S_a^{2m+1} = (-1)^m\langle a, \overline{a}\rangle^m S_a. \tag{3.4}$$

Proof: The proof goes by induction on m.

For (3.3), by 10. in Proposition 3.1, the base case holds. Also from 10. in Proposition 3.1 and the induction hypothesis, we have

$$\begin{aligned}S_a^{2(m+1)} &= S_a^{2m}S_a^2\\ &= [(-1)^{m+1}\langle a, \overline{a}\rangle^{m-1}aa^T + (-1)^m\langle a, \overline{a}\rangle^m I_7](aa^T - \langle a, \overline{a}\rangle I_7)\\ &= (-1)^{m+1}\langle a, \overline{a}\rangle^m aa^T - (-1)^{m+1}\langle a, \overline{a}\rangle^m aa^T\\ &\quad +(-1)^m\langle a, \overline{a}\rangle^m aa^T - (-1)^m\langle a, \overline{a}\rangle^{m+1} I_7\\ &= (-1)^{m+2}\langle a, \overline{a}\rangle^m aa^T + (-1)^{m+1}\langle a, \overline{a}\rangle^{m+1} I_7,\end{aligned}$$

and the induction step holds too.

For (3.4), by 11. in Proposition 3.1, it is straighforward to see that the base case holds. As for the induction step, by 10. in Proposition 3.1 and the induction hypothesis, we obtain

$$\begin{aligned}S_a^{2m+3} &= S_a^{2m+1}S_a^2\\ &= (-1)^m\langle a, \overline{a}\rangle^m S_a(aa^T - \langle a, \overline{a}\rangle I_7)\\ &= (-1)^m\langle a, \overline{a}\rangle^m (S_a a)a^T + (-1)^{m+1}\langle a, \overline{a}\rangle^{m+1} S_a.\end{aligned}$$

From here, taking into account 6. in Proposition 3.1, the second part of the result follows. ■

Theorem 3.3. *Let $a \in \mathbb{C}^7$ such that $\langle a, \overline{a}\rangle \neq 0$. For $m \in \mathbb{N}$, $\operatorname{tr}(S_a^{2m+1}) = 0$ and*

$$\operatorname{tr}(S_a^{2m}) = 6(-1)^m \langle a, \overline{a}\rangle^m. \tag{3.5}$$

Proof: From (3.4) in Lemma 3.2, it is clear that

$$\operatorname{tr}(S_a^{2m+1}) \;=\; (-1)^m \langle a, \overline{a}\rangle^m \operatorname{tr}(S_a) = 0.$$

From (3.3) in Lemma 3.2, taking into account aa^T written for the proof of 10. in Proposition 3.1,

$$\begin{aligned}\operatorname{tr}(S_a^{2m}) &= (-1)^{m+1}\langle a, \overline{a}\rangle^{m-1}\operatorname{tr}(aa^T) + (-1)^m \langle a, \overline{a}\rangle^m \operatorname{tr}(I_7)\\ &= -(-1)^m \langle a, \overline{a}\rangle^m + 7(-1)^m \langle a, \overline{a}\rangle^m,\end{aligned}$$

and the expression for the trace of S_a^{2m} in (3.5) is obtained. ■

The following results are devoted to generalized inverses, invertibility and inverses of matrices related to the matrices defined in (3.1)-(3.2).

Theorem 3.4. *Let $a \in \mathbb{C}^7$ such that $\langle a, \overline{a}\rangle \neq 0$. A generalized inverse of S_a is*

$$S_a^{(1)} = -\langle a, \overline{a}\rangle^{-1} S_a. \tag{3.6}$$

Proof: With $\langle a, \overline{a}\rangle \neq 0$, 11. in Proposition 3.1 leads to (3.6) since

$$S_a\left(-\langle a, \overline{a}\rangle^{-1} S_a\right) S_a = -\langle a, \overline{a}\rangle^{-1} S_a^3 = S_a.$$ ■

Proposition 3.5. *Let $a, b \in \mathbb{C}^7$ and $\gamma \in \mathbb{C}$. The matrix $\gamma S_a + S_b$ is singular.*

Proof: As S_a and S_b are skew-symmetric matrices, then, for any $\gamma \in \mathbb{C}$, $\gamma S_a + S_b$ is also skew-symmetric of odd order. Hence, $\det(\gamma S_a + S_b) = 0$. ■

Lemma 3.6. *Let $a \in \mathbb{C}^7$ and $\alpha \in \mathbb{C}$. The matrix $S_a + \alpha I_7$ is non-singular if and only if $\alpha \neq 0$ and α is not a square root of $-\langle a, \overline{a}\rangle$.*

Proof: A straightforward calculation of $\det(S_a + \alpha I_7)$ leads to $\alpha(\alpha^2 + \langle a, \overline{a}\rangle)^3$. In the stated conditions, $\det(S_a + \alpha I_7) = 0$ if and only if $\alpha = 0$ or $\alpha^2 = -\langle a, \overline{a}\rangle$. ■

Theorem 3.7. *Let $a \in \mathbb{C}^7$. Let $\alpha \in \mathbb{C}\backslash\{0\}$ such that α is not a square root of $-\langle a, \overline{a}\rangle$. Then*

$$(S_a + \alpha I_7)^{-1} = -(\alpha^2 + \langle a, \overline{a}\rangle)^{-1}(S_a - \alpha I_7 - \alpha^{-1} aa^T). \tag{3.7}$$

Proof: From Lemma 3.6, $S_a + \alpha I_7$ is invertible. Invoking properties 6. and 10. of Proposition 3.1, we get

$$\begin{aligned}
&(S_a + \alpha I_7)(-(\alpha^2 + \langle a, \overline{a}\rangle)^{-1}(S_a - \alpha I_7 - \alpha^{-1} a a^T))\\
&= -(\alpha^2 + \langle a, \overline{a}\rangle)^{-1}(S_a^2 - \alpha S_a - \alpha^{-1} S_a a a^T + \alpha S_a - \alpha^2 I_7 - a a^T)\\
&= -(\alpha^2 + \langle a, \overline{a}\rangle)^{-1}(-\langle a, \overline{a}\rangle I_7 - \alpha^2 I_7)\\
&= I_7.
\end{aligned}$$

■

In the last results of the present section, the indexes of matrices related to the matrices defined in (3.1)-(3.2) are determined.

Theorem 3.8. *Let $a \in \mathbb{C}^7$ such that $\langle a, \overline{a}\rangle \neq 0$. Then* $\operatorname{Ind}(S_a) = 1$.

Proof: The matrix S_a has index 1 if $\mathbb{C}^7 = R(S_a) \oplus N(S_a)$.

First of all, from 10. in Proposition 3.1, every $x \in \mathbb{C}^7$ can be written as $x = \langle a, \overline{a}\rangle^{-1}(a a^T x - S_a^2 x)$. Clearly, $S_a^2 x \in R(S_a)$. By 6. in Proposition 3.1, $a a^T x \in N(S_a)$ since $S_a(a a^T x) = (S_a a)(a^T x) = 0$.

Secondly, let $x \in R(S_a) \cap N(S_a)$. As $x \in R(S_a)$, there exists $y \in \mathbb{C}^7$ such that $x = S_a y$. In addition, $x \in N(S_a)$ which, together with 11. in Proposition 3.1, allows to write $0 = S_a^2 x = S_a^3 y = -\langle a, \overline{a}\rangle S_a y$. Consequently, $y \in N(S_a)$, which implies $x = 0$. ■

Theorem 3.9. *Let $u, v \in \mathbb{C}^7$ such that $\langle u, \overline{u}\rangle \neq 0$. Let $\alpha \in \mathbb{C}\backslash\{0\}$ such that α is not a square root of $-\langle v, \overline{v}\rangle$. Then* $\operatorname{Ind}((S_v + \alpha I_7)^{-1} S_u) = 1$.

Proof: By Lemma 3.6, $S_v+\alpha I_7$ is non-singular. Notice that $N((S_v+\alpha I_7)^{-1}S_u) \subseteq N(((S_v + \alpha I_7)^{-1} S_u)^2)$. Suppose that

$$N((S_v + \alpha I_7)^{-1} S_u) \subsetneq N(((S_v + \alpha I_7)^{-1} S_u)^2).$$

Hence, there exists $x \in \mathbb{C}^7\backslash\{0\}$ such that $((S_v + \alpha I_7)^{-1} S_u)^2 x = 0$ and $(S_v + \alpha I_7)^{-1} S_u x \neq 0$. It is clear that $N((S_v + \alpha I_7)^{-1} S_u) = N(S_u)$. By 13. of Proposition 3.1, as $(S_v + \alpha I_7)^{-1} S_u x \in N((S_v + \alpha I_7)^{-1} S_u)$,

$$(S_v + \alpha I_7)^{-1} S_u x = \delta u$$

for some $\delta \in \mathbb{C}\backslash\{0\}$, that is, $S_u x = \delta(S_v u + \alpha u)$. This implies that $\delta\alpha u = u \times x - \delta v \times u$ and so, $\langle \delta\alpha u, \overline{u}\rangle = \langle u \times x - \delta v \times u, \overline{u}\rangle = 0$, that is, $\delta\alpha\langle u, \overline{u}\rangle = 0$, which is a contradiction. Thus, $N((S_v + \alpha I_7)^{-1} S_u) = N(((S_v + \alpha I_7)^{-1} S_u)^2)$. Finally, $N(((S_v+\alpha I_7)^{-1}S_u)^0) \neq N((S_v+\alpha I_7)^{-1}S_u)$. The result is proved. ■

4. Equations

4.1. Vector Cross Product Equations. In the following results, some vector cross product equations in $\mathbb{C}^7$ are presented.

Lemma 4.1. *Let $a, b \in \mathbb{C}^7$ such that $\langle a, \overline{a}\rangle \neq 0$. Then, the equation $a \times x = b$ is consistent in $\mathbb{C}^7$ if and only if $\langle a, \overline{b}\rangle = 0$.*

Proof: Observe that, from (3.1), the matrix form of the equation $a \times x = b$ is $S_a x = b$. By Theorem 3.4, $S_a^{(1)} = -\langle a, \overline{a}\rangle^{-1} S_a$ is a generalized inverse of S_a. Invoking Theorem 2.1, $S_a x = b$ is consistent if and only if $S_a S_a^{(1)} b = b$. Finally, applying 10. in Proposition 3.1, notice that

$$S_a S_a^{(1)} b = b \Leftrightarrow -\langle a, \overline{a}\rangle^{-1} a(a^t b) + b = b \Leftrightarrow -\langle a, \overline{b}\rangle \langle a, \overline{a}\rangle^{-1} a = 0 \Leftrightarrow \langle a, \overline{b}\rangle = 0.$$

■

Theorem 4.2. *Let $a, b \in \mathbb{C}^7$ such that $\langle a, \overline{a}\rangle \neq 0$ and $\langle a, \overline{b}\rangle = 0$. Then, the solutions in $\mathbb{C}^7$ of the equation $a \times x = b$ are*

$$-\langle a, \overline{a}\rangle^{-1} S_a b + \lambda a, \tag{4.1}$$

where $\lambda \in \mathbb{C}$.

Proof: From Lemma 4.1, the equation $a \times x = b$ is consistent. Furthermore, from (3.1), the matrix form of the equation $a \times x = b$ is $S_a x = b$. Let a_0 be a particular solution of this equation. Then, for each $\lambda \in \mathbb{C}$, $a_0 + \lambda a$ is a solution of the same equation since, by 6. in Proposition 3.1,

$$S_a(a_0 + \lambda a) = S_a a_0 + \lambda S_a a = b.$$

Observe that there are no other solutions for the considered equation. In fact, if a_1 and a_2 are two solutions then $S_a(a_1 - a_2) = 0$. Hence, $a_1 - a_2 \in N(S_a)$ and, by 13. of Proposition 3.1, $a_1 = a_2 + \beta a$ for some $\beta \in \mathbb{C}$.

From the calculations in the proof of Lemma 4.1 with $\langle a, \overline{b}\rangle = 0$, as

$$S_a \left(-\langle a, \overline{a}\rangle^{-1} S_a\right) b = S_a S_a^{(1)} b = b$$

then $-\langle a, \overline{a}\rangle^{-1} S_a b$ is a solution of the equation in question. Therefore, the solutions in $\mathbb{C}^7$ of this equation are the ones stated in (4.1). ■

Corollary 4.3. *Let $a, b \in \mathbb{C}^7$ such that $\langle a, \overline{a}\rangle \neq 0$ and $\langle a, \overline{b}\rangle = 0$. Then, the solutions in $\mathbb{C}^7$ of the equation $x \times a = b$ are*

$$\langle a, \overline{a}\rangle^{-1} S_a b + \lambda a, \tag{4.2}$$

where $\lambda \in \mathbb{C}$.

Proof: In order to obtain (4.2), after observing that $S_x a = b \Leftrightarrow S_a x = -b$ by 5. in Proposition 3.1, it suffices to apply Theorem 4.2. ■

Corollary 4.4. *Let $a, b \in \mathbb{C}^7$ such that $\langle a-b, \overline{a-b}\rangle \neq 0$. The solutions in $\mathbb{C}^7$ of the equation $a \times x = b \times x$ are $\lambda(a-b)$ with $\lambda \in \mathbb{C}$.*

Proof: As, by 1. in Proposition 3.1, $S_a x = S_b x \Leftrightarrow S_{(a-b)} x = 0$, then the corollary follows invoking Theorem 4.2. ■

Corollary 4.5. *Let $a, b \in \mathbb{C}^7$ such that $\langle a+b, \overline{a+b}\rangle \neq 0$. The solutions in $\mathbb{C}^7$ of the equation $a \times x = x \times b$ are $\lambda(a+b)$ with $\lambda \in \mathbb{C}$.*

Proof: As, by 1. and 5. in Proposition 3.1, $S_a x = S_x b \Leftrightarrow S_{(a+b)} x = 0$, then the result follows from Theorem 4.2. ■

Corollary 4.6. *Let $a, b, c \in \mathbb{C}^7$ such that $\langle a-b, \overline{a-b}\rangle \neq 0$ and $\langle a-b, \overline{c}\rangle = 0$. Then, the solutions in $\mathbb{C}^7$ of the equation $a \times x = b \times x + c$ are*

$$-\langle a-b, \overline{a-b}\rangle^{-1} S_{a-b} c + \lambda(a-b), \tag{4.3}$$

where $\lambda \in \mathbb{C}$.

Proof: From 1. in Proposition 3.1, $S_a x = S_b x + c \Leftrightarrow S_{a-b} x = c$. Thus, by Theorem 4.2, (4.3) is obtained. ■

Corollary 4.7. *Let $a, b, c \in \mathbb{C}^7$ such that $\langle a+b, \overline{a+b}\rangle \neq 0$ and $\langle a+b, \overline{c}\rangle = 0$. Then, the solutions in $\mathbb{C}^7$ of the equation $a \times x = x \times b + c$ are*

$$-\langle a+b, \overline{a+b}\rangle^{-1} S_{a+b} c + \lambda(a+b), \tag{4.4}$$

where $\lambda \in \mathbb{C}$.

Proof: By 1. and 5. in Proposition 3.1, $S_a x = S_x b + c \Leftrightarrow S_{a+b} x = c$. Thus, (4.4) is got from Theorem 4.2. ■

In the next corollary, denote the composition of $2m+1$ functions equal to $a_\times$ by $a_\times^{\circ(2m+1)}$.

Corollary 4.8. *Let $a, b \in \mathbb{C}^7$ such that $\langle a, \overline{a}\rangle \neq 0$ and $\langle a, \overline{b}\rangle = 0$. Then, the solutions in $\mathbb{C}^7$ of the equation $a_\times^{\circ(2m+1)} x = b$ are*

$$(-1)^{m+1} \langle a, \overline{a}\rangle^{-m-1} S_a b + \lambda a, \tag{4.5}$$

where $\lambda \in \mathbb{C}$.

Proof: By (3.4) in Lemma 3.2, $S_a^{2m+1} x = b \Leftrightarrow S_a x = (-1)^m \langle a, \overline{a}\rangle^{-m} b$. Hence, Theorem 4.2 allows to arrive at (4.5). ■

The last result of the present section gives a characterization of the column space of S_a.

Corollary 4.9. *Let $a \in \mathbb{C}^7$ such that $\langle a, \overline{a} \rangle \neq 0$. The column space of S_a is $R(S_a) = \{y \in \mathbb{C}^7 : \langle y, \overline{a} \rangle = 0\}$.*

Proof: ($\supseteq$) Let $y \in \mathbb{C}^7$ such that $\langle y, \overline{a} \rangle = 0$. As $\langle a, \overline{y} \rangle = \langle y, \overline{a} \rangle$, then, from 5. in Proposition 3.1 and the proof of Theorem 4.2, $S_a \left(\langle a, \overline{a} \rangle^{-1} S_y a \right) = S_a \left(-\langle a, \overline{a} \rangle^{-1} S_a y \right) = y$. Thus, $y \in R(S_a)$.

($\subseteq$) Let $b \in R(S_a)$ such that $\langle b, \overline{a} \rangle \neq 0$. There is $d \in \mathbb{C}^7$ such that $S_a d = b$, which means that d is a solution of $S_a x = b$. By Lemma 4.1, this equation is consistent in $\mathbb{C}^7$ if and only if $\langle a, \overline{b} \rangle = 0$, that is, $\langle b, \overline{a} \rangle = 0$ – a contradiction. ■

4.2. Vector Cross Product Differential Equations. In the present section, some vector cross product differential equations in $\mathbb{C}^7$ are considered. First, we introduce a technical result.

Lemma 4.10. *Let $b \in \mathbb{C}^7$ such that $\langle b, \overline{b} \rangle \neq 0$. Then*

$$\mathrm{e}^{-tS_b} = \cos(\beta t) I_7 - \frac{\sin(\beta t)}{\beta} S_b + \frac{1 - \cos(\beta t)}{\beta^2} b b^T, \tag{4.6}$$

where $\beta = |\langle b, \overline{b} \rangle|^{1/2} \mathrm{e}^{i\frac{\theta}{2}}$, with θ an argument of $\langle b, \overline{b} \rangle$.

Proof: Using Lemma 3.2, the expression of e^{-tS_b} as a infinite power series can be written in the following way

$$\begin{aligned} \mathrm{e}^{-tS_b} \;=\; & (1 - \tfrac{\langle b, \overline{b} \rangle}{2} t^2 + \tfrac{\langle b, \overline{b} \rangle^2}{4!} t^4 + \cdots) I_7 \\ & -(t - \tfrac{\langle b, \overline{b} \rangle}{3!} t^3 + \tfrac{\langle b, \overline{b} \rangle^2}{5!} t^5 - \cdots) S_b \\ & +(\tfrac{1}{2} t^2 - \tfrac{\langle b, \overline{b} \rangle}{4!} t^4 + \tfrac{\langle b, \overline{b} \rangle^2}{6!} t^6 - \cdots) b b^T. \end{aligned}$$

From the infinite power series expansions of the complex functions $\cos(\beta t)$ and $\sin(\beta t)$, (4.6) is obtained. ■

Theorem 4.11. *Let $b \in \mathbb{C}^7$ such that $\langle b, \overline{b} \rangle \neq 0$ and let $x = x(t)$ be an unknown $\mathbb{C}^7$-valued function of the real variable t. The unique solution of the vector cross product differential equation*

$$\dot{x} + b \times x = 0, \tag{4.7}$$

with initial condition $x(t_0) = x_0$, *is*

$$x(t) = \cos(\beta(t-t_0))x_0 - \frac{\sin(\beta(t-t_0))}{\beta}S_b x_0 + \frac{1-\cos(\beta(t-t_0))}{\beta^2}bb^T x_0, \quad (4.8)$$

where $\beta = |\langle b, \overline{b}\rangle|^{1/2}\mathrm{e}^{i\frac{\theta}{2}}$, *with* θ *an argument of* $\langle b, \overline{b}\rangle$.

Proof: From (3.1), equation (4.7) assumes the form $\dot{x} + S_b x = 0$, which is a tractable equation by Theorem 2.2. In fact, from Lemma 3.6, $(\lambda I_7 + S_b)^{-1}$ exists for every $\lambda \in \mathbb{C}\backslash\{0\}$ such that $\lambda^2 \neq -\langle b, \overline{b}\rangle$. As the coefficient of the term in $\dot{x}$ is a non-singular matrix, the classical theory recalled in [9, p.171] applies to the homogeneous initial value problem $\dot{x} + S_b x = 0, x(t_0) = x_0$. Its unique solution is given by

$$x(t) = \mathrm{e}^{-(t-t_0)S_b}x_0.$$

Invoking Lemma 4.10, we obtain (4.8). ∎

Theorem 4.12. *Let* $b \in \mathbb{C}^7$ *such that* $\langle b, \overline{b}\rangle \neq 0$, *let* $f = f(t)$ *be a* $\mathbb{C}^7$*-valued function of the real variable* t, *continuous in some interval containing* t_0, *and let* $x = x(t)$ *be an unknown* $\mathbb{C}^7$*-valued function of the real variable* t. *The unique solution of the vector cross product differential equation*

$$\dot{x} + b \times x = f, \quad (4.9)$$

with initial condition $x(t_0) = x_0$, *is*

$$x(t) = \cos(\beta(t-t_0))x_0 - \frac{\sin(\beta(t-t_0))}{\beta}S_b x_0 + \frac{1-\cos(\beta(t-t_0))}{\beta^2}bb^T x_0 + \int_{t_0}^{t}\left(\cos(\beta(t-s)) - \frac{\sin(\beta(t-s))}{\beta}S_b + \frac{1-\cos(\beta(t-s))}{\beta^2}bb^T\right) f(s)ds, \quad (4.10)$$

where $\beta = |\langle b, \overline{b}\rangle|^{1/2}\mathrm{e}^{i\frac{\theta}{2}}$, *with* θ *an argument of* $\langle b, \overline{b}\rangle$.

Proof: Again by (3.1), we can rewrite equation (4.9) as $\dot{x} + S_b x = f$, where the coefficient of the term in $\dot{x}$ is a non-singular matrix. Thus, the classical theory applies to the inhomogeneous initial value problem $\dot{x} + S_b x = f, x(t_0) = x_0$. Its unique solution is given by

$$x(t) = \mathrm{e}^{-(t-t_0)S_b}x_0 + \int_{t_0}^{t}\mathrm{e}^{-(t-s)S_b}f(s)ds.$$

From Lemma 4.10, we obtain (4.10). ∎

Theorem 4.13. *Let $a, b \in \mathbb{C}^7 \backslash \{0\}$ and let $x = x(t)$ be an unknown $\mathbb{C}^7$-valued function of the real variable t. The vector cross product differential equation*

$$a \times \dot{x} + b \times x = 0 \tag{4.11}$$

is not tractable.

Proof: From (3.1), the rewriting of equation (4.11) leads to $S_a\dot{x} + S_b x = 0$. By Proposition 3.5, for any $\lambda \in \mathbb{C}$, $\lambda S_a + S_b$ is a singular matrix and the result follows from Theorem 2.2. ∎

Taking into account the previous result, the remaining part of the section is devoted to the study of differential equations which can be considered as perturbations of (4.11).

Theorem 4.14. *Let $a \in \mathbb{C}^7$ with $\langle a, \overline{a}\rangle \neq 0$, $b \in \mathbb{C}^7 \backslash \{0\}$ and $\alpha \in \mathbb{C}\backslash\{0\}$ such that α is not a square root of $-\langle b, \overline{b}\rangle$. Let $x = x(t)$ an unknown $\mathbb{C}^7$-valued function of the real variable t. A vector $x_0 \in \mathbb{C}^7$ is a consistent initial vector for the vector cross product differential equation*

$$a \times \dot{x} + b \times x + \alpha x = 0 \tag{4.12}$$

if and only if x_0 is of the form

$$x_0 = \hat{S}_a \hat{S}_a^D q, \tag{4.13}$$

for some $q \in \mathbb{C}^7$, where

$$\hat{S}_a = -(\alpha^2 + \langle b, \overline{b}\rangle)^{-1} \left(S_b - \alpha I_7 - \alpha^{-1} b b^T\right) S_a. \tag{4.14}$$

Moreover, if $x_0 \in \mathbb{C}^7$ is a consistent initial vector for (4.12), then the unique solution of (4.12), with initial condition $x(t_0) = x_0$, is

$$x(t) = \mathrm{e}^{-\hat{S}_a^D (t-t_0)} \hat{S}_a \hat{S}_a^D x_0. \tag{4.15}$$

Proof: According to (3.1), equation (4.12) assumes the form $S_a\dot{x} + (S_b + \alpha I_7)x = 0$ where $\alpha \in \mathbb{C}\backslash\{0\}$ is such that $\alpha^2 \neq -\langle b, \overline{b}\rangle$. Let us denote $S_b + \alpha I_7$ by B, matrix which, due to Lemma 3.6, is non-singular. Thus, $(\lambda S_a + B)^{-1}$ exists for $\lambda = 0$ and, by Theorem 2.2, $S_a\dot{x} + Bx = 0$ is a tractable equation.

Following the notation in [9], let

$$\hat{S}_{a,\lambda} = (\lambda S_a + B)^{-1} S_a \text{ and } \hat{B}_\lambda = (\lambda S_a + B)^{-1} B,$$

where $\lambda \in \mathbb{C}$ is such that $\lambda S_a + B$ is non-singular. By [9, Theorem 9.2.2, p. 174], the consistency of an initial vector for (4.12) and its general solution are independent of the used λ. Hence, in what follows, we drop the subscripts λ and take $\lambda = 0$.

From Theorem 3.9, $\mathrm{Ind}(\hat{S}_a) = 1$. Invoking [9, Theorem 9.2.3, p. 175], we obtain the necessary and sufficient condition $x_0 \in R(\hat{S}_a) = R(\hat{S_a^D}\hat{S}_a)$ for a vector $x_0 \in \mathbb{C}^7$ to be a consistent initial vector for (4.12). Since $\hat{S_a^D}\hat{S}_a = \hat{S}_a\hat{S_a^D}$, we get (4.13). As $\hat{S}_a = B^{-1}S_a$, then, by (3.7) in Theorem 3.7, we obtain (4.14).

Assume now that $x_0 \in \mathbb{C}^7$ is a consistent initial vector for (4.12). As $\hat{B} = I_7$, once again from [9, Theorem 9.2.3], the unique solution of the homogeneous initial value problem $S_a\dot{x} + Bx = 0, x(t_0) = x_0$, is given by (4.15). ■

Theorem 4.15. *Let $a \in \mathbb{C}^7$ with $\langle a, \overline{a}\rangle \neq 0$, $b \in \mathbb{C}^7\backslash\{0\}$ and $\alpha \in \mathbb{C}\backslash\{0\}$ such that α is not a square root of $-\langle b, \overline{b}\rangle$. Let $f = f(t)$ be a $\mathbb{C}^7$-valued function of the real variable t, continuously differentiable around t_0, and let $x = x(t)$ an unknown $\mathbb{C}^7$-valued function of the real variable t. A vector $x_0 \in \mathbb{C}^7$ is a consistent initial vector for the vector cross product differential equation*

$$a \times \dot{x} + b \times x + \alpha x = f \tag{4.16}$$

if and only if x_0 is of the form

$$x_0 = (I - \hat{S}_a\hat{S}_a^D)\hat{f}(t_0) + \hat{S}_a\hat{S}_a^D q, \tag{4.17}$$

for some vector $q \in \mathbb{C}^7$, where

$$\hat{S}_a = -(\alpha^2 + \langle b, \overline{b}\rangle)^{-1}\left(S_b - \alpha I_7 - \alpha^{-1}bb^T\right)S_a \tag{4.18}$$

and

$$\hat{f} = -(\alpha^2 + \langle b, \overline{b}\rangle)^{-1}\left(S_b - \alpha I_7 - \alpha^{-1}bb^T\right)f. \tag{4.19}$$

Moreover, if $x_0 \in \mathbb{C}^7$ is a consistent initial vector for (4.16), then the unique solution of (4.16), with initial condition $x(t_0) = x_0$, is

$$x(t) = \mathrm{e}^{-\hat{S}_a^D(t-t_0)}\hat{S}_a\hat{S}_a^D x_0 + \mathrm{e}^{-\hat{S}_a^D t}\int_{t_0}^{t}\mathrm{e}^{\hat{S}_a^D s}\hat{S}_a^D\hat{f}(s)\,ds + (I_7 - \hat{S}_a\hat{S}_a^D)\hat{f}(t). \tag{4.20}$$

Proof: By (3.1), we can rewrite equation (4.16) as $S_a\dot{x} + (S_b + \alpha I_7)x = f$, where $\alpha \in \mathbb{C}\backslash\{0\}$ is such that $\alpha^2 \neq -\langle b, \overline{b}\rangle$. As in the proof of Theorem 4.14, let $B = S_b + \alpha I_7$, $\hat{S}_a = B^{-1}S_a$, $\hat{B} = I_7$, $\hat{f} = B^{-1}f$.

Taking into account Theorem 3.9, $\mathrm{Ind}(\hat{S}_a) = 1$. The necessary and sufficient condition $x_0 \in \{(I_7 - \hat{S}_a\hat{S}_a^D)\hat{f}(t_0) + R(\hat{S_a^D}\hat{S}_a)\}$ for a vector $x_0 \in \mathbb{C}^7$ to be a consistent initial vector for (4.16) comes from [9, Theorem 9.2.3, p. 175], which leads to (4.17). By (3.7) in Theorem 3.7, we obtain (4.18) and (4.19).

Suppose now that $x_0 \in \mathbb{C}^7$ is a consistent initial vector for (4.16). Once again from [9, Theorem 9.2.3], the unique solution of the inhomogeneous initial value problem $S_a\dot{x} + Bx = f, x(t_0) = x_0$, is given by (4.20). ■

4.3. Vector Cross Product Difference Equations. In the present section, some vector cross product difference equations in $\mathbb{C}^7$ are studied.

Theorem 4.16. *Let $b \in \mathbb{C}^7$ such that $\langle b, \overline{b}\rangle \neq 0$ and let $x^{(k)} \in \mathbb{C}^7$ be the k-th term of an unknown sequence of vectors, $k = 0, 1, 2, ...$ The unique solution of the vector cross product difference equation*

$$x^{(k+1)} = b \times x^{(k)}, \tag{4.21}$$

with initial condition $x^{(0)} = x_0$, is

$$x^{(k)} = \begin{cases} x_0, & k = 0 \\ (-1)^{\frac{k-1}{2}}\beta^{k-1}S_b x_0, & k \in \mathbb{N}, odd \\ \left((-1)^{\frac{k}{2}+1}\beta^{k-2}bb^T + (-1)^{\frac{k}{2}}\beta^k I_7\right)x_0, & k \in \mathbb{N}, even \end{cases} \tag{4.22}$$

where $\beta = |\langle b, \overline{b}\rangle|^{1/2}e^{i\frac{\theta}{2}}$, with θ an argument of $\langle b, \overline{b}\rangle$.

Proof: Due to (3.1), equation (4.21) assumes the form $x^{(k+1)} = S_b x^{(k)}$, which is a tractable equation by Theorem 2.3. In fact, from Lemma 3.6, $(\lambda I_7 + S_b)^{-1}$ exists for every $\lambda \in \mathbb{C}\backslash\{0\}$ that is not a square root of $-\langle b, \overline{b}\rangle$. Taking into account the recurrence relation, the unique solution of the homogeneous initial value problem $x^{(k+1)} = S_b x^{(k)}$, $k = 0, 1, 2, \ldots$, $x^{(0)} = x_0$, is given by

$$x^{(k)} = S_b^k x_0,\ k = 0, 1, 2, ...$$

From Lemma 3.2, we arrive at (4.22). ■

Theorem 4.17. *Let $b \in \mathbb{C}^7$ such that $\langle b, \overline{b}\rangle \neq 0$. Let $f^{(k)} \in \mathbb{C}^7$ be the k-th term of a sequence of vectors, $k = 0, 1, 2, ...$, and let $x^{(k)} \in \mathbb{C}^7$ be the k-th term of an unknown sequence of vectors, $k = 0, 1, 2, ...$ The unique solution of the vector cross product difference equation*

$$x^{(k+1)} = b \times x^{(k)} + f^{(k)}, \tag{4.23}$$

with initial condition $x^{(0)} = x_0$, *is*

$$x^{(k)} = \begin{cases} x_0, & k = 0 \\ (-1)^{\frac{k-1}{2}} \beta^{k-1} S_b x_0 + \sum_{i=0}^{k-1} S_b^{k-1-i} f^{(i)}, & k \in \mathbb{N}, odd \\ \left((-1)^{\frac{k}{2}+1} \beta^{k-2} b b^T + (-1)^{\frac{k}{2}} \beta^k I_7 \right) x_0 + \sum_{i=0}^{k-1} S_b^{k-1-i} f^{(i)}, & k \in \mathbb{N}, even \end{cases} \tag{4.24}$$

where $\beta = |\langle b, \overline{b} \rangle|^{1/2} \mathrm{e}^{i\frac{\theta}{2}}$, *with* θ *an argument of* $\langle b, \overline{b} \rangle$.

Proof: Again by (3.1), equation (4.23) assumes the form $x^{(k+1)} = S_b x^{(k)} + f^{(k)}$. The recurrence relation allows to obtain the unique solution of the inhomogeneous initial value problem $x^{(k+1)} = S_b x^{(k)} + f^{(k)}$, $k = 0, 1, 2, \ldots$, $x^{(0)} = x_0$, given by

$$x^{(k)} = S_b^k x_0 + \sum_{i=0}^{k-1} S_b^{k-1-i} f^{(i)}, \ k = 1, 2, ... \tag{4.25}$$

From Lemma 3.2, we obtain (4.24). ■

Corollary 4.18. *Let* $b \in \mathbb{C}^7$ *such that* $\langle b, \overline{b} \rangle \neq 0$, $c \in \mathbb{C}^7$ *and let* $x^{(k)} \in \mathbb{C}^7$ *be the* k*-th term of an unknown sequence of vectors,* $k = 0, 1, 2, ...$. *The unique solution of the vector cross product difference equation*

$$x^{(k+1)} = b \times x^{(k)} + c, \tag{4.26}$$

with initial condition $x^{(0)} = x_0$, *is*

$$x^{(k)} = \begin{cases} x_0, & k = 0 \\ (-1)^{\frac{k-1}{2}} \beta^{k-1} S_b x_0 + \sum_{i=0}^{k-1} S_b^i c, & k \in \mathbb{N}, odd \\ \left((-1)^{\frac{k}{2}+1} \beta^{k-2} b b^T + (-1)^{\frac{k}{2}} \beta^k I_7 \right) x_0 + \sum_{i=0}^{k-1} S_b^i c, & k \in \mathbb{N}, even \end{cases} \tag{4.27}$$

where $\beta = |\langle b, \overline{b} \rangle|^{1/2} \mathrm{e}^{i\frac{\theta}{2}}$, *with* θ *an argument of* $\langle b, \overline{b} \rangle$.

Proof: A particular case of the previous result, putting c instead of the sequence $\left(f^{(k)} \right)_{k \in \mathbb{N}_0}$. ■

Theorem 4.19. *Let $a, b \in \mathbb{C}^7 \backslash \{0\}$ and let $x^{(k)} \in \mathbb{C}^7$ be the k-th term of an unknown sequence of vectors, $k = 0, 1, 2, ...$ The vector cross product difference equation*

$$a \times x^{(k+1)} = b \times x^{(k)} \tag{4.28}$$

is not tractable.

Proof: From (3.1), the rewriting of equation (4.28) leads to $S_a x^{(k+1)} = S_b x^{(k)}$. From Proposition 3.5, for any $\lambda \in \mathbb{C}$, $\lambda S_a + S_b$ is a singular matrix and the result follows from Theorem 2.3. ■

Similarly to subsection 4.2, due to the previous result, perturbed versions of the difference equation (4.28) are now studied.

Theorem 4.20. *Let $a \in \mathbb{C}^7$ with $\langle a, \overline{a} \rangle \neq 0$, $b \in \mathbb{C}^7 \backslash \{0\}$ and $\alpha \in \mathbb{C} \backslash \{0\}$ such that α is not a square root of $-\langle b, \overline{b} \rangle$. Let $x^{(k)} \in \mathbb{C}^7$ be the k-th term of an unknown sequence of vectors, $k = 0, 1, 2, ...$ A vector $x_0 \in \mathbb{C}^7$ is a consistent initial vector for the vector cross product difference equation*

$$a \times x^{(k+1)} = b \times x^{(k)} + \alpha x^{(k)} \tag{4.29}$$

if and only if x_0 is of the form

$$x_0 = \hat{S}_a \hat{S}_a^D q, \tag{4.30}$$

for some $q \in \mathbb{C}^7$, where

$$\hat{S}_a = -(\alpha^2 + \langle b, \overline{b} \rangle)^{-1} \left(S_b - \alpha I_7 - \alpha^{-1} b b^T \right) S_a. \tag{4.31}$$

Moreover, if $x_0 \in \mathbb{C}^7$ is a consistent initial vector for (4.29), then the unique solution of (4.29), with initial condition $x^{(0)} = x_0$, is

$$x^{(k)} = \left(\hat{S}_a^D \right)^k x_0, \ \ k = 0, 1, 2, \ldots \tag{4.32}$$

Proof: From (3.1), equation (4.29) assumes the form $S_a x^{(k+1)} = B x^{(k)}$ where $B = S_b + \alpha I_7$ with $\alpha \in \mathbb{C} \backslash \{0\}$ such that α is not a square root of $-\langle b, \overline{b} \rangle$. By Lemma 3.6, B is non-singular. Owed to this fact, $\lambda S_a + B$ is also a non-singular matrix if $\lambda = 0$ and, by Theorem 2.3, (4.29) is a tractable equation.

Following the notation in [9], let

$$\hat{S}_{a,\lambda} = (\lambda S_a + B)^{-1} S_a \text{ and } \hat{B}_\lambda = (\lambda S_a + B)^{-1} B,$$

where $\lambda \in \mathbb{C}$ is such that $\lambda S_a + B$ is non-singular. By [9, Theorem 9.2.2, p. 174], the consistency of an initial vector for (4.29) and its general solution are independent of the used λ. Hence, in what follows, we drop the subscripts λ and take $\lambda = 0$.

By Theorem 3.9, $\mathrm{Ind}(\hat{S}_a) = 1$. Invoking [9, Theorem 9.3.2, p. 182-183], we get the necessary and sufficient condition $x_0 \in R(\hat{S}_a) = R(\hat{S_a^D}\hat{S}_a)$ for a vector $x_0 \in \mathbb{C}^7$ to be a consistent initial vector for (4.29). As $\hat{S_a^D}\hat{S}_a = \hat{S}_a\hat{S_a^D}$, we obtain (4.30). Since $\hat{S}_a = B^{-1}S_a$, then, by (3.7) of Theorem 3.7, we arrive at (4.31).

Suppose now that $x_0 \in \mathbb{C}^7$ is a consistent initial vector for (4.29). Since $\hat{B} = I_7$, once again from [9, Theorem 9.3.2], the unique solution of the homogeneous initial value problem $S_a x^{(k+1)} = Bx^{(k)}$, $k = 0, 1, \ldots$, $x^{(0)} = x_0$, is given by (4.32). ■

Theorem 4.21. *Let $a \in \mathbb{C}^7$ with $\langle a, \overline{a}\rangle \neq 0$, $b \in \mathbb{C}^7\backslash\{0\}$ and $\alpha \in \mathbb{C}\backslash\{0\}$ such that α is not a square root of $-\langle b, \overline{b}\rangle$. Let $f^{(k)} \in \mathbb{C}^7$ be the k-th term of a sequence of vectors, $k = 0, 1, 2, \ldots$, and let $x^{(k)} \in \mathbb{C}^7$ the k-th term of an unknown sequence of vectors, $k = 0, 1, 2, \ldots$. A vector $x_0 \in \mathbb{C}^7$ is a consistent initial vector for the vector cross product difference equation*

$$a \times x^{(k+1)} = b \times x^{(k)} + \alpha x^{(k)} + f^{(k)}, \ k = 0, 1, 2, \ldots, \tag{4.33}$$

if and only if x_0 is of the form

$$x_0 = -\left(I_7 - \hat{S}_a\hat{S}_a^D\right)\hat{f}^{(0)} + \hat{S}_a\hat{S}_a^D q, \tag{4.34}$$

for some $q \in \mathbb{C}^7$, where

$$\hat{S}_a = -(\alpha^2 + \langle b, \overline{b}\rangle)^{-1}\left(S_b - \alpha I_7 - \alpha^{-1}bb^T\right)S_a \tag{4.35}$$

and

$$\hat{f}^{(k)} = -(\alpha^2 + \langle b, \overline{b}\rangle)^{-1}\left(S_b - \alpha I_7 - \alpha^{-1}bb^T\right)f^{(k)}. \tag{4.36}$$

Moreover, if $x_0 \in \mathbb{C}^7$ is a consistent initial vector for (4.33), then the unique solution of (4.33), with initial condition $x^{(0)} = x_0$, is $x^{(k)}$ given by

$$\begin{cases} x_0, & k = 0 \\ \left(\hat{S}_a^D\right)^k \hat{S}_a\hat{S}_a^D x_0 + \hat{S}_a^D \sum_{i=0}^{k-1}\left(\hat{S}_a^D\right)^{k-i-1}\hat{f}^{(i)} - \left(I_7 - \hat{S}_a\hat{S}_a^D\right)\hat{f}^{(k)}, & k = 1, 2, \ldots \end{cases} \tag{4.37}$$

Proof: By (3.1), the rewriting of equation (4.33) leads to $S_a x^{(k+1)} = Bx^{(k)} + f^{(k)}$, where $B = S_b + \alpha I_7$ with $\alpha \in \mathbb{C}\backslash\{0\}$ such that $\alpha^2 \neq -\langle b, \overline{b}\rangle$. As in the proof of Theorem 4.20, let $\hat{S}_a = B^{-1}S_a$, $\hat{B} = I_7$, $\hat{f}^{(k)} = B^{-1}f^{(k)}$.

From Theorem 3.9, $\text{Ind}(\hat{S}_a) = 1$. The necessary and sufficient condition $x_0 \in \{-(I_7 - \hat{S}_a\hat{S}_a^D)\hat{f}^{(0)} + R(\hat{S_a^D}\hat{S}_a)\}$ for a vector $x_0 \in \mathbb{C}^7$ to be a consistent initial vector for (4.33) comes from [9, Theorem 9.3.2, p. 182-183]. Thus, we obtain (4.34). By (3.7), we get (4.35) and (4.36).

Assume now that $x_0 \in \mathbb{C}^7$ is a consistent initial vector for (4.33). Once again from [9, Theorem 9.3.2], the unique solution of the inhomogeneous initial value problem $S_a x^{(k+1)} = Bx^{(k)} + f^{(k)}$, $k = 0, 1, 2, \ldots$, $x^{(0)} = x_0$, is given by (4.37). ■

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