Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 22–13

# SYMMETRIC SPACES ROLLING ON FLAT SPACES

V. JURDJEVIC, I. MARKINA AND F. SILVA LEITE

ABSTRACT: The objective of the current paper is essentially twofold. Firstly, to make clear the difference between two notions of rolling a Riemannian manifold over another, using a language accessible to a wider audience, in particular to readers with interest in applications. Secondly, we concentrate on rolling an important class of Riemannian manifolds. In the first part of the paper, the relation between intrinsic and extrinsic rollings is explained in detail, while in the second part we address rollings of symmetric spaces on flat spaces and complement the theoretical results with illustrative examples.

KEYWORDS: Semi-Riemannian manifolds, symmetric spaces, group actions, Cartan decomposition, intrinsic rolling, extrinsic rolling, no-slip, no-twist, Stiefel manifolds.

## 1. Introduction

In the contemporary literature, there exist two notions of rolling a Riemannian manifold over another, which more recently have also been extended to the semi-Riemannian case. One of these notions is *intrinsic rolling*, that doesn't require that the Riemannian manifolds are embedded. This concept uses the intrinsic geometry of the manifolds only, and for Riemannian surfaces was introduced by Agrachev and Sachkov in [1] and by Bryant and Hsu in [2], and later studied for manifolds of higher dimensions, for instance, in [3], [4] and [8]. Extension to the semi-Riemannian situation appeared in [21].

Another definition of rolling initiated by K. Nomizu in [23] and presented more formally by R.W. Sharpe in [25] is the *extrinsic rolling*, which makes use of the isometric embedding of the manifolds in an ambient (semi)-Euclidean vector space $V$, so that the rolling is described in terms of the action of the group $\mathrm{SE}(V)$ of oriented isometries of $V$. More recent works that use the extrinsic rolling are, for instance, [26, 12, 16, 11, 6, 20, 18, 22]. As far as we know, only in [8] and [21] both notions of rolling were addressed, the first for the Riemannian case and the second for the semi-Riemannian case. These two works are rather theoretical for researchers interested in applications

Received May 27, 2022.

of rolling motions but that do not have a strong background in differential geometry.

The purpose of the current paper is essentially twofold. Firstly, we want to elucidate the difference between the two notions of rolling using a language that is more accessible to those with practical interest in rolling motions but less familiar with semi-Riemannian geometry. Secondly, we make transparent the relation between the geometry of the symmetric spaces and its rolling on flat spaces. Our main message is that the transitive action $\tau$ of a Lie group on a symmetric space completely defines its rolling along a chosen curve on the manifold. The differential map $d\tau$ is the isometry between the tangent spaces (after an identification of related vector spaces), that also matches the parallel vector fields on the rolling curves. Examples of semi-Riemannian symmetric spaces are provided, together with how to construct both types of rollings. It is always assumed throughout the paper that nonholonomic constraints of no-slip and no-twist are required in both situations, and the rollings are confined to semi-Riemannian manifolds.

The organization of the paper is the following. After setting the notation, we discuss in Section 3 the intrinsic rolling versus extrinsic. Section 4 is dedicated to rolling of symmetric spaces on flat spaces. Finally, we include Section 5 with the rolling of Stiefel manifolds, in order to illustrate the difference in the construction of rolling motions for a reductive homogeneous manifold that is not a symmetric semi-Riemannian manifold.

## 2. Background and notations

In this section we revisit the most important known concepts and results that will be used in the paper, and introduce the necessary notations. The main reference is the book of O'Neill [24], where the reader may find further details.

**2.1. Semi-Riemannian manifolds.** A semi-Riemannian manifold $M$ is a smooth manifold endowed with a non-degenerate symmetric tensor $g(.\,,.)$. We write $n = \dim M$, and denote by $p$ the number of positive eigenvalues of the tensor $g$, so that $n-p$ is the number of negative eigenvalues of $g$. The crucial example of a semi-Riemannian manifold is the semi-Euclidean vector space $\mathbb{R}^{p,n-p}$ with the semi-Euclidean product

$$\langle x, y\rangle_{p,n-p} = \sum_{k=1}^{p} x_k y_k - \sum_{k=p+1}^{n} x_k y_k, \quad x, y \in \mathbb{R}^{p,n-p}.$$

Another important example is a vector space $V$ with a bilinear symmetric non-degenerate form $(.\,,.)_{p,n-p}$. We will often refer to $(.\,,.)_{p,n-p}$ as a scalar product and simply write $(.\,,.)$ in case there is no need to specify the signature.

Let $(V, (.\,,.)_{p,n-p})$ be a semi-Riemannian vector space. The isometric embedding map will be denoted by

$$\iota\colon M \to V.$$

On existence of such an embedding see [5]. For the moment, we will identify the manifold $M$ with its image under the embedding, that is, notationally, $\iota(M) = M$. The semi-Riemannian metric $g(.\,,.)$ on the embedded manifold $M$ is inherited from the semi-Riemannian product $(.\,,.)_{p,n-p}$ in the ambient space $V$.

The isometric embedding of $M$ into $V$ splits the tangent space of $V$, at a point $m \in M$, into a direct sum:

$$T_m V = T_m M \oplus (T_m M)^{\perp}, \quad m \in M, \tag{1}$$

where $^{\perp}$ denotes the orthogonal complement with respect to $(.\,,.)_{p,n-p}$. Note that the tangent space $T_mM$ and the normal space $(T_mM)^{\perp}$ are nondegenerate subspaces of $(V, (.\,,.)_{p,n-p})$. According to this, any vector $v \in T_mV$, $m \in M$, can be written uniquely as the sum $v = v^{\top} + v^{\perp}$, where $v^{\top} \in T_mM$, $v^{\perp} \in (T_mM)^{\perp}$.

In what follows, $\overline{\nabla}$ denotes the Levi-Civita connection on the ambient space $V$, and $\nabla$ for the Levi-Civita connection on $M$. If $X$ and $Y$ are tangent vector fields on $M$, and $\Upsilon$ is a normal vector field on $M$, then

$$\nabla_X Y(p) = \left(\overline{\nabla}_{\bar{X}} \bar{Y}(p)\right)^{\top}, \ \nabla^{\perp}_X \Upsilon(p) = \left(\overline{\nabla}_{\bar{X}} \bar{\Upsilon}(p)\right)^{\perp}, \ p \in M,$$

where $\bar{X}$, $\bar{Y}$ and $\bar{\Upsilon}$ are any local extensions to $V$ of the vector fields $X$, $Y$ and $\Upsilon$, respectively. If $Z(t)$ and $\Upsilon(t)$ are vector fields along a curve $\alpha(t)$, we use $\frac{D_{\alpha(t)}}{dt}Z(t)$ to denote the covariant derivative of $Z(t)$ along $\alpha(t)$ and $\frac{D^{\perp}_{\alpha(t)}}{dt}\Upsilon(t)$ for the normal covariant derivative of $\Upsilon(t)$ along $\alpha(t)$ (these notations are according to [24, p. 119]). Again, to simplify notations, in cases where it is clear what is the curve along which the covariant derivative is considered, we may simply write $\frac{D}{dt}$ and $\frac{D^\perp}{dt}$ instead of the above. Observe that an isometric imbedding of $M$ into $V$ induces the equalities

$$\frac{D}{dt}Z(t) = \left(\frac{d}{dt}Z(t)\right)^\top, \quad \frac{D^\perp}{dt}\Upsilon(t) = \left(\frac{d}{dt}\Upsilon(t)\right)^\perp. \tag{2}$$

A tangent vector field $Y(t)$ along an absolutely continuous curve $\alpha(t)$ is *tangent parallel* if $\frac{D}{dt}Z(t) = 0$, for almost every $t$. Similarly, a normal vector field $\Upsilon(t)$ along $\alpha(t)$ is *normal parallel* if $\frac{D^\perp}{dt}\Upsilon(t) = 0$, for almost every $t$.

From now on we assume that all curves are absolutely continuous on some real interval $I = [0, T]$, $T > 0$ and, even if not said, conditions involving derivatives are valid only for values of the parameter $t$ for which they are well defined.

We denote by $\mathrm{SE}(V)$ the Lie group of semi-Riemannian isometries of the space $(V, (\,.\,,.)_{p,n-p})$. It can be shown that $SE(V) = \mathrm{SO}(V) \ltimes V$, where, by abuse of notation, $V$ is the subgroup of translations on the vector space $V$, and $\mathrm{SO}(V)$ is the connected component containing identity $e$ of the group of isometries $O(V)$, preserving the orientation of both positive definite and negative definite subspaces of $V$. Elements in $\mathrm{SE}(V)$ will be represented by pairs $g = (R, s)$, $R \in \mathrm{SO}(V)$, $s \in V$, and in this representation the action of $\mathrm{SE}(V)$ on $V$ is denoted by $(g, v) \mapsto g.v := R.v + s$, $v \in V$, where $(R, v) \mapsto R.v$ denotes the action of $\mathrm{SO}(V)$ on $V$. The group product in $\mathrm{SE}(V)$ is defined as $(R_2, s_2)(R_1, s_1) = (R_2R_1, s_2 + R_2.s_1)$. It then follows that $(e, 0)$ is the group identity in $\mathrm{SE}(V)$, and $(R, s)^{-1} = (R^{-1}, -R^{-1}.s)$.

# 3. Intrinsic versus extrinsic rolling

We want to recall the definition of a rolling of a semi-Riemannian manifold $M$ over a semi-Riemannian manifold $\widehat{M}$ along a given curve $\alpha\colon I \to M$ with the restrictions of no-slip and no-twist. There are two notions of such a rolling, commonly referred to as "intrinsic" and "extrinsic", that currently exist in the literature. Intrinsic rolling of the Riemannian manifold, introduced in [1, 2], and also used, for instance, in [3, 8]. The intrinsic rolling of semi-Riemannian manifolds was studied in [21], and the extrinsic rolling of particular families of semi-Riemannian manifolds was treated in [6, 16, 17, 22]. The difference between the two definitions is that an "intrinsic" rolling doesn't require that semi-Riemannian manifolds $M$ and $\widehat{M}$ are isometrically embedded into $(V, (\,.\,,.)_{p,n-p})$, meanwhile an "extrinsic" rolling presumes such an embedding.

In the following definitions, the semi-Riemannian manifolds $(M, g)$ and $(\widehat{M}, \widehat{g})$ have equal dimension and the semi-Riemannian metric tensors have equal signature. We call an isometry $A\colon T_mM \to T_{\widehat{m}}\widehat{M}$ oriented if it preserves the orientation of the positive definite and the negative definite subspaces of $T_mM$ and $T_{\widehat{m}}\widehat{M}$.

**Definition 1. Intrinsic rolling.** *A curve $\alpha(t)$ on $M$ is said to roll on a curve $\widehat{\alpha}(t)$ on $\widehat{M}$ if there exists an oriented isometry $A(t) : T_{\alpha(t)}M \to T_{\widehat{\alpha}(t)}\widehat{M}$ such that*

$$\dot{\widehat{\alpha}}(t) = A(t)\dot{\alpha}(t), \quad \textit{and} \tag{3}$$

*$A(t)X(t)$ is a parallel vector field in $\widehat{M}$ along $\widehat{\alpha}(t)$ if and only if $X(t)$ is a parallel vector field in $M$ along $\alpha(t)$.*

*The triplet $(\alpha(t), \widehat{\alpha}(t), A(t))$ is called a rolling curve.*

In the Riemannian case, the definition of "extrinsic" rolling initiated by K. Nomizu [23] and presented more formally by R.W. Sharpe [25], makes use of the isometric embedding of $M$ and $\widehat{M}$ in an ambient space. Here we use a definition of extrinsic rolling that is more general then that used by [25]. This extended class of rollings is better suited for making the bridge with control theory and also for comparison with Definition 1. It includes the presence of a semi-Riemannian vector space $(V, (.\,,.)_{p,n-p})$, orientability for the group of semi-Riemannian motions $\mathrm{SE}(V)$, and the replacement of piecewise continuous curves by absolutely continuous curves, see also [8, 21].

**Definition 2. Extrinsic Rolling.** *An absolutely continuous curve $g(t)$ in $\mathrm{SE}(V)$, defined on an interval $I = [0, T]$, is said to roll a curve $\alpha(t)$ in $M$ onto a curve $\widehat{\alpha}(t)$ in $\widehat{M}$, without slipping and without twisting, if*

1 *$g(t)\alpha(t) = \widehat{\alpha}(t)$, for all $t \in I$,*
2 *$d_{\alpha(t)}g(t)\, T_{\alpha(t)}M = T_{\widehat{\alpha}(t)}\widehat{M}$, for all $t \in I$.*
3 *No-slip condition:*

$$\dot{\widehat{\alpha}}(t) = d_{\alpha(t)}g(t)\, \dot{\alpha}(t), \ \textit{for almost every } t;$$

4 *No-twist condition (tangential part)*

$$d_{\alpha(t)}g(t)\, \frac{D}{dt}\, Z(t) = \frac{D}{dt}\, d_{\alpha(t)}g(t)\, Z(t),$$

*for any tangent vector field $Z(t)$ along $\alpha(t)$ and almost every $t$;*

5 *No-twist condition (normal part)*

$$d_{\alpha(t)}g(t)\,\frac{D^{\perp}}{dt}\,\Psi(t)=\frac{D^{\perp}}{dt}\,d_{\alpha(t)}g(t)\,\Psi(t),$$

*for any normal vector field* $\Psi(t)$ *along* $\alpha(t)$ *and almost every* $t$*;*

6 $d_{\alpha(t)}g(t)|_{T_{\alpha(t)}M}:T_{\alpha(t)}M\to T_{\widehat{\alpha}(t)}\widehat{M}$ *is orientation preserving.*

*The curve* $g(t)$ *that satisfies the above conditions is called* rolling map *along the curve* $\alpha(t)$ *(also called* rolling curve*), and* $\widehat{\alpha}(t)$ *is the* development *of* $\alpha(t)$ *on* $\widehat{M}$*.*

From now on, we may refer to rolling without slipping and without twisting simply as "rolling". The first two conditions in Definition 2 are called "rolling conditions". Notice that the second rolling condition and the splitting (1) also implies:

$$d_{\alpha(t)}g(t)(T_{\alpha(t)}M)^{\perp}=(T_{\widehat{\alpha}(t)}\widehat{M})^{\perp}. \tag{4}$$

The no-slip and no-twist conditions can be seen as nonholonomic constraints. They give rise to equations for the velocity of the rolling map, usually called the *kinematic equations of rolling.*

At first glance, the no-slip and no-twist conditions in Definition 2 may look different from those in [25], however they are equivalent, as proven in [8, 21]. When dealing with concrete examples these nonholonomic constraints are easier to handle when written as in [25]. For that reason, after introducing some necessary notations, we rewrite conditions 3, 4 and 5 in Definition 2 using the terminology in [25].

For each action $g(t)=(R(t),s(t))\in \mathrm{SE}(V)$ on $V$, defined by $g(t).p=R(t).p+s(t)$, the differential (or tangent map) of $g(t)$ at $p\in V$ is given by

$$d_pg(t)v:=\frac{d}{d\epsilon}g(t).p(\epsilon)\Big|_{\epsilon=0}=R(t).v, \tag{5}$$

where $\epsilon\mapsto p(\epsilon)$ is a curve in $V$ satisfying $p(0)=p$, $\frac{dp}{d\epsilon}(0)=v$. If $\dot g(t)$ denotes the time derivative of the curve $g(t)$, i.e.,

$$\dot g(t).p=\frac{d}{d\epsilon}g(\epsilon).p\Big|_{\epsilon=t}=\dot R(t).p+\dot s(t),$$

then, since $g^{-1}=(R^{-1},-R^{-1}.s)$, we can define

$$\begin{aligned}\left(\dot g(t)g^{-1}(t)\right).p: &= \tfrac{d}{d\epsilon}g(\epsilon).(g^{-1}(t).p)\big|_{\epsilon=t}\\ &= \dot R(t)R^{-1}(t).(p-s(t))+\dot s(t),\end{aligned} \tag{6}$$

so that

$$\begin{aligned} d_p\left(\dot{g}(t)g^{-1}(t)\right).v : &= \tfrac{d}{d\epsilon}\left(\dot{g}(t)g^{-1}(t)\right).p(\epsilon)\big|_{\epsilon=0} \\ &= \dot{R}(t)R^{-1}(t).v. \end{aligned} \tag{7}$$

**Proposition 1.** *Conditions* 3, 4 *and* 5 *in Definition 2 are, respectively, equivalent to:*

3' *No-slip condition:*

$$(\dot{g}(t)g^{-1}(t)).\widehat{\alpha}(t) = 0, \text{ for almost every } t;$$

4' *No-twist condition (tangential part):*

$$d_{\widehat{\alpha}(t)}(\dot{g}(t)g^{-1}(t))\, T_{\widehat{\alpha}(t)}\widehat{M} \subset (T_{\widehat{\alpha}(t)}\widehat{M})^{\perp}, \text{ for almost every } t;$$

5' *No-twist condition (normal part):*

$$d_{\widehat{\alpha}(t)}(\dot{g}(t)g^{-1}(t))\, (T_{\widehat{\alpha}(t)}\widehat{M})^{\perp} \subset T_{\widehat{\alpha}(t)}\widehat{M}, \text{ for almost every } t;$$

It was proved in [25] that given a curve $\alpha(t)$ in $M$ there always exists a unique rolling map $g(t)$ that rolls a Riemannian manifold $M$ on a Riemannian manifold $\widehat{M}$ along $\alpha$. The proof can be literally extended to the rolling of semi-Riemannian manifolds, since the arguments in [25] do not rely on the positive definite property of the metric tensor, but rather on being non-degenerate.

**Remark 1.** *The definition of rolling map doesn't exclude the possibility that $g(t)$ is the identity in* $\mathrm{SE}(V)$*, otherwise the existence of a rolling map for each curve in $M$ would not be guaranteed. This is clearly seen, for instance, in the system consisting of a cylinder rolling on the tangent plane at a point, when the rolling curve lies in the straight line of intersection of the two manifolds.*

The no-twist conditions in Definition 2 can also be rewritten in terms of parallel vector fields as follows. This is particularly important for the comparison with the intrinsic rolling.

**Proposition 2.** *Conditions* 4 *and* 5 *of Definition 2 are, respectively, equivalent to:*

4" *No-twist condition (tangential part): A vector field $Z(t)$ is tangent parallel along the curve $\alpha(t)$ if, and only if, $d_{\alpha(t)}g(t)(Z(t))$ is tangent parallel along $\widehat{\alpha}(t)$.*

5" *No-twist condition (normal part): A vector field $\Psi(t)$ is normal parallel along the curve $\alpha(t)$ if, and only if, $d_{\alpha(t)}g(t)(\Psi(t))$ is normal parallel along $\widehat{\alpha}(t)$.*

*Proof*: We prove the tangential part only. The proof of the normal part can be done similarly. It is clear that $\frac{D}{dt}Z(t)=0$ if, and only if, $\frac{D}{dt}\left(d_{\alpha(t)}g(t)(Z(t))\right)=0$. Consequently, condition 4 of Definition 2 implies condition 4" above.

To prove that condition 4" implies condition 4 of Definition 2, let $Z(t)$ be an arbitrary tangent vector field along $\alpha(t)$ and $\{E_1(t),\ldots,E_n(t)\}$, $n=\dim(M)$, be a parallel tangent frame along $\alpha(t)$, so that

$$Z(t)=\sum_{i=1}^{n} z_i(t)E_i(t) \quad \text{and} \quad \frac{D}{dt}Z(t)=\sum_{i=1}^{n}\dot{z}_i(t)E_i(t).$$

Now define $\widehat{E}_i(t):=d_{\alpha(t)}g(t)(E_i(t))$. Taking into account assumption $4'$, we can guarantee that $\{\widehat{E}_1(t),\ldots,\widehat{E}_n(t)\}$ is a parallel tangent frame along the development curve $\widehat{\alpha}(t)$. Since $d_{\alpha(t)}g(t)$ is a linear isomorphism, using properties of the covariant derivative we obtain

$$\begin{array}{rcl} d_{\alpha(t)}g(t)\left(\frac{D}{dt}Z(t)\right) &=& \sum_{i=1}^{n}\dot{z}_i(t)d_{\alpha(t)}g(t)(E_i(t)) \\ &=& \sum_{i=1}^{n}\dot{z}_i(t)\widehat{E}_i(t) \end{array}$$

and

$$\frac{D}{dt}\left(d_{\alpha(t)}g(t)(Z(t))\right)=\frac{D}{dt}\left(\sum_{i=1}^{n} z_i(t)\widehat{E}_i(t)\right)=\sum_{i=1}^{n}\dot{z}_i(t)\widehat{E}_i(t).$$

Therefore, condition 4 in Definition 2 follows. ■

**Remark 2.** *It is clear from Proposition 2 that the tangent part of the no-twist condition is always satisfied when the manifolds $M$ and $\widehat{M}$ are one-dimensional, and the normal no-twist condition is always satisfied when those manifolds have co-dimension one.*

In order to relate the two seemingly very different definitions of rolling, when $M$ and $\widehat{M}$ are isometrically embedded in the semi-Riemannian vector space $(V,(.\,,.)_{p,n-p})$, we also need to compare $A(t)$, the part responsible for the isometric mapping of the tangent spaces in the intrinsic rolling definition, with the rolling map $g(t)=(R(t),s(t))\in\mathrm{SE}(V)$, in the extrinsic definition.

Since $\mathrm{SE}(V)=\mathrm{SO}(V)\ltimes V$ and $\mathrm{SO}(V)$ both act on $V$, if $\alpha(t)$ is a curve in $V$, then, for any $g(t)=(R(t),s(t))\in\mathrm{SE}(V)$ and any tangent vector field $Z(t)$ along $\alpha(t)$ we have

$$d_{\alpha(t)}g(t).Z(t)=d_{\alpha(t)}R(t)Z(t)=R(t).Z(t). \tag{8}$$

**Remark 3.** *According to* (8)*, we can refer to the restriction of* $d_{\alpha(t)}g(t)$ *to* $T_{\alpha(t)}M$*, which is the same as the restriction of* $d_{\alpha(t)}R(t)$ *to* $T_{\alpha(t)}M$*, as the restriction of* $R(t)$ *to* $T_{\alpha(t)}M$*. This abuse of terminology simplifies the exposition that follows.*

The following proposition provides a relationship between the intrinsic and the extrinsic rolling when $M$ and $\widehat{M}$ are isometrically embedded in $V$.

**Proposition 3.** *Assume that* $(\alpha(t), \widehat{\alpha}(t), A(t))$ *is a rolling curve in the sense of Definition 1. Let* $g(t) = (R(t), s(t))$ *be a curve in* $SE(V)$ *such that the restriction of* $R(t)$ *to* $T_{\alpha(t)}M$ *is equal to* $A(t)$ *and* $s(t) = \widehat{\alpha}(t) - R(t).\alpha(t)$*. Then* $g(t)$ *satisfies conditions 1 through 4 and 6 of Definition 2.*

*Conversely, if* $g(t) = (R(t), s(t))$ *is a curve in* $SE(V)$ *that satisfies conditions 1 through 4 and 6 of Definition 2, then* $(\alpha(t), \widehat{\alpha}(t), A(t))$ *is an intrinsic rolling curve, where* $A(t)$ *is the restriction of* $R(t)$ *to* $T_{\alpha(t)}M$*. This happens, in particular, if* $g(t)$ *is a rolling map along* $\alpha(t)$*.*

The previous statement is completely obvious in view of Proposition 2, since the tangential no-twist condition in Definition 2 is equivalent to the parallel transport condition required by the intrinsic rolling. According to the last statement of Proposition 3, if $g(t) = (R(t), s(t))$ is a rolling map along $\alpha(t)$ with development $\widehat{\alpha}(t)$, we say that $R(t)$ defines the intrinsic rolling curve $(\alpha(t), \widehat{\alpha}(t), A(t))$, where $A(t) := R(t)|_{T_{\alpha(t)}M}$.

**Remark 4.** *We now also see precisely the difference between the rolling of Definition 1 and the rolling of Definition 2. A curve* $\alpha(t)$ *in* $M$ *rolls on a curve* $\widehat{\alpha}(t)$ *in* $\widehat{M}$ *independently of the definition used. However, in the absence of the normal no-twist condition, the lifting of the isometry* $A(t)$ *to an isometry* $d_{\alpha(t)}g(t)$ *in* $T_{\alpha(t)}V$ *is not one to one since the latter can be completely arbitrary on the orthogonal complement* $(T_{\alpha(t)}M)^{\perp}$*. If* $A^{\perp}(t) : (T_{\alpha(t)}M)^{\perp} \to (T_{\widehat{\alpha}(t)}\widehat{M})^{\perp}$ *is a map such that any normal parallel vector field along* $\alpha(t)$ *maps to a normal parallel vector field along* $\widehat{\alpha}(t)$*, then* $g(t)$ *is completely and uniquely defined by*

$$d_{\alpha(t)}g(t)|_{T_{\alpha(t)}M} = A(t) \quad \text{and} \quad d_{\alpha(t)}g(t)|_{(T_{\alpha(t)}M)^{\perp}} = A^{\perp}(t).$$

*The arbitrariness of* $A(t)$*, which due to Remark 3 can be seen as an arbitrariness of* $R(t)$*, is removed by adding the normal part of the no-twist condition, for then there is a one to one correspondence between* $A(t)$ *that rolls* $\alpha(t)$ *onto* $\widehat{\alpha}(t)$ *in the sense of Definition 1 and the rolling map* $g(t) = (R(t), s(t))$

*in $SE(V)$ that rolls $\alpha(t)$ onto $\widehat{\alpha}(t)$ in the sense of Definition 2. In fact, $d_{\alpha(t)}(g(t))$ is equal to $A(t)$ on the tangent space $T_{\alpha(t)}M$, and is uniquely determined on the orthogonal complement by the normal no-twist condition.*

We have already seen in Proposition 3 that if $g(t) = (R(t), s(t))$ is a rolling map along a curve $\alpha(t)$ with development $\widehat{\alpha}(t)$, then the triple $(\alpha(t), \widehat{\alpha}(t), A(t))$, where for $A(t) := R(t)|_{T_{\alpha(t)}M}$, is an intrinsically rolling curve. In other words, each extrinsic rolling map determines a unique intrinsic rolling curve.

However, we may perturb $R(t)$ so that the normal part of the no-twist condition is violated and still obtain an intrinsic rolling curve of $\alpha(t)$ on $\widehat{\alpha}(t)$. The next proposition makes this statement clear. We use the symbol $\circ$ to denote the composition of linear maps.

**Proposition 4.** *Suppose that $g(t) = (R(t), s(t))$ is a rolling map along the curve $\alpha(t)$ with development $\widehat{\alpha}(t)$, and $\dot{R}(t) = \Omega(t) \circ R(t)$, with $\Omega(t) \in \mathfrak{so}(V)$. Let $\widetilde{R}(t)$ be the solution of*

$$\dot{\widetilde{R}}(t) = (\Omega(t) + \Omega_0(t)) \circ \widetilde{R}(t), \quad \widetilde{R}(0) = R(0), \tag{9}$$

*where $\Omega_0(t) \in \mathfrak{so}(V)$ satisfies*

$$\Omega_0(t)(T_{\widehat{\alpha}(t)}\widehat{M}) = 0, \quad \Omega_0(t)(T_{\widehat{\alpha}(t)}\widehat{M})^{\perp} \subseteq (T_{\widehat{\alpha}(t)}\widehat{M})^{\perp}. \tag{10}$$

*Then $R(t)$ and $\widetilde{R}(t)$ define the same intrinsic rolling, given by the triple $(\alpha(t), \widehat{\alpha}(t), A(t))$.*

*Proof*: We already know that $R(t)$ defines the intrinsic rolling curve

$$(\alpha(t), \widehat{\alpha}(t), A(t)),$$

where $A(t) = R(t)|_{T_{\alpha(t)}M}$. In order to prove the statement it is enough to show that $\widetilde{R}(t)|_{T_{\alpha(t)}M} = A(t)$. For that, first notice that $R$ and $\widetilde{R}$ are related by $\widetilde{R}(t) = R(t) \circ S(t)$, where $S(t) \in SO(V)$ is the solution of

$$\dot{S}(t) = (R^{-1}(t) \circ \Omega_0(t) \circ R(t)) \circ S(t), \quad S(0) = I. \tag{11}$$

Indeed, since $S(t) \in SO(V)$, we have $\dot{S}(t) = \Lambda(t) \circ S(t)$, for some $\Lambda(t) \in \mathfrak{so}(V)$. And so,

$$\begin{aligned} \dot{\widetilde{R}}(t) &= \dot{R}(t) \circ S(t) + R(t) \circ \dot{S}(t) \\ &= (\Omega(t) + R(t) \circ \Lambda(t) \circ R^{-1}(t)) \circ \widetilde{R}(t), \end{aligned}$$

which according to the assumption $\dot{\widetilde{R}}(t) = (\Omega(t) + \Omega_0(t)) \circ \widetilde{R}(t)$ implies $\Lambda(t) = R^{-1}(t) \circ \Omega_0(t) \circ R(t)$. Since by assumption $R(t)$ satisfies

$$R(t)(T_{\alpha(t)}M) = T_{\widehat{\alpha}(t)}\widehat{M},\ R(t)(T_{\alpha(t)}M)^{\perp} = (T_{\widehat{\alpha}(t)}M)^{\perp}, \tag{12}$$

and $\Omega_0(t)$ satisfies (10), we conclude that for $\Lambda(t) := R^{-1}(t) \circ \Omega_0(t) \circ R(t)$,

$$\begin{aligned} \Lambda(t)(T_{\alpha(t)}M) &= R^{-1}(t) \circ \Omega_0(t)(T_{\widehat{\alpha}(t)}\widehat{M}) = 0, \\ \Lambda(t)(T_{\alpha(t)}M)^{\perp} &= R^{-1}(t) \circ \Omega_0(t)(T_{\widehat{\alpha}(t)}\widehat{M})^{\perp} \\ &\subseteq R^{-1}(t)(T_{\widehat{\alpha}(t)}\widehat{M})^{\perp} = (T_{\alpha(t)}M)^{\perp}. \end{aligned}$$

Now we are going to choose a system of coordinates so that $S(t)|_{T_{\alpha(t)}M}$ becomes the identity map; that is $S(t)Z(t) = Z(t)$, for every $Z(t) \in T_{\alpha(t)}M$. Let $\{b_1(t), \dots, b_N(t)\}$ be an orthonormal frame in $V$ along $\alpha(t)$ such that, for every $t$, $\{b_1(t), \dots b_n(t)\}$ and $\{b_{n+1}(t), \dots, b_N(t)\}$ are basis for $T_{\alpha(t)}M$ and $(T_{\alpha(t)}M)^{\perp}$, respectively. In this system of coordinates, $\Lambda(t)(T_{\alpha(t)}M)$ is represented by the block matrix $\begin{pmatrix} 0_{n,n} \\ 0_{N-n,n} \end{pmatrix}$, where $0_{m,n}$ denotes the zero matrix of size $m \times n$, while $\Lambda(t)(T_{\alpha(t)}M)^{\perp}$ is represented by the block matrix $\begin{pmatrix} 0_{n,N-n} \\ \Lambda_0(t) \end{pmatrix}$, where $\Lambda_0(t)$ is the projection of $\Lambda(t)(T_{\alpha(t)}M)^{\perp}$ on $(T_{\alpha(t)}M)^{\perp}$. As a consequence, $\Lambda(t)$ is represented by the block matrix

$$\Lambda(t) = \begin{pmatrix} 0_{n,n} & 0_{n,N-n} \\ 0_{N-n,n} & \Lambda_0(t) \end{pmatrix}.$$

So, since $\dot{S}(t) = \Lambda(t) \circ S(t)$, we must have

$$S(t) = \begin{pmatrix} I_n & 0_{n,N-n} \\ 0_{N-n,n} & S_4(t) \end{pmatrix},$$

from what follows that $S(t)Z(t) = Z(t)$, for every $Z(t) \in T_{\alpha(t)}M$, and, consequently,

$$\widetilde{R}(t)|_{T_{\alpha(t)}M} = R(t)|_{T_{\alpha(t)}M} = A(t). \qquad \blacksquare$$

These subtle differences between various notions of rollings are best illustrated through the comparison of the rolling of a semi-Riemannian manifold $M$ over a flat manifold $\widehat{M}$, versus the rolling of $M$ on its affine tangent space $\widehat{M}$ at a fixed point, when $M$ and $\widehat{M}$ are isometrically embedded in $(V, (\,.\,,.\,)_{p,n-p})$. We start by revising general facts about a rolling on affine tangent spaces.

The following important properties of the rolling map of Riemannian manifolds have also been proved in [25]. The proof uses the arguments involving the group properties of $SE(V)$, that are also true for the semi-Riemannian vector space $(V, (\,.\,,\,.\,)_{p,n-p})$.

**Proposition 5.** *Let $M$, $M_1$ and $M_2$ be manifolds of the same dimension, isometrically embedded in $V$, and $\alpha(t)$, $\alpha_1(t)$ and $\alpha_2(t)$ curves in $M$, $M_1$ and $M_2$ respectively, defined in the real interval $I$, that satisfy $\alpha(0) = \alpha_1(0) = \alpha_2(0)$.*

- *Symmetric property of rolling*
  *If $g(t) \in \mathrm{SE}(V)$ is a rolling map of $M$ on $M_1$ along the rolling curve $\alpha(t)$ with development curve $\alpha_1(t)$, then $g^{-1}(t) \in \mathrm{SE}(V)$ is a rolling map of $M_1$ on $M$, along the rolling curve $\alpha_1(t)$ with development curve $\alpha(t)$.*
- *Transitive property of rolling*
  *If $g(t) \in \mathrm{SE}(V)$ is a rolling map of $M$ on $M_1$ along the rolling curve $\alpha(t)$ with development curve $\alpha_1(t)$, and $g_1(t) \in \mathrm{SE}(V)$ is a rolling map of $M_1$ on $M_2$ along the rolling curve $\alpha_1(t)$ with development curve $\alpha_2(t)$, then $g(t)g_1(t) \in \mathrm{SE}(V)$, is a rolling map of $M$ on $M_2$, with rolling curve $\alpha(t)$ and development curve $\alpha_2(t)$.*

**Remark 5.** *Using these two properties, one can reduce the study of rolling a manifold on another to the simpler situation when the second manifold is the affine tangent space at a point of the first. Properties above have been used in [20] to derive the kinematic equations of a sphere rolling on another sphere of the same dimension, using the equations of spheres rolling on affine tangent spaces at a point. Also in the semi-Riemannian case, these properties have been used in [22] to derive the kinematic equations for rolling a hyperbolic sphere over another.*

## 4. Rolling of symmetric spaces on flat manifolds

We start from setting the notation and recalling useful information about symmetric spaces based on [24].

### 4.1. Symmetric spaces.

**Definition 3.** *A connected semi-Riemannian manifold* $\big(M, g\big)$ *is called a semi-Riemannian symmetric space if for each* $o \in M$ *there exists a diffeomorphic isometric map* $\zeta_o \colon M \to M$*, called the global isometry of* $M$ *at* $o$*, such that* $d_o\zeta_o = -\mathrm{Id}$ *on the vector space* $T_oM$*.*

The symmetric semi-Riemannian spaces have close relation to Lie groups. The connected identity componnet $G$ of the isometry group acts transitively on $M$. Let $H$ be the isotropy subgroup of a point $o \in M$. Then $M$ can be identified with the homogeneous space $G/H$. Note that the isotropy subgroups of different points are conjugate subgroups of $G$. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of the Lie groups $G$ and $H$ respectively. Then, the following Cartan decomposition holds,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}, \quad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h}. \tag{13}$$

We denote by $\tau \colon G \times M \to M$, $(q, m) \mapsto \tau(q, m) = q.m$ the action of $G$ on $M$. Then, for any fixed $q \in G$, $\tau_q \colon M \to M$ is a diffeomorphism of $M$. Recall that a metric tensor $g(.\,,.)$ on $M$ is said to be $G$-*invariant* if

$$g(X(m), Y(m)) = g(d_m\tau_q(X), d_m\tau_q(Y)),$$

for $q \in G$, and vector fields $X, Y$ on $M$. A scalar product $\langle .\,, . \rangle$ in $\mathfrak{g}$ is said to be $\mathrm{Ad}_H$-*invariant* if

$$\langle \mathrm{Ad}_h X, \mathrm{Ad}_h Y \rangle = \langle X, Y \rangle, \quad h \in H, \quad X, Y \in \mathfrak{g}.$$

Let $\pi$ denote the projection of $G$ on the coset manifold, i.e., $\pi \colon G \to G/H = M$, $g \mapsto g.o = m$ . If $e$ is the identity of $G$, then the map $\pi$ and the differential map

$$d_e\pi \colon T_eG = \mathfrak{g} \to T_oM \tag{14}$$

have the following properties, see [24, Chapter 11]:

*1.* $\pi \colon G \to G/H = M$ is a submersion, such that $d_e\pi(\mathfrak{h}) = \{0\} \subset T_oM$, and $d_e\pi \colon \mathfrak{p} \to T_oM$ is an isomorphism;

*2.* $d_e\pi$ makes one-to-one correspondence between $Ad_H$-invariant scalar products on $\mathfrak{p}$ and $G$-invariant metrics on $M$.

**Definition 4.** *Let* $(M, g^M)$ *and* $(N, g^N)$ *be two semi-Riemannian manifolds and* $\pi \colon N \to M$ *a submersion such that* $T_nN = \mathcal{V}_n \oplus \mathcal{H}_n$*, with* $\mathcal{V}_n = \ker(d_n\pi)$*. Then* $\pi$ *is called a semi-Riemannian submersion if* $\pi^{-1}(m)$ *is a Riemannian submanifold of* $N$ *and the direct sum* $\mathcal{V}_n \oplus \mathcal{H}_n$ *is orthogonal at each* $n \in N$*.*

Let $M = G/H$ be a semi-Riemannian symmetric space with $G$-invariant metric $g(.\,,.)$ corresponding to an $\mathrm{Ad}_H$-invariant scalar product $\langle .\,,.\rangle$ on $\mathfrak{p}$, as mentioned in property *2* before Definition 4. We extend $\langle .\,,.\rangle$ to the entire Lie algebra $\mathfrak{g}$ such that the direct sum $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{h}$ becomes orthogonal. We denote by $L_q h = qh$ the multiplication from the left on $G$. We then define the vertical left invariant distribution $\mathcal{V}$ by $\mathcal{V}_q = d_e L_q(\mathfrak{h})$ and the horizontal distribution $\mathcal{H}$ by $\mathcal{H}_q = d_e L_q(\mathfrak{p})$. We keep the notation $\langle .\,,.\rangle$ for the left-invariant $\mathrm{Ad}_H$-invariant metric on $G$ induced by the extended scalar product on $\mathfrak{g}$. Under these conditions the projection map $\pi\colon G \to M$ is a semi-Riemannian submersion.

We say that a vector field $X$ on $G$ is horizontal if $X(q) \in \mathcal{H}_q$ for any $q \in G$. An absolutely continuous curve $q\colon I \to G$ on $G$ is horizontal if $\dot{q}(t) \in \mathcal{H}_{q(t)}$ for almost every $t \in I$, or equivalently, if $q^{-1}(t)\dot{q}(t) \in \mathcal{H}_{q(0)}$.

For a vector field $Y$ on $M$ there is a horizontal vector field $\widetilde{Y}$ on $G$ such that $d_q\pi(\widetilde{Y}(q)) = Y_{\pi(q)}$. In particular, this implies that for any absolutely continuous curve $\alpha\colon I \to M$ there is a horizontal curve $q\colon I \to G$ such that $\pi(q(t)) = \alpha(t)$ and $d_{q(t)}\pi(\dot{q}(t)) = \dot{\alpha}(t)$ for almost every $t \in I$. We call $\widetilde{Y}$ and $q(t)$ the horizontal lifts of $Y$ and $\alpha(t)$, respectively. A horizontal lift $q(t)$ of a curve $\alpha(t)$ is unique, if we specify the initial value $q(0)$.

The following result will be useful later on.

**Lemma 1.** *Let $\pi\colon G \to M$ be a semi-Riemannian submersion onto a semi-Riemannian symmetric space as above. Let $\alpha\colon [0,T] \to M$ be an absolutely continuous curve and $Y$ be a vector field along $\alpha$. Let $q\colon [0,T] \to G$ be a horizontal lift of $\alpha$ and $\widetilde{Y}$ a horizontal lift of $Y$ along $q$. Then*

$$\frac{D^M_{\alpha(t)}}{dt} Y(t) = d_{q(t)}\pi\Big(\sum_{j=1}^{k} \frac{dy_j(t)}{dt} A_j\Big), \tag{15}$$

*where $d_{q(t)}L_{q^{-1}(t)}\widetilde{Y}(t) = \sum_{j=1}^{k} y_j(t)A_j$ is written in a left invariant basis $\{A_1, \dots, A_k\}$ of $\mathfrak{p}$.*

*Proof*: We denote by $\nabla^G$ the Levi-Civita connection on $G$. First we show that

$$\nabla^G_V W = \frac{1}{2}[V, W] \tag{16}$$

for left invariant vector fields $V, W \in \mathfrak{p}$. Since $\mathfrak{p}$ and the metric on $G$ are $\mathrm{Ad}_H$-invariant, then

$$\langle [V,Z], W\rangle = \langle V, [Z,W]\rangle, \quad V, W \in \mathfrak{p}, \quad Z \in \mathfrak{h}, \tag{17}$$

see, for instance [24, Lemma 3, Chapter 11]. Then for any $Z \in \mathfrak{g}$ and $V, W \in \mathfrak{p}$ we have

$$2\langle \nabla^G_V W, Z\rangle = -\langle V, [W,Z]\rangle + \langle W, [Z,V]\rangle + \langle Z, [V,W]\rangle \tag{18}$$

by Koszul formula. If $Z \in \mathfrak{h}$, then the first two terms on the right hand side are cancelled by (17). If $Z \in \mathfrak{p}$, then the first two terms on the right-hand side vanish by $[\mathfrak{p},\mathfrak{p}] \in \mathfrak{h}$ and the orthogonality of $\mathfrak{p}$ and $\mathfrak{h}$. It shows (16).

Let $\pi\colon G \to M$ be a Riemannian submersion, $X$, $Y$ vector fields on $M$, and $\widetilde{X}$, $\widetilde{Y}$ their horizontal lifts to $G$. We denote $\mathrm{pr}_{\mathcal{H}_q}\colon T_qG \to \mathcal{H}_q$ the orthogonal projection onto a horizontal sub-bundle $\mathcal{H}$ at $q \in G$. We recall that the Levi-Civita connections $\nabla^M$ on $M$ and $\nabla^G$ on $G$ are related by

$$\nabla^M_X Y = d_q\pi(\mathrm{pr}_{\mathcal{H}_q} \nabla^G_{\widetilde{X}} \widetilde{Y}), \tag{19}$$

see [24, Lemma 45, Chapter 7]. We write the horizontal lifts $\widetilde{X}$ and $\widetilde{Y}$, in the left invariant basis $\{A_1, \dots, A_k\}$ of $\mathfrak{p}$ by

$$d_{q(t)}L_{q^{-1}(t)}\widetilde{X} = \sum_{i=1}^k x_i A_i, \quad d_{q(t)}L_{q^{-1}(t)}\widetilde{Y} = \sum_{j=1}^k y_j A_j$$

Then,

$$\begin{aligned}
\nabla^G_{\widetilde{X}}\widetilde{Y} &= \nabla^G_{\sum_{i=1}^k x_i A_i} \sum_{j=1}^k y_j A_j = \sum_{i,j=1}^k x_i \Big(\nabla^G_{A_i} y_j A_j\Big) \\
&= \sum_{i,j=1}^k x_i \Big(A_i(y_j)A_j + y_j \nabla^G_{A_i} A_j\Big) \\
&= \sum_{j=1}^k \Big(\sum_{i=1}^k (x_i A_i)(y_j)\Big) A_j + \sum_{i,j=1}^k x_i y_j \nabla^G_{A_i} A_j \\
&= \sum_{j=1}^k \Big(\sum_{i=1}^k (x_i A_i)(y_j)\Big) A_j + \frac{1}{2}\sum_{i,j=1}^k x_i y_j [A_i, A_j].
\end{aligned}$$

Since $M$ is a semi-Riemannian symmetric space we have $[A_i, A_j] \subset \mathfrak{h}$ and therefore

$$\mathrm{pr}_{\mathcal{H}_q} \nabla^G_{\widetilde{X}} \widetilde{Y} = \sum_{j=1}^{k} \Big( \sum_{i=1}^{k} (x_i(q) A_i)(y_j(q)) \Big) A_j.$$

Now, we set

$$d_{q(t)} L_{q^{-1}(t)} \widetilde{X}(t) = d_{q(t)} L_{q^{-1}(t)} \dot{q}(t) = \sum_{i=1}^{k} x_i(q(t)) A_i,$$

and obtain

$$\begin{aligned} \mathrm{pr}_{\mathcal{H}_{q(t)}} \frac{D^G_{\dot{q}(t)} \widetilde{Y}(t)}{dt} &= \mathrm{pr}_{\mathcal{H}_{q(t)}} \nabla^G_{\dot{q}(t)} \widetilde{Y}(t) = \sum_{j=1}^{k} \Big( \dot{q}(t)(y_j(q(t))) \Big) A_j \\ &= \sum_{j=1}^{k} \frac{d y_j(q(t))}{dt} A_j. \end{aligned}$$

We set $X(t) = \dot{\alpha}(t)$ in formula (19) and obtain (15). ∎

**4.2. Intrinsic rolling of symmetric spaces on flat manifolds.** The definition of a symmetric space is intimately related to the rolling on a flat space. We aim to construct an intrinsic rolling of a semi-Riemannian symmetric manifold $M$ on the tangent space $T_oM = \widehat{M}$. Namely, we will find the triplet $\big(\alpha(t), \widehat{\alpha}(t), A(t)\big)$ satisfying Definition 1 by using only the data of the symmetric manifold.

The main properties that result from assuming that $M$ is a symmetric space can be summarised in the following commutative diagrams,

$$\begin{array}{ccc} G & \xrightarrow{\pi} & M \\ {\scriptstyle L_q}\downarrow & & \downarrow{\scriptstyle \tau_q} \\ G & \xrightarrow[\pi]{} & M \end{array} \qquad\qquad \begin{array}{ccc} \mathfrak{g} = \mathfrak{p} \oplus \mathfrak{h} & \xrightarrow{d_e\pi} & T_oM \\ {\scriptstyle d_e(L_q)}\downarrow & & \downarrow{\scriptstyle d_o(\tau_q)} \\ T_qG = \mathcal{H}_q \oplus \mathcal{V}_q & \xrightarrow[d_q\pi]{} & T_{\tau_q(o)}M \end{array} \qquad (20)$$

Thus we conclude that

$$\pi(q) = \pi(L_q(e)) = \tau_q(\pi(e)) = \tau_q(o), \quad q \in G. \qquad (21)$$

We also recall that, $\forall q \in G$,

$$\mathcal{V}_q = \ker(d_q\pi), \quad \mathfrak{h} = \ker(d_e\pi),$$

and the map $d_q\pi\colon \mathcal{H}_q \to T_{\tau_q(o)}M$ is an isometry. Let $U \in \mathfrak{p}$. Then, from the second diagram in (20), it follows that

$$d_q\pi(d_eL_q(U)) = d_o\tau_q(d_e\pi(U)). \tag{22}$$

Now, choose an absolutely continuous curve $\alpha\colon [0,T] \to M$ such that $\alpha(0) = o$. Then, there exists a horizontal curve $q\colon [0,T] \to G$ that projects to $\alpha$. More precisely

L1 $\pi(q(t)) = \alpha(t) = \tau_{q(t)}(o)$. In the last equality we used (21);
L2 $d_{q(t)}\pi(\dot q(t)) = \dot\alpha(t)$;
L3 $\dot q(t) = d_eL_{q(t)}U(t)$, for some curve $U \in \mathfrak{p}$.

Combining $L3, L2$ and (22) we obtain

$$\begin{aligned}\dot\alpha(t) \;&\overset{L2}{=}\; d_{q(t)}\pi(\dot q(t)) \overset{L3}{=} d_{q(t)}\pi(d_eL_{q(t)}U(t))\\ &\overset{(22)}{=}\; d_0\tau_{q(t)}(d_e\pi(U(t))),\end{aligned} \tag{23}$$

and emphasize that both maps $d_{q(t)}\pi$ and $d_0\tau_{q(t)}$ are isometries between the corresponding spaces, for any $t \in [0,T]$.

Now we define the curve $\widehat\alpha(t) \in T_oM$. For the curve $U(t) = d_{q(t)}L_{q^{-1}(t)}(\dot q(t))$ on $\mathfrak{p}$, we write $d_e\pi(U(t)) \in T_oM$, and by solving the Cauchy problem

$$\begin{cases}\frac{d\widehat\alpha}{dt} = d_e\pi(U(t)) = \dot{\widehat\alpha}\\ \widehat\alpha(0) = 0,\end{cases} \tag{24}$$

we obtain $\widehat\alpha(t)$ Here we implicitly identified the vector space $T_oM$ and $T_{\widehat\alpha(t)}(T_oM)$ by an isometric orientation preserving map $j$.

We also define $A(t)\colon T_{\alpha(t)}M \to T_{\widehat\alpha(t)}(T_oM)$ as a composition of the maps

$$T_{\alpha(t)}M \overset{L1}{=} T_{\tau_{q(t)}(o)}M \xrightarrow[\text{by (20)}]{(d_0\tau_{q(t)})^{-1}} T_oM \xrightarrow{j} T_{\widehat\alpha(t)}(T_oM). \tag{25}$$

Note that, by the commutative diagram (20), the map $A(t)$ can be defined alternatively by the composition of the following isometric maps,

$$T_{\alpha(t)}M \xrightarrow{(d_{q(t)}\pi)^{-1}} \mathcal{H}_{q(t)} \xrightarrow{d_{q(t)}L_{q^{-1}(t)}} \mathfrak{p} \xrightarrow{d_e\pi} T_oM \xrightarrow{j} T_{\widehat\alpha(t)}(T_oM). \tag{26}$$

**Proposition 6.** *If $\alpha(t)$, $\widehat\alpha(t)$ and $A(t)\colon T_{\alpha(t)}M \to T_{\widehat\alpha(t)}(\widehat M)$ are defined as in Section 4.2, with $\widehat M = T_oM$, the triple $(\alpha(t), \widehat\alpha(t), A(t))$ is a rolling curve for the intrinsic rolling of the manifold $M$ on $\widehat M$, i.e., it satisfies conditions in Definition 1.*

*Proof*: By the construction of $A$, the condition (3) in the intrinsic rolling for the isometry $A(t)\colon T_{\alpha(t)}M \to T_{\widehat{\alpha}(t)}(T_oM)$ is fulfilled. We need to verify that a parallel vector field $Y$ along $\alpha$ is mapped to a parallel vector field $\widehat{Y}$ along $\widehat{\alpha}$.

Let $Y$ be a vector field along $\alpha(t) = \tau_{q(t)}(o)$, where $q(t)$ is a horizontal lift of $\alpha(t)$. Then $\widehat{Y} = A(Y)$ is given by

$$\widehat{Y}(t) = d_e\pi \circ d_{q(t)}L_{q^{-1}(t)}(\widetilde{Y}(t)) = d_e\pi\Big(\sum_{j=1}^{k}\widetilde{y}(t)A_j\Big), \tag{27}$$

where we used (26). We also denoted by $\widetilde{Y}(t) \in \mathcal{H}_{q(t)}$ the horizontal lift of $Y(t)$, and write $d_{q(t)}L_{q^{-1}(t)}\widetilde{Y}(t) = \sum_{j=1}^{k}\widetilde{y}(t)A_j$, where $\{A_1, \dots, A_k\}$ is a basis for $\mathfrak{p}$. Assume that $Y(t)$ is a parallel vector field along $\alpha$. Then, using the identity (15) in Lemma 1, $d_{q(t)}\pi\Big(\sum_{j=1}^{k}\frac{d\widetilde{y}(t)}{dt}A_j\Big) = 0$. Since $d_{q(t)}\pi\colon \mathcal{H}_{q(t)} \to T_{\alpha(t)}M$ is a bijection, we conclude that $\frac{d\widetilde{y}_j(t)}{dt} = 0$ for all $j = 1, \dots, k$. Then, (27) shows that

$$\frac{d\widehat{Y}(t)}{dt} = d_e\pi\Big(\sum_{j=1}^{k}\frac{d\widetilde{y}(t)}{dt}A_j\Big) = 0.$$

Thus $\widehat{Y}$ is a parallel vector field along $\widehat{\alpha}$ on $T_oM$. ∎

**4.3. Extrinsic rolling of symmetric spaces on flat manifolds.** In the present section we describe the rolling of a semi-Riemannian symmetric manifold $M$ on the flat manifold which is the affine tangent space $\widehat{M} = T_o^{\mathrm{aff}}M$ at $o \in M$. We will connect the intrinsic rolling, described in Section 4.2 to the extrinsic rolling, by choosing an isometric embedding into a vector space $V$. Let $M = G/H$ be a semi-Riemannian symmetric manifold. Let

$$\iota\colon M \to V, \quad \overline{M} = \iota(M) \tag{28}$$

be an isometric embedding. In the present section all the objects related to $V$ will be marked by a line on top, like the image $\overline{M} = \iota(M) \subset V$ of $M$ in $V$, or $\overline{o} = \iota(o)$ the image of the isotropy point in $V$. The map $d_o\iota$ is a linear isometry and

$$d_o\iota(T_oM) = T_{\iota(o)}\overline{M} = T_{\overline{o}}\overline{M} \subset V. \tag{29}$$

We define

$$\overline{\widehat{M}} := T_{\overline{o}}^{\mathrm{aff}}\overline{M} = \overline{o} + T_{\overline{o}}\overline{M}. \tag{30}$$

Note that diagram (20) implies that any $\overline{W} \in T_{\overline{o}}\overline{M}$ can be written as

$$\overline{W} = d_o\iota(W) = d_o\iota(d_e\pi(U)), \quad \text{where} \quad W \in T_oM,\ U \in \mathfrak{p}. \tag{31}$$

We assume that $\rho\colon G \to \mathrm{GL}(V)$ is a representation of $G$ on $V$, and define

$$\overline{G} := \rho(G). \tag{32}$$

The action of $\overline{G}$ on $\overline{M}$ is denoted by $\overline{q}.\overline{m}$, with $\overline{q} = \rho(q) \in \overline{G}$ and $\overline{m} \in V$, to emphasize that the group $\overline{G}$ acts on both $\overline{M}$ and $\widehat{\overline{M}}$ as it does on vectors in $V$. We keep writing $\tau_q$ for the action of $q \in G$ on $M$. Moreover, we assume that the imbedding map $\iota$ is *equivariant* under these actions, i.e.,

$$\iota(\tau_q(m)) = \overline{q}.(\iota(m)) = \overline{q}.\overline{m}. \tag{33}$$

We know that $G$ acts on the symmetric space $M$ by isometries and $\iota : M \to V$ is an isometric embedding. So, since the metric on $\overline{M}$ is the restriction of the metric on $V$, $\overline{G} = \rho(G)$ must preserve the metric on $V$. As a consequence, $\overline{G} \subset \mathrm{SO}(V)$.

The group representation $\rho$ induces the Lie algebra representation $d_e\rho$ that maps $A \in \mathfrak{g}$ to $\bar{A} \in \bar{\mathfrak{g}} \subset \mathfrak{so}(V)$. Let $(.\,,.)$ denote the scalar product on $\bar{\mathfrak{g}}$, defined by

$$(\bar{A}, \bar{B}) := \langle A, B\rangle.$$

Then $\bar{\mathfrak{p}} = d_e\rho(\mathfrak{p})$ is the orthogonal complement to $\bar{\mathfrak{h}} = d_e\rho(\mathfrak{h})$ in $\bar{\mathfrak{g}}$ relative to $(.\,,.)$, and $\bar{\mathfrak{g}} = \bar{\mathfrak{h}} \oplus \bar{\mathfrak{p}}$ is a Cartan decomposition.

Define the map $\mathbb{P} : \overline{G} \to \overline{M}$ by

$$\mathbb{P}(\rho(q)) = \iota(\pi(q)). \tag{34}$$

The map $\mathbb{P}$ is smooth, as a composition of smooth maps.

Differentiating (33), we get that the lower part of the following diagram commutes, while differentiating (34) we also get that the upper part of this diagram commutes.

$$\begin{tikzcd}
\mathfrak{p} \arrow[r, "d_e\rho"] \arrow[d, "d_e\pi"'] & \bar{\mathfrak{p}} \arrow[d, "p = d_e\mathbb{P}"] \\
T_oM \arrow[r, "d_o\iota"] \arrow[d, "d_o\tau_q"'] & T_{\overline{o}}\overline{M} \arrow[d, "d_{\overline{o}}\overline{q}"] \\
T_{\tau_q(o)}M \arrow[r, "d_{\tau_q(o)}\iota"] & T_{\overline{q}.\overline{o}}\overline{M}
\end{tikzcd} \tag{35}$$

Here all the linear maps are bijective isometries.

We take an absolutely continuous curve $\alpha\colon [0,T] \to M$, $\alpha(0) = o$, and its horizontal lift

$$q\colon [0,T] \to G, \quad d_{\dot{q}(t)} L_{q^{-1}(t)} \dot{q}(t) = U(t) \in \mathfrak{p}, \tag{36}$$

such that $\alpha(t) = \tau_{q(t)}(o)$. This, together with the equivariance of $\iota$ given by (33), imply that

$$\overline{\alpha}(t) = \iota(\alpha(t)) = \iota(\tau_{q(t)}(o)) = \overline{q}(t).\overline{o}, \tag{37}$$

where $\overline{q}\colon [0,T] \to \overline{G}$ is horizontal, i.e., $\bar{q}^{-1}\dot{\bar{q}} \in \bar{\mathfrak{p}}$. But then,

$$\begin{aligned} \dot{\overline{\alpha}}(t) &= d_{\alpha(t)}\iota(\dot{\alpha}(t)) = d_{\alpha(t)}\iota \circ d_o\tau_q(t)(d_e\pi(U)) \\ &= d_{\overline{o}}\overline{q}(t)(p \circ d_e\rho(U)) = d_{\overline{o}}\overline{q}(t)(\overline{W}) \end{aligned} \tag{38}$$

where $U \in \mathfrak{p}$ from (36), and $\overline{W} := p \circ d_e\rho(U) \in T_{\overline{o}}\overline{M}$, by diagram (35). Since $\overline{q}(t)$ is a linear map, then $d_{\overline{o}}\overline{q}(t) = \overline{q}(t)$ and we simply write $\overline{W}(t) = \overline{q}(t)^{-1}.\,\dot{\overline{\alpha}}(t)$ for $\overline{q}(t)$ from (37).

We are ready to define $\widehat{\overline{\alpha}} \in \widehat{\overline{M}} = T^{\mathrm{aff}}_{\overline{o}}\overline{M}$. For that we find a curve $\overline{s}\colon [0,T] \to T_{\overline{o}}\overline{M}$ as the solution of the Cauchy problem

$$\begin{cases} \dot{\overline{s}}(t) = \overline{W}(t) = \overline{q}(t)^{-1}.\dot{\overline{\alpha}}(t) \\ \overline{s}(0) = 0 \end{cases}, \tag{39}$$

and set $\widehat{\overline{\alpha}}(t) = \overline{o} + \overline{s}(t) \in T^{\mathrm{aff}}_{\overline{o}}\overline{M}$.

**Proposition 7.** *In the notation of Section 4.3, there is $\overline{R}\colon [0,T] \to \mathrm{SO}(V)$ such that $\overline{g}(t) = (\overline{R}(t), \overline{s}(t)) \in \mathrm{SE}(V)$ is a rolling map that rolls the curve $\overline{\alpha}(t) \in \overline{M}$ onto the curve $\widehat{\overline{\alpha}}(t) \in \widehat{\overline{M}} = T^{\mathit{aff}}_{\overline{o}}\overline{M}$, where $\widehat{\overline{\alpha}}(t) = \overline{o} + \overline{s}(t)$.*

We emphasise that for the rolling map $\overline{g}(t)$ in the statement of this proposition, one has the freedom to define $\overline{R}(t)|_{T^{\perp}\overline{M}}$ such that $g(t)$ satisfies the normal no-twist condition. The no-slip and tangent no-twist conditions are determined by the intrinsic rolling, the latter being the nature of symmetric spaces.

*Proof*: The proof is constructive. Since $\overline{s}$ is the solution of (39), it is enough to define $\overline{R}(t) \in \mathrm{SO}(V)$ so that $\overline{g}(t) = (\overline{R}(t), \overline{s}(t)) \in \mathrm{SE}(V)$ satisfies the conditions in Definition 2.

Let $\overline{R}(t)$ be such that

$$d_{\overline{\alpha}(t)}\overline{R}(t)|_{T_{\overline{\alpha}(t)}\overline{M}}(\overline{X}(t)) = \overline{q}(t)^{-1}.\,\overline{X}(t), \tag{40}$$

for any tangent vector field $\overline{X}(t)$ along $\overline{\alpha}(t)$ and $\overline{R}(0) = e$.
Using (38) and (39), we see that $g(t)$ satisfies the no-slip condition

$$d_{\overline{\alpha}(t)}\overline{g}(t)(\dot{\overline{\alpha}}(t)) = \overline{q}(t)^{-1}.\dot{\overline{\alpha}}(t) = \dot{\overline{s}}(t) = \dot{\overline{\widehat{\alpha}}}(t).$$

Now, we show that if $\overline{R}(t)$ is defined as above, then $\overline{g}(t) = (\overline{R}(t), \overline{s}(t))$ satisfies the tangent no-twist condition given in Proposition 2. Let $\overline{X}(t)$ be a tangent parallel vector field along $\overline{\alpha}(t)$. Notice that, from the bottom of diagram (35), $d_\alpha \iota$ is a bijective isometry between $T_\alpha M$ and $T_{\overline{\alpha}}\overline{M}$. Since, according to [24, Proposition 3.59], the covariant derivative on $M$ is the pullback of the covariant derivative on $\overline{M}$ under the isometries, the vector field $X(t) = \big(d_{\alpha(t)}\iota\big)^{-1}(\overline{X}(t))$ is parallel along $\alpha(t)$ on $M$.

Moreover, diagram (35) shows that the parallel vector field $X(t)$ is mapped to the parallel tangent vector field $\overline{\overline{X}}(t)$ along $\overline{s}(t)$ on $T_{\overline{o}}\overline{M}$ due to Lemma 1 and the isometric embedding. As a consequence, the vector field $\overline{\widehat{X}}(t)$ along $\overline{\widehat{\alpha}}$ on $\overline{\widehat{M}} = T^{\mathrm{aff}}_{\overline{o}}\overline{M}$ is parallel.

As was mentioned earlier on, the condition (40) on $\overline{R}(t)$ still leaves freedom on how $d_{\overline{\alpha}(t)}\overline{R}(t)$ acts on the normal space $T^{\perp}_{\overline{\alpha}(t)}\overline{M}$. In order to guarantee that $\overline{g}(t) = (\overline{R}(t), \overline{s}(t)) \subset \mathrm{SE}(V)$ also satisfies the normal no-twist condition, we define the (unique) map $\overline{R}(t)$ along $\overline{\alpha}$ on $\overline{M}$ such that the differential $d_{\overline{\alpha}(t)}\overline{R}(t)|_{T^{\perp}_{\overline{\alpha}(t)}\overline{M}}$ maps the normal parallel vector fields along $\overline{\alpha}$ to the normal parallel vector fields along $\overline{\widehat{\alpha}}$ on $\overline{\widehat{M}} = T^{\mathrm{aff}}_{\overline{o}}\overline{M}$. ■

**Remark 6.** *In relation to the last part of the proof of Proposition 7, we point out that in* [8, Section 3.3] *a complete answer was given to the problem of extending intrinsic rollings to extrinsic ones. We also refer to* [18] *for non-twist conditions in the case of embedded sub-Euclidean manifolds.*

**Corollary 1.** *If $\overline{M}$ has co-dimension 1, let $\overline{\alpha}(t) = \overline{q}(t).\overline{o}$ be a curve in $\overline{M}$, satisfying $\overline{\alpha}(0) = \overline{o}$, where $\overline{q}(t)$ is a horizontal curve in $\overline{G}$ and $\dot{\overline{q}} = \overline{q}\,.\overline{U}(t)$. Then, $(\overline{R}(t), \overline{s}(t)) \in \mathrm{SE}(V)$ is a rolling map of $\overline{M}$ on $\overline{\widehat{M}}$ along $\overline{\alpha}(t)$, with development $\overline{\widehat{\alpha}}(t) = \overline{s}(t) + \overline{o}$, where $\overline{R}(t) = \overline{q}(t)^{-1}$ and $\overline{s}(t)$ satisfies the Cauchy problem $\dot{\overline{s}}(t) = \overline{U}(t).\overline{o}$, $\overline{s}(0) = 0$. Moreover,*

$$\begin{cases} \dot{\overline{R}}(t) = -\overline{U}(t).\overline{R}(t) \\ \dot{\overline{s}}(t) = \overline{U}(t).\overline{o} \end{cases}, \tag{41}$$

*are the corresponding kinematic equations.*

*Proof*: For manifolds of co-dimension 1, the normal non-twist condition is always satisfied. So, taking into account (40) and Remark 3, we have $\overline{R}(t) = \overline{q}(t)^{-1}$. So, $\dot{\overline{R}}(t) = -\overline{U}(t)\overline{R}(t)$. According to (39), $\dot{\overline{s}}(t) = \overline{q}(t)^{-1}.\dot{\overline{\alpha}}(t)$. But $\dot{\overline{\alpha}}(t) = \dot{\overline{q}}(t).\overline{o}$, so it follows that $\dot{\overline{s}}(t) = \overline{q}(t)^{-1}\dot{\overline{q}}(t).\overline{o} = \overline{U}(t).\overline{o}$. ■

**4.4. Examples.** We will exemplify the results of Sections 4.2 and 4.3.

**4.4.1.** *Rolling the* 2*-dimensional hyperbolic space.* First we describe the hyperbolic disc as a symmetric manifold, and construct the intrinsic and extrinsic rolling on the corresponding flat spaces. We refer to [19] for more details about hyperbolic spaces and the relationship between two of its equivalent models, which will be used in this section.

Let $\mathcal{D}$ be the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ in $\mathbb{R}^2$, with the hyperbolic metric given in coordinates $(x_1, x_2)$ by $h^2 = 4\frac{(dx_1)^2+(dx_2)^2}{(1-(x_1^2+x_2^2))^2}$. $\mathcal{D}$ is also known as the Poincaré ball model. The Lie group

$$G = SU(1,1) = \left\{ g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : \ a, b \in \mathbb{C}, \ |a|^2 - |b|^2 = 1 \right\}$$

acts transitively on $\mathcal{D}$ via the Möebius transformations, i.e.,

$$\tau_g(z) = \frac{az+b}{\bar{b}z+\bar{a}}, \quad \tau_g(0) = \frac{b}{\bar{a}}.$$

Let $H := \left\{ \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}, |a|^2 = 1 \right\}$ be the isotropy subgroup of $0 \in \mathcal{D}$. The projection map is

$$\begin{array}{rccl} \pi : & G & \to & \mathcal{D} = G/H \\ & g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} & \mapsto & \tau_g(0) = \frac{b}{\bar{a}} \end{array} \tag{42}$$

The Lie algebra $\mathfrak{g}$ of $G$ is given by

$$\mathfrak{g} = su(1,1) = \left\{ \begin{pmatrix} iv & u_1 + iu_2 \\ u_1 - iu_2 & -iv \end{pmatrix} : \ v, u_1, u_2 \in \mathbb{R} \right\}.$$

We endow $\mathfrak{g}$ with an $Ad_G$-invariant semi-Riemannian metric defined by

$$\langle X, Y \rangle = 2\,\mathrm{tr}(XY) = \frac{1}{2}B(X,Y),$$

$B(.,.)$ being the Killing form. The matrices

$$A_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, A_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, A_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \tag{43}$$

form an orthonormal basis of $\mathfrak{g}$.

The Lie algebra $\mathfrak{h}$ of the isotropy subgroup $H$ is spanned by $A_1$ and its orthogonal complement $\mathfrak{p}$ is spanned by $A_2$ and $A_3$. Note that the restriction of $\langle . , . \rangle$ to $\mathfrak{p}$ is positive definite. From the commutation relations, we conclude that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$.

A curve $z(t)$ in $\mathcal{D}$ lifts to a horizontal curve

$$g(t) = \frac{1}{\sqrt{1 - |z(t)|^2}} \begin{pmatrix} 1 & z(t) \\ \bar{z}(t) & 1 \end{pmatrix} e^{\theta(t) A_1} \in G,$$

when $\dot{\theta} = \frac{-2}{1-|z|^2}(x_1\dot{x}_2 - \dot{x}_1 x_2)$. In such case,

$$g^{-1}\dot{g} = \frac{1}{1 - |z|^2} \begin{pmatrix} 0 & \dot{z}\, e^{-i\theta} \\ \dot{\bar{z}}\, e^{i\theta} & 0 \end{pmatrix}. \tag{44}$$

The proof of these two facts regarding lifts of curves can be found in [15, pages 97, 98], modulo minor obvious missprints.

• Intrinsic rolling of $\mathcal{D}$ on $T_0\mathcal{D}$.

We are now in conditions to apply the theory developed at the beginning of this section for the intrinsic rolling of $M = \mathcal{D}$ on $\widehat{M} = T_0\mathcal{D}$. Let $\alpha(t)$ be a curve in $\mathcal{D}$ satisfying $\alpha(0) = 0$. Define $u(t) := \dfrac{\dot{\alpha}(t)\, e^{-i\theta(t)}}{1 - |\alpha(t)|^2}$, so that the horizontal lift of $\alpha$ to $G$ satisfies $g^{-1}\dot{g} = \begin{pmatrix} 0 & u(t) \\ \overline{u}(t) & 0 \end{pmatrix} =: U(t),\ g(0) = I$. Notice that $\alpha(t) = \tau_{g(t)}(0) = \frac{b(t)}{\bar{a}(t)}$. According to (24) the curve $\widehat{\alpha}(t)$ is the solution of the initial value problem $\dot{\widehat{\alpha}}(t) = d_e\pi(U(t)) = u(t),\ \widehat{\alpha}(0) = 0$.

The isometry $A\colon T_{\alpha(t)}M \to T_{\hat{\alpha}(t)}\widehat{M}$ is obtained explicitly using (25) and it is given by.

$$A(t)v(t) = \left(d_o\tau_{g(t)}\right)^{-1} = \overline{a}(t)^2 v(t), \quad v(t) \in T_{\alpha(t)}M.$$

• Extrinsic rolling of $\mathcal{D}$ on $T_0^{\mathrm{aff}}\mathcal{D}$

The rollings of $\mathcal{D}$ can be also represented "extrinsically" after embedding $\mathcal{D}$ in a vector space and defining an appropriate representation of $G$. Here, we consider the embedding of $\mathcal{D}$ in $V = \mathbb{R}^{1,2}$, which is $\mathbb{R}^3$ equipped with the Minkowski metric $dm^2 = -(dx_1)^2 + (dx_2)^2 + (dx_3)^2$. $V$ is isometric to

$(su(1,1), \langle .,. \rangle)$. The isometric diffeomorphism

$$\begin{array}{llll} \iota: & \mathcal{D} & \to & \mathbb{R}^{1,2} \\ & z = x_2 + ix_3 & \mapsto & \iota(z) = \left(\frac{1+|z|^2}{1-|z|^2}, \frac{2x_3}{1-|z|^2}, \frac{-2x_2}{1-|z|^2}\right) \end{array} \tag{45}$$

is obtained via the hyperbolic stereographic projection, through the point $(-1,0,0)$, and an appropriate change of coordinates. Then

$$\iota(\mathcal{D}) = \mathcal{H}^2 = \{(x_1,x_2,x_3) : \, x_1^2 = 1 + x_2^2 + x_3^2, \, x_1 > 0\}. \tag{46}$$

Now define $\overline{G} = \mathrm{Ad}_G$. It is known that $\overline{G} \subset \mathrm{SO}(V) = \mathrm{SO}^+(1,2)$,[9, 24], which is the connected identity component of

$$\mathrm{SO}(1,2) = \{X \in \mathrm{SL}(3, \mathbb{R} : X^T I_{1,2} X = I_{1,2}\},$$

where

$$I_{1,2} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Indeed, calculating $gA_jg^{-1}$, $j = 1,2,3$, with $g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$, $|a|^2 - |b|^2 = 1$, we obtain

$$Ad_g = \left[\begin{array}{c|cc} |a|^2 + |b|^2 & 2Im(\bar{a}b) & -2Re(a\bar{b}) \\ \hline 2Im(ab) & Re(a^2 - b^2) & -Im(a^2 + b^2) \\ -2Re(ab) & Im(a^2 - b^2) & Re(a^2 + b^2) \end{array}\right]. \tag{47}$$

It can be shown that $\mathrm{Ad}_g \, I_{1,2} \, \mathrm{Ad}_g = I_{1,2}$, for all $g \in G$, and the determinant of the diagonal blocks is in both cases equal to $|a|^2 + |b|^2 > 0$, so $\mathrm{Ad}_g \in \mathrm{SO}^+(1,2)$. It follows that $\dot{g}(0) \mapsto ad_{\dot{g}(0)}$ defines is a Lie algebra isomorphism $d_e\rho$, between $\mathfrak{g} = su(1,1)$ and $\bar{\mathfrak{g}} = \mathfrak{so}(1,2)$. Since $[A_1, A_2] = A_3$, $[A_1, A_3] = -A_2$, and $[A_2, A_3] = -A_1$, an easy calculation yields

$$A = \frac{1}{2}\begin{pmatrix} iv & u \\ \bar{u} & -iv \end{pmatrix} \mapsto ad_A = \begin{pmatrix} 0 & u_2 & -u_1 \\ u_2 & 0 & -v \\ -u_1 & v & 0 \end{pmatrix}, u = u_1 + iu_2. \tag{48}$$

We also have the Cartan decomposition $\mathfrak{so}(1,2) = \overline{\mathfrak{h}} \oplus \overline{\mathfrak{p}}$, where

$$\overline{\mathfrak{h}} = \mathrm{span}\left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\},$$

$$\overline{\mathfrak{p}} = \operatorname{span}\left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$\overline{\mathfrak{h}}$ is the Lie algebra of the isotropy subgroup of $\mathrm{SO}(1,2)$ at the point $e_1$.

We also need to guarantee that the embedding $\iota$ is equivariant relative to $\overline{G}$, i.e., $\iota(\tau_g(z)) = \mathrm{Ad}_g(\iota(z))$, for every $z \in \mathcal{D}$ and $g \in G$. We first show that this identity is true for $z = 0$, and then use the transitive action of $G$ on $\mathcal{D}$ to prove the general case.

$$\begin{aligned} \iota(\tau_g(0)) &= \iota(\tfrac{b}{\overline{a}}) = \iota(\tfrac{1}{|a|^2} Re(ab), \tfrac{1}{|a|^2} Im(ab)) \\ &= \big(|a|^2 + |b|^2, 2Im(ab), -2Re(ab)\big) \\ &= \mathrm{Ad}_g(e_1) = \mathrm{Ad}_g(\iota(0)). \end{aligned} \tag{49}$$

Now, let $h := \frac{1}{\sqrt{1-|z|^2}} \begin{pmatrix} 1 & z \\ \bar{z} & 1 \end{pmatrix} \in \mathrm{SU}(1,1)$, so that $z = \tau_h(0)$. Using this and the identity (49), we can write, for each $g \in G$ and $z \in \mathcal{D}$,

$$\begin{aligned} \iota(\tau_g(z)) &= \iota(\tau_g(\tau_h(0))) = \iota(\tau_{gh}(0)) = \mathrm{Ad}_{gh}(\iota(0)) \\ &= \mathrm{Ad}_g(\mathrm{Ad}_h(\iota(0))) = \mathrm{Ad}_g(\iota(\tau_h(0)) = \mathrm{Ad}_g(\iota(z)). \end{aligned} \tag{50}$$

We are finally in conditions to deal with the extrinsic rolling of the hyperboloid $\mathcal{H}$ on its affine tangent space at $e_1$, resulting from the action of $\mathrm{SO}^+(1,2)$. Since $\mathcal{H}^2$ is co-dimension 1, Corollary 1 applies and $(\overline{R}(t), \overline{s}(t))$ is a rolling map along the curve $\overline{\alpha}(t) = \overline{g}(t)e_1$. The kinematic equations for the extrinsic rolling of $\mathcal{H}^2$ on $T^{\mathrm{aff}}_{e_1}\mathcal{H}$ are,

$$\left\{ \begin{array}{l} \dot{\overline{R}}(t) = -\overline{U}(t)\overline{R}(t) \\ \\ \dot{\overline{s}}(t) = \overline{U}(t)e_1 \end{array} \right., \qquad \overline{U} = \begin{pmatrix} 0 & u_1 & u_2 \\ u_1 & 0 & 0 \\ u_2 & 0 & 0 \end{pmatrix}. \tag{51}$$

This agrees with the results reported in [16].

**4.4.2.** *The projective complex plane and the Riemann sphere.* This is another example where the natural geometry on $M$ is induced by the structure of $G$. The rollings of the projective space $\mathbb{C}\mathcal{P}^1$, identified with the extended complex plane $\mathbb{C} \cup \infty$, on its tangent planes can be obtained essentially in the same way as in the case of the Poincaré disk, with obvious adaptations. For this reason we omit certain details here.

Consider the projective plane $M = \mathbb{C}\mathcal{P}^1$ with the elliptic metric given in coordinates $(x_1, x_2)$ by $l^2 = 4\frac{(dx_1)^2+(dx_2)^2}{(1+(x_1^2+x_2^2))^2}$. The Lie group $G = \mathrm{SU}(2)$ acts transitively on $M$. The isotropy group of the origin $z = 0$ is $H = \{\begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}, |a| = 1\}$. We endow $\mathfrak{g} = \mathfrak{su}(2)$ with the metric defined by $\langle X, Y\rangle = -2\,\mathrm{tr}(XY)$. Relative to this metric, the matrices

$$A_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, A_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, A_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{52}$$

form an orthonormal basis of $\mathfrak{g}$. The Lie algebra $\mathfrak{h}$ and the complementary space $\mathfrak{p}$ are given by

$$\mathfrak{h} = \{\frac{1}{2}\begin{pmatrix} iv & 0 \\ 0 & -iv \end{pmatrix} : v \in \mathbb{R}\}, \quad \mathfrak{p} = \{\frac{1}{2}\begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix}, u \in \mathbb{C}\}.$$

The horizontal lift of a curve $\alpha(t) = x_1 + ix_2$ in $\mathbb{C}\mathcal{P}^1$ to $\mathrm{SU}(2)$ is given by $g(t) = \frac{1}{\sqrt{1+|\alpha(t)|^2}}\begin{pmatrix} 1 & \alpha(t) \\ -\bar{\alpha}(t) & 1 \end{pmatrix} e^{\theta(t)A_1}$, with $\theta$ being a solution of $\dot{\theta} = \frac{2}{1+|\alpha|^2}(x_1\dot{x}_2 - \dot{x}_1 x_2)$.

• Intrinsic rolling of $\mathbb{C}\mathcal{P}^1$ on $T_0\mathbb{C}\mathcal{P}^1$

We are ready to deal with the intrinsic rolling of $M = \mathbb{C}\mathcal{P}^1$ on its tangent space at $z = 0$. Let $\alpha(t)$ be a curve in $\mathbb{C}\mathcal{P}^1$ satisfying $\alpha(0) = 0$, and define $u(t) := \dfrac{\dot{\alpha}(t)\, e^{-i\theta(t)}}{1 + |\alpha(t)|^2}$, so that the horizontal lift $g(t) \in \mathrm{SU}(2)$ of $\alpha$ satisfies $g^{-1}\dot{g} = \begin{pmatrix} 0 & u(t) \\ -\overline{u}(t) & 0 \end{pmatrix} =: U(t) \in \mathfrak{p}$, $g(0) = I$. So, according to Proposition 6, the curve $\alpha(t)$ in $\mathbb{C}\mathcal{P}^1$ rolls on the curve $\widehat{\alpha}(t)$ in $T_0\mathbb{C}\mathcal{P}^1$ which is the solution of $\dot{\widehat{\alpha}}(t) = u(t)$, $\widehat{\alpha}(0) = 0$.

The isometry (that preserves the elliptic metric) is given explicitly by

$$A = (d_0\tau_{g(t)})^{-1} : \begin{array}{ccc} T_{\alpha(t)}M & \to & T_{\widehat{\alpha}(t)}\widehat{M} \\ v(t) & \mapsto & v(t)\overline{a}(t)^2 \end{array}.$$

So, $(\alpha, \widehat{\alpha}, A)$ is a rolling curve for the intrinsic rolling of $\mathbb{C}\mathcal{P}^1$ on its tangent space at 0.

• Extrinsic rolling of $\mathbb{C}\mathcal{P}^1$ on $T_0^{\mathrm{aff}}\mathbb{C}\mathcal{P}^1$

For the extrinsic rolling, we embed $\mathbb{C}\mathcal{P}^1$ in the 3-dimensional Euclidean space, through the passage to the Riemann sphere $S^2$ via the inverse of the stereographic projection and a change of coordinates. This isometric embedding is defined by

$$\begin{array}{rccl} \iota: & \mathbb{C}\mathcal{P}^1 & \to & \mathbb{R}^3 \\ & z = x+iy & \mapsto & \iota(z) = (\dfrac{-2x}{1+|z|^2}, \dfrac{|z|^2-1}{1+|z|^2}, \dfrac{-2y}{1+|z|^2}) \end{array}. \tag{53}$$

Clearly $\iota(\mathbb{C}\mathcal{P}^1) = S^2$, and $\infty$ is mapped to the north pole of $S^2$.
In this case

$$\mathrm{Ad}_g = \left[\begin{array}{ccc} |a|^2-|b|^2 & -2Im(\bar{a}b) & 2Re(\bar{a}b) \\ 2Im(ab) & Re(a^2+b^2) & -Im(a^2-b^2) \\ -2Re(ab) & Im(a^2+b^2) & Re(a^2-b^2) \end{array}\right] \in \mathrm{SO}(3), \tag{54}$$

so, we define $\rho(\mathrm{SU}(2)) = \overline{G} = \mathrm{Ad}_G = \mathrm{SO}(3)$. The Lie algebra isomorphism $d_e\rho\colon \mathfrak{su}(2) \to \mathfrak{so}(3,\mathbb{R})$ is defined by

$$A = \frac{1}{2}\begin{pmatrix} iu_1 & u_2+iu_3 \\ -u_2+iu_3 & -iu_1 \end{pmatrix} \mapsto ad_A = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}. \tag{55}$$

Clearly, $\overline{\mathfrak{p}} = span\left\{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\right\}$. Since the embedding defined in (53) is equivariant relative to the adjoint group SO(3), we can finally apply Corollary 1 to obtain the extrinsic rolling of the Riemann sphere on its affine tangent space at the south pole $-e_3$, along the curve $\overline{\alpha}(t) = \overline{g}(t)(-e_3)$, where $\overline{g}(t)$ is horizontal. Assume that

$$\overline{g}(t)^{-1}\dot{\overline{g}}(t) = \overline{U}(t) = \begin{pmatrix} 0 & 0 & u_1(t) \\ 0 & 0 & u_2(t) \\ -u_1(t) & -u_2(t) & 0 \end{pmatrix}$$

Then, the kinematic equations are:

$$\left\{\begin{array}{rcl} \dot{\overline{R}}(t) & = & -\overline{U}(t)\overline{R}(t) \\ \\ \dot{\overline{s}}(t) & = & -\overline{U}(t)\, e_3 \end{array}\right., \tag{56}$$

with $\overline{U}$ as above. These equations are the same as the equations for the ball-plate problem [13], or the equations for the sphere rolling on a plane [12, 14].

**4.4.3.** *Rolling semi-Riemannian orthogonal groups.* Here we consider $M$ to be the connected component containing the identity of the semi-Riemannian orthogonal group $O(p, n-p)$, $1 \le p \le n-1$, consisting of invertible $n \times n$ real matrices $P$, satisfying $P^J P = I_n$, with $J = \mathrm{diag}(I_p, -I_{n-p})$, and $P^J := J^T P^T J$. The Lie algebra of $O(p, n-p)$, denoted by $\mathfrak{so}(p, n-p)$, consists of $n \times n$ matrices $B$ satisfying $B^J = -B$. If we consider $P \in O(p, n-p)$ partitioned as $P = \left[\begin{array}{c|c} P_1 & P_2 \\ \hline P_3 & P_4 \end{array}\right]$, where $P_1$ is $p \times p$, then

$$\begin{array}{rcl} M & = & \mathrm{SO}^+(p, n-p) \\ & = & \{P \in O(p, n-p) : \det(P) = 1, \det(P_1) > 0, \det(P_2) > 0\}\,. \end{array} \tag{57}$$

We consider $M$ equipped with the semi-Riemannian metric defined by

$$\langle B, C \rangle_J := \mathrm{tr}\,(B^J C). \tag{58}$$

Consider the Lie group $G := \mathrm{SO}^+(p, n-p) \times \mathrm{SO}^+(p, n-p)$, equipped with the natural semi-Riemannian metric induced by (58) on each component, which is bi-invariant. $G$ acts transitively on $M$ with action

$$\begin{array}{rccc} \tau : & G \times M & \to & M \\ & ((Q_1, Q_2), P) & \mapsto & Q_1 P Q_2^{-1} \end{array}. \tag{59}$$

Fixing a point $P_0 \in M$, the projection $\pi : G \to M$ maps $(Q_1, Q_2)$ to $Q_1 P_0 Q_2^{-1}$. The isotropy subgroup at $P_0$ is

$$H = \left\{(Q_1, Q_2) \in G : Q_1 P_0 Q_2^{-1} = P_0\right\}, \tag{60}$$

and $M = G/H$. Of course, the semi-Riemannian metric (58) on $M$ is also $Ad_H$-invariant. The Lie algebra $\mathfrak{g} = \mathfrak{so}(p, n-p) \oplus \mathfrak{so}(p, n-p)$ splits as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, where

$$\begin{array}{l} \mathfrak{h} = \left\{(B, P_0^{-1} B P_0) : B \in \mathfrak{so}(p, n-p)\right\} \\ \mathfrak{p} = \left\{(C, -P_0^{-1} C P_0) : C \in \mathfrak{so}(p, n-p)\right\} \end{array}, \tag{61}$$

and this orthogonal splitting satisfies (13).

• Intrinsic rolling of the manifold $M = \mathrm{SO}^+(p, n-p)$ on $\widehat{M} = T_{P_0}\,\mathrm{SO}^+(p, n-p)$.

We now apply the results obtained in Section 4.2 for the intrinsic rolling of $M = \mathrm{SO}^+(p, n-p)$ on its tangent space at the point $P_0$. Note that the differential of $\pi$ at $(e, e)$, the identity in $G$, is given by

$$\begin{array}{rccc} d_{(e,e)}\pi : & \mathfrak{g} & \to & T_{P_0} M \\ & (U_1, U_2) & \mapsto & U_1 P_0 - P_0 U_2 \end{array}, \tag{62}$$

and the kernel of $d_{(e,e)}\pi$ is $\mathfrak{h}$. So, $d_{(e,e)}\pi$ defines an isomorphism between $\mathfrak{p}$ and $T_{P_0}M$, mapping $(U, -P_0^{-1}UP_0)$ to $2UP_0$.

Let $\alpha(t)$ be a curve in $M$ satisfying $\alpha(0) = P_0$, and $Q(t)$ a horizontal lift of $\alpha(t)$ to $G$, i.e., $\pi(Q(t)) = \alpha(t)$ and $Q^{-1}\dot{Q} = (U(t), -P_0^{-1}U(t)P_0)$, for some curve $U(t) \in \mathfrak{so}(p, n-p)$. Then, according to (24), Section 4.2, the curve $\alpha(t) \in M$ rolls on the curve $\widehat{\alpha}(t) \in \widehat{M}$ defined by

$$\dot{\widehat{\alpha}}(t) = 2U(t)P_0, \qquad \widehat{\alpha}(0) = 0,$$

and the isometry $A(t)$ is defined in (25) as the inverse of $d_{P_0}\tau_{Q(t)}$. Since for $Q = (Q_1, Q_2)$,

$$\begin{array}{rccl} d_{P_0}\tau_Q: & T_{P_0}M & \to & T_{Q_1P_0Q_2^{-1}}M \\ & CP_0 & \mapsto & Q_1CP_0Q_2^{-1} = Q_1CQ_1^{-1}Q_1P_0Q_2^{-1} \end{array},$$

where $C \in \mathfrak{so}(p, n-p)$, with the identification of the vector spaces $T_{P_0}M$ and $T_{\widehat{\alpha}(t)}M(T_{P_0}M)$, we finally obtain

$$\begin{array}{rccl} A(t): & T_{\alpha(t)}M & \to & T_{\widehat{\alpha}(t)}\widehat{M} \\ & DQ_1P_0Q_2^{-1} & \mapsto & Q_2^{-1}DQ_2P_0, \quad D \in \mathfrak{so}(p, n-p) \end{array}.$$

In conclusion, the triple $(\alpha(t), \widehat{\alpha}(t), A(t))$ is a rolling curve in the sense of Definition 1.

• Extrinsic rolling of the manifold $M = \mathrm{SO}^+(p, n-p)$ on $\widehat{M} = T_{P_0}^{\mathrm{aff}}\mathrm{SO}^+(p, n-p)$.
We isometrically embed $M$ and $\widehat{M}$ on the semi-Euclidean vector space $V = (\mathfrak{gl}(p, n-p), \langle . , . \rangle_J)$, and identify $\iota(M)$ and $\iota(\widehat{M})$ with $M$ and $\widehat{M}$ respectively. In this case, also the representation $\rho$ of $G$ on $V$ is the identity map, so we can write everything in Proposition 7 without using overlines. The equivariance property (33) is also trivially satisfied, and the action of $G$ on $V$ is simply the extension of the action (59) from $M$ to the embedding space $V$. Notice that

$$\begin{aligned} T_{P_0}\mathrm{SO}^+(p, n-p) &= \{BP_0,\ B^J = -B\}, \\ T_{P_0}^{\perp}\mathrm{SO}^+(p, n-p) &= \{CP_0,\ C^J = C\}. \end{aligned} \tag{63}$$

We now find the rolling map $g(t) = (R(t), s(t)) \in SE(V)$ along the curve $\alpha(t) = Q_1(t)P_0Q_2^{-1}(t)$ in $M$, satisfying $\alpha(0) = P_0$ where $Q(t) = (Q_1(t), Q_2(t))$ is a horizontal lift of $\alpha(t)$ to $G$. So, $Q^{-1}\dot{Q} \in \mathfrak{p}$, that is,

$$Q^{-1}\dot{Q} = (Q_1^{-1}\dot{Q}_1, -P_O^{-1}Q_1^{-1}\dot{Q}_1P_0).$$

Defining $U := Q_1^{-1}\dot{Q}_1$, we have $Q^{-1}\dot{Q} = (U, -P_O^{-1}UP_0)$, and after a few simple calculations, we get $\dot{\alpha}(t) = Q_1(t)(2U(t)P_0)Q_2^{-1}(t)$. So, according to Proposition 7, $s(t)$ is the only solution of

$$\dot{s}(t) = 2U(t)P_0, \qquad s(0) = 0.$$

We also know that, for every tangent vector field $X(t)$ along $\alpha(t)$

$$d_{\alpha(t)}R(t)|_{T_{\alpha(t)}M}(X(t)) = Q(t)^{-1}\,X(t),$$

and the tangent no-twist condition is satisfied. For a general symmetric space this is not enough to define a rolling map in the sense of Definition 2, because the normal no-twist condition also requires that we know how to define $d_{\alpha(t)}R(t)|_{T^{\perp}_{\alpha(t)}M}$. However, for this particular example, it turns out that if $d_{\alpha(t)}R(t) = Q(t)^{-1}$, the normal no-twist condition is also satisfied.

To show this we rewrite the normal no-twist condition 5 of Definition 1 in its equivalent form given in 5' of Proposition 1. Taking into consideration that

$$T_{\widehat{\alpha}(t)}\widehat{M} = T_{P_0}\,\mathrm{SO}^+(p, n-p), \quad T^{\perp}_{\widehat{\alpha}(t)}\widehat{M} = T^{\perp}_{P_0}\,\mathrm{SO}^+(p, n-p),$$

and using (63), the normal no-twist condition is equivalent to prove that for every $B = -B^J$, $(R^{-1}\dot{R})(BP_0)$ is always of the form $CP_0$, for some matrix $C$ satisfying $C = C^J$. But $R^{-1}\dot{R} = (U, -P_O^{-1}UP_0)$ and $U^J = -U$, so

$$\begin{aligned} (R^{-1}\dot{R}).(BP_0) &= UBP_0 + BP_0P_O^{-1}UP_0 \\ &= (UB + BU)P_0 = CP_0, \end{aligned}$$

where $C = UB + BU = (UB + BU)^J = C^J$. Writing $R = (R_1, R_2)$, the kinematic equations are:

$$\left\{\begin{array}{lcl} \dot{s}(t) &=& 2U(t)P_0 \\ \dot{R}_1(t) &=& -U(t)R_1 \\ \dot{R}_2(t) &=& P_O^{-1}U(t)P_0R_2(t) \end{array}\right. , \qquad (64)$$

with initial conditions $s(0) = 0$, $R(0) = (e, e)$. This coincides with the results in [6].

## 5. Rolling of Stiefel manifolds on the affine tangent space

We will now narrow our discussion to the Stiefel manifolds $\mathrm{St}_{nk}$ equipped with the Riemannian metric inherited from the ambient Euclidean vector space $\mathcal{M}_{nk}$ consisting of $n \times k$ matrices. Rolling motions of Stiefel manifolds were already studied in [10], but here we present an alternative approach which is coordinates free, and also a different representation of Stiefel as a homogeneous space is used.

There are two compelling reasons for including the Stiefels in this paper. Firstly, because it is the only case outside of the $G$-invariant Riemannian manifolds where the rolling equations are explicitly calculated, and secondly because it illustrates the relevance of the normal no-twist condition for rolling manifolds that are homogeneous spaces but not symmetric spaces.

**5.1. Stiefel manifold.** Let $\mathbb{R}^n$ be the Euclidean vector space with its standard scalar product. We denote by $\mathfrak{gl}(n)$ the vector space of all real $n \times n$ matrices endowed with the positive definite scalar product $\langle N, M\rangle = \operatorname{tr}(N^T M)$. We induce this metric on the subspace $V = \mathcal{M}_{nk} \subset \mathfrak{gl}(n)$ of $n \times k$ matrices. We also consider the groups $\mathrm{GL}(n)$ and $\mathrm{SO}(n)$ as submanifolds of $\mathfrak{gl}(n)$. We define the action of $\mathfrak{gl}(n)$ on $V$ through the linear isometric homomorphism

$$\begin{array}{cccc} \rho & : & \mathfrak{gl}(n) & \to & \mathfrak{gl}(V) \\ & & A & \mapsto & \rho_A \end{array}$$

with $\rho_A(M) = AM$, $M \in V$. Under this convention, we obtain $\frac{d}{dt}\rho_{A(t)} = \rho_{\dot{A}(t)}$ for any smooth enought curve $A(t)$ in $\mathfrak{gl}(n)$. Moreover, the restriction of $\rho$ on the group $\mathrm{SO}(n)$ is a group homomorphism $\rho\colon \mathrm{SO}(n) \to \mathrm{SO}(V)$ meaning that $\rho_{Q^{-1}} = (\rho_Q)^{-1}$ and $\rho_{Q_1Q_2} = \rho_{Q_1} \circ \rho_{Q_2}$.

The Stiefel manifold $\mathrm{St}_{nk}$ consists of ordered sets of $k$-orthonormal vectors in $\mathbb{R}^n$. Any ordered set $m_1, \dots, m_k$ of orthonormal vectors can be identified with a matrix $M$ whose columns are $m_1, \dots, m_k$. Any such matrix $M$ satisfies $M^T M = I_k$, where $I_k$ is the $k$-dimensional identity matrix, and $M^T$ is the matrix transpose of $M$. This matrix representation realizes $\mathrm{St}_{nk}$ as an isometrically embedded compact submanifold of the Euclidean vector space $V = \mathcal{M}_{nk}$, which we continue to denote by $\mathrm{St}_{nk}$.

The Stiefel manifold can be also viewed as a homogeneous space. In what follows $\{e_1, \dots, e_n\}$ denotes the standard basis in $\mathbb{R}^n$ and $E$ denotes the matrix with columns $e_1, \dots, e_k$. The group $\mathrm{SO}(n)$ acts transitively on $\mathrm{St}_{nk}$. Thus $\mathrm{St}_{nk}$ can be identified with the orbit $\{\rho_Q(E) : Q \in \mathrm{SO}(n)\}$. The isotropy subgroup $H = \{Q \in \mathrm{SO}(n) : \rho_Q(E) = E\}$ reduces to matrices $Q = \begin{pmatrix} I_k & 0 \\ 0 & X \end{pmatrix}$, with $X \in \mathrm{SO}(n-k)$. Evidently $H$ is isomorphic to $\mathrm{SO}(n-k)$ and consequently

$\mathrm{St}_{nk} = \mathrm{SO}(n)/\mathrm{SO}(n-k)$. It follows that $\mathfrak{so}(n) = \mathfrak{p} \oplus \mathfrak{h}$, where

$$
\begin{aligned}
\mathfrak{h} &= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}, \, C \in \mathfrak{so}(n-k) \right\}, \\
\mathfrak{p} &= \left\{ \begin{pmatrix} A & -B^T \\ B & 0 \end{pmatrix}, \, A \in \mathfrak{so}(k), \, B \in \mathcal{M}_{(n-k)k} \right\}.
\end{aligned}
\tag{65}
$$

One can easily verify that $\mathfrak{h}$ is the Lie algebra of $H$, $\mathfrak{p}$ is the orthogonal complement to $\mathfrak{h}$ relative to the trace metric $\langle M_1, M_2 \rangle = -\operatorname{tr}(M_1 M_2)$, and

$$
[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{p}, \mathfrak{h}] = \mathfrak{p}, \quad \mathfrak{h} \subset [\mathfrak{p}, \mathfrak{p}]. \tag{66}
$$

The latter algebraic properties show that the Stiefel manifold $\mathrm{St}_{nk}$ is not a symmetric space. It does not allow to use the general aproach for the construction of the rolling, which we developed for symmetric spaces in Section 4.2. The projection map $\pi\colon \mathrm{SO}(n) \to \mathrm{St}_{nk} = \mathrm{SO}(n)/\mathrm{SO}(n-k)$ is given by $\pi(Q) = QE = \rho_Q(E)$. It is a submersion, and $d_e\pi\colon \mathfrak{p} \to T_E St_{nk}$ is an isomorphism, mapping $U \in \mathfrak{p}$ to $UE$. The group homomorphism $\rho\colon \mathrm{SO}(n) \to \mathrm{SO}(V)$ induces the Lie algebra homomorphism $\mathfrak{so}(n) \to \mathfrak{sl}(V)$. We will use the same notation $\mathfrak{p}$, $\mathfrak{h}$ for the images of $\mathfrak{p}$, $\mathfrak{h}$ under the Lie algebra homomorphism.

It follows, see for instance [7], that the tangent to $\mathrm{St}_{nk}$ at a point $P \in \mathrm{St}_{nk}$ is given by

$$
T_P \mathrm{St}_{nk} = \{ W \in \mathcal{M}_{nk} : \, W^T P + P^T W = 0 \}. \tag{67}
$$

**Remark 7.** *A simple calculation using* (67) *shows that for $Q \in G$, we have $T_{QP}\mathrm{St}_{nk} = Q T_P \mathrm{St}_{nk}$.*

In particular,

$$
T_E \mathrm{St}_{nk} = \left\{ \begin{pmatrix} A \\ B \end{pmatrix}, \, A \in \mathfrak{so}(k), B \in \mathcal{M}_{(n-k)k} \right\} \subset \mathcal{M}_{nk}. \tag{68}
$$

Hence, its orthogonal complement in $\mathcal{M}_{nk}$ is

$$
T_E^{\perp} \mathrm{St}_{nk} = \left\{ \begin{pmatrix} S \\ 0 \end{pmatrix}, \, S \in \mathcal{M}_{kk}, \, S^T = S \right\}. \tag{69}
$$

The orthogonal complement is further decomposed as

$$
T_E^{\perp} \mathrm{St}_{nk} = V_E \oplus sl_{kk}, \tag{70}
$$

where $V_E$ is the linear span of $E$, and its orthogonal complement, denoted by $sl_{kk}$ is defined as

$$sl_{kk} = \left\{ \begin{pmatrix} S \\ 0 \end{pmatrix},\ S \in \mathcal{M}_{kk},\ S^T = S,\ \mathrm{tr}(S) = 0 \right\}.$$

**5.2.Extrinsic rolling of** $\mathrm{St}_{nk}$**.** We will now turn our attention to the rollings of curves in $M = \mathrm{St}_{nk}$ on $\widehat{M} = T_E^{\mathrm{aff}}\mathrm{St}_{nk} := E + T_E\mathrm{St}_{nk}$ which is the affine tangent space at $E$. According to Definition 2, curves $\alpha(t)$ in $\mathrm{St}_{nk}$ are rolled on curves $\widehat{\alpha}(t)$ in $\widehat{M}$ by rolling maps $g(t) = (R(t), s(t))$ in $\mathrm{SE}(V) = \mathrm{SO}(V) \ltimes V$ under the action $g(t)(\alpha(t)) = R(t)(\alpha(t)) + s(t) = \widehat{\alpha}(t)$. The fact that $\mathrm{SO}(n)$ acts transitively on $\mathrm{St}_{nk}$ implies that there is a unique horizontal curve $Q(t) \in \mathrm{SO}(n)$, $Q(0) = I$, that projects on $\alpha(t)$, that is, $\rho_{Q(t)}(E) := Q(t)E = \alpha(t)$, and $Q^{-1}(t)\dot{Q}(t) \in \mathfrak{p}$.

We will now assume that $R(t)^{-1} = \rho_{Q(t)} \circ S(t)$ for some curve $S(t)$ in the isotropy group $K = \{S \in \mathrm{SO}(V) :\ S(E) = E\}$. The choice of $Q(t)$ as a horizontal lift of $\alpha(t)$ allows particularly satisfy the first rolling condition in Definition 2. Namelly: $R(t)^{-1}(E) = \rho_{Q(t)} \circ S(t)(E) = \rho_{Q(t)}(E) = \alpha(t)$, and therefore $R(t)(\alpha(t)) = E$. But then $g(t)(\alpha(t)) = R(t)(\alpha(t)) + s(t) = E + s(t) = \widehat{\alpha}(t)$ by the reguirement of the first rolling condition in Definition 2. It implies that

$$s(t) \in T_E\mathrm{St}_{nk}, \quad \text{and so} \quad \dot{s}(t) = \dot{\widehat{\alpha}}(t). \tag{71}$$

In what follow we will find the condition on $S(t)$ such that $g(t) = \big(R(t), s(t)\big)$ satisfies the no-slip and both no-twist constrains. According to the second rolling condition in Definition 2,

$$d_{\alpha(t)}g(t)(T_{\alpha(t)}\mathrm{St}_{nk}) = R(t)(T_{\alpha(t)}\mathrm{St}_{nk}) = T_E\mathrm{St}_{nk}. \tag{72}$$

Remark 7 and $\rho_{Q(t)}(E) = \alpha(t)$ lead to

$$d_E\rho_{Q(t)}(T_E\mathrm{St}_{n,k}) = \rho_{Q(t)}(T_E\mathrm{St}_{n,k}) = T_{\alpha(t)}\mathrm{St}_{n,k}.$$

Therefore,

$$\begin{aligned} T_E\mathrm{St}_{nk} &= R(t)(T_{\alpha(t)}\mathrm{St}_{nk}) \\ &= S^{-1} \circ \rho_{Q^{-1}}(T_{\alpha(t)}\mathrm{St}_{nk}) = S^{-1}(T_E\mathrm{St}_{n,k}). \end{aligned} \tag{73}$$

Hence, $S(T_E\mathrm{St}_{nk}) = T_E\mathrm{St}_{nk}$, and since $S$ is an isometry in $V$, we also have $S(T_E^{\perp}\mathrm{St}_{nk}) = T_E^{\perp}\mathrm{St}_{nk}$. So,

$$S(T_E\mathrm{St}_{nk}) = T_E\mathrm{St}_{nk}, \quad S(T_E^{\perp}\mathrm{St}_{nk}) = T_E^{\perp}\mathrm{St}_{nk}. \tag{74}$$

Moreover, since $S(E) = E$ and $S$ is an orthogonal transformation,

$$S(V_E) = V_E, \qquad S(sl_{kk}) = sl_{kk}. \tag{75}$$

From now on, we use the notation $\dot{R}(t)$ for the time derivative of $R(t) \in \mathrm{SO}(V)$, and similarly for the time derivative of any other curves in $\mathrm{SO}(V)$. We also recall from the beginning of this section that $\rho_{\dot{Q(t)}} := \rho_{\dot{Q}(t)}$ for $Q(t) \in \mathrm{SO}(n)$.

The no-slip condition requires that $\dot{R}(t)(\alpha(t)) + \dot{s}(t) = 0$, or,

$$\dot{R}(t) \circ R(t)^{-1}(E) = -\dot{s}(t). \tag{76}$$

Since $R = S^{-1} \circ \rho_{Q^{-1}}$, we have

$$\begin{aligned} \dot{R} \circ R^{-1} &= (\dot{\overline{S^{-1}}} \circ \rho_{Q^{-1}} + S^{-1} \circ \rho_{\dot{\overline{Q^{-1}}}}) \circ \rho_Q \circ S \\ &= \dot{\overline{S^{-1}}} \circ S + S^{-1} \circ \rho_{\dot{\overline{Q^{-1}}}} \circ \rho_Q \circ S \end{aligned} \tag{77}$$

Since $\dot{\overline{S^{-1}}} \circ S = -S^{-1} \circ \dot{S}$ and $\rho_{\dot{\overline{Q^{-1}}}} \circ \rho_Q = -\rho_{Q^{-1}} \circ \rho_{\dot{Q}} = -\rho_{Q^{-1}\dot{Q}}$, the above can be rewritten as

$$\dot{R} \circ R^{-1} = -S^{-1} \circ \dot{S} - S^{-1} \circ (\rho_{Q^{-1}\dot{Q}}) \circ S. \tag{78}$$

Note that $S(E) = E$ implies $\dot{S}(E) = 0$, and $\rho_{Q^{-1}\dot{Q}}(E) = Q^{-1}\dot{Q}E = UE$, for $U \in \mathfrak{p}$. Taking into consideration the structure of elements in $\mathfrak{p}$, appearing in (65), $Q^{-1}(t)\dot{Q}(t) = \begin{pmatrix} A(t) & -B^T(t) \\ B(t) & 0 \end{pmatrix}$, and consequently the no-slip condition requires that

$$\dot{s}(t) = -\dot{R}(t)R^{-1}(t)(E) = S^{-1}\Big( \begin{pmatrix} A(t) \\ B(t) \end{pmatrix} \Big). \tag{79}$$

We will now choose $S(t) \in K$, or equivalently $\Omega(t) = \dot{S} \circ S^{-1} \in \mathfrak{so}(V)$ so that $R(t)$ satisfies the no-twist conditions.

Since $\Omega(t)(E) = 0$. Therefore

$$\begin{aligned} \dot{S} \circ S^{-1}(T_E\mathrm{St}_{nk}) = \Omega(t)(T_E\mathrm{St}_{nk}) \subset T_E\mathrm{St}_{nk}, \\ \dot{S} \circ S^{-1}(T_E^{\perp}\mathrm{St}_{nk}) = \Omega(t)(T_E^{\perp}\mathrm{St}_{nk}) \subset T_E^{\perp}\mathrm{St}_{nk}. \end{aligned} \tag{80}$$

$$\Omega\,(V_E) = 0, \quad \Omega\,(sl_{kk}) \subset sl_{kk}. \tag{81}$$

due to (74) and (75).

Now, since $T_{\widehat{\alpha}(t)}\widehat{M} = T_E\mathrm{St}_{nk}$, and similarly $(T_{\widehat{\alpha}(t)}\widehat{M})^{\perp} = T_E^{\perp}\mathrm{St}_{nk}$, the tangential no-twist condition 4' given in Proposition 1, requires that

$$\dot{g}(t) \circ g^{-1}(t)(T_E\mathrm{St}_{nk}) \subset T_E^{\perp}\mathrm{St}_{nk}. \tag{82}$$

Since $\dot{g} \circ g^{-1}(T_E\mathrm{St}_{nk}) = \dot{R} \circ R^{-1}(T_E\mathrm{St}_{nk})$, taking into account (78), we can write

$$\dot{g} \circ g^{-1}(T_E\mathrm{St}_{nk}) = -S^{-1} \circ (\dot{S} \circ S^{-1} + \rho_{Q^{-1}\dot{Q}}) \circ S\,(T_E\mathrm{St}_{nk}),$$

and using (74), the tangential no-twist condition (82) can be written as

$$\dot{S} \circ S^{-1}\,(T_E\mathrm{St}_{nk}) + \rho_{Q^{-1}\dot{Q}}\,(T_E\mathrm{St}_{nk}) \subset T_E^{\perp}\mathrm{St}_{nk},$$

or, equivalently,

$$\Omega\,(T_E\mathrm{St}_{nk}) = -\Pi\left(\rho_{Q^{-1}\dot{Q}}\,(T_E\mathrm{St}_{nk})\right), \tag{83}$$

where $\Pi$ denotes the orthogonal projections of $V$ onto $T_E\mathrm{St}_{nk}$.

We now impose the normal no-twist condition

$$\dot{g}(t) \circ g^{-1}(t)(T_E^{\perp}\mathrm{St}_{nk}) \subset T_E\mathrm{St}_{nk}, \tag{84}$$

and similarly to the previous calculations, we obtain a second restriction on $S$:

$$\Omega\,(T_E^{\perp}\mathrm{St}_{nk}) = -\Pi^{\perp}\left(\rho_{Q^{-1}\dot{Q}}\,(T_E^{\perp}\mathrm{St}_{nk})\right), \tag{85}$$

where $\Pi^{\perp}$ denotes the orthogonal projections of $V$ onto $T_E^{\perp}\mathrm{St}_{nk}$.

To make sure that the previous condition can be fulfilled, we must show that the righthand side of (85) is according to the action (81) of $\Omega$ on each subspace of the direct decomposition of $T_E^{\perp}\mathrm{St}_{nk}$ in (70). For that, we compute the product of the matrix $Q^{-1}\dot{Q}$ by elements in $T_E^{\perp}\mathrm{St}_{nk}$, using the fact that $Q^{-1}\dot{Q} \in \mathfrak{p}$ and the structure of the matrices in these subspaces, given in (65) and (69).

Assume that

$$Q^{-1}\dot{Q} = \begin{pmatrix} A & -B^T \\ B & 0 \end{pmatrix},\ A = -A^T,\ \text{ and take }\ X = X^T.$$

Then,

$$Q^{-1}\dot{Q}\begin{pmatrix} X \\ 0 \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} AX + XA \\ 2BX \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} AX - XA \\ 0 \end{pmatrix}.$$

Notice that $AX - XA$ is symmetric with trace zero, and when $X = I_k$, $AX - XA = 0$. Therefore, as required,

$$\Pi^{\perp}\left(\rho_{Q^{-1}\dot{Q}}\,(V_E)\right) = 0, \qquad \Pi^{\perp}\left(\rho_{Q^{-1}\dot{Q}}\,(sl_{kk})\right) \subset sl_{kk}.$$

**We now summarize how to find the rolling map** $(R(t), s(t)) \in \mathrm{SE}(V)$, for rolling $\mathrm{St}_{nk}$ on $T_E^{\mathrm{aff}}\mathrm{St}_{nk}$, along a curve $\alpha(t)$, $\alpha(0) = E$.

1. Find the horizontal lift $Q(t)$ of $\alpha(t)$, s.t. $Q(0) = e_G$. We know that $Q^{-1}\dot{Q} = \begin{pmatrix} A & -B^T \\ B & 0 \end{pmatrix}$, $A = -A^T$.
2. Find $S(t)$ using the no-twist conditions (83), (85), with $S(0) = e_{\mathrm{SO}(V)}$. Those conditions can be rewritten as:
$$\begin{cases} \dot{S} \circ S^{-1}(v^\top) = -\Pi(Q^{-1}\dot{Q}\, v^\top), & \forall v^\top \in T_E\mathrm{St}_{nk} \\ \dot{S} \circ S^{-1}(v^\perp) = -\Pi^\perp(Q^{-1}\dot{Q}\, v^\perp), & \forall v^\perp \in T_E^\perp\mathrm{St}_{nk} \end{cases}.$$
3. Find $R = S^{-1} \circ \rho_{Q^{-1}}$.
4. Find $s(t)$ by solving equation (79), resulting from the no-slip condition, with $s(0) = 0$:
$$\dot{s}(t) = S^{-1}\Big( \begin{pmatrix} A(t) \\ B(t) \end{pmatrix} \Big).$$


## Acknowledgments

The work of the second and third authors was partially supported by the project Pure Mathematics in Norway, funded by Trond Mohn Foundation and Tromsø Research Foundation. The third author also thanks Fundação para a Ciência e Tecnologia (FCT) and COMPETE 2020 program for the financial support to the project UIDB/00048/2020.


## References


[1] A. A. Agrachev and Y. L. Sachkov, *Control theory from the geometric viewpoint*, vol. 87 of Encyclopaedia of Mathematical Sciences, Springer-Verlag, Berlin, 2004. Control Theory and Optimization, II.

[2] R. L. Bryant and L. Hsu, *Rigidity of integral curves of rank 2 distributions*, Invent. Math., 114 (1993), pp. 435–461.

[3] Y. Chitour, M. Godoy Molina, and P. Kokkonen, *The rolling problem: overview and challenges*, in Geometric control theory and sub-Riemannian geometry, vol. 5 of Springer INdAM Ser., Springer, Cham, 2014, pp. 103–122.

[4] Y. Chitour and P. Kokkonen, *Rolling manifolds on space forms*, Ann. Inst. H. Poincaré C Anal. Non Linéaire, 29 (2012), pp. 927–954.

[5] C. J. S. Clarke, *On the global isometric embedding of pseudo-Riemannian manifolds*, Proc. Roy. Soc. London Ser. A, 314 (1970), pp. 417–428.

[6] P. Crouch and F. S. Leite, *Rolling motions of pseudo-orthogonal groups*, in 2012 IEEE 51st IEEE Conference on Decision and Control (CDC), IEEE, 2012, pp. 7485–7491.

[7] A. Edelman, T. A. Arias, and S. T. Smith, *The geometry of algorithms with orthogonality constraints*, SIAM J. Matrix Anal. Appl., 20 (1999), pp. 303–353.

[8] M. Godoy Molina, E. Grong, I. Markina, and F. Silva Leite, *An intrinsic formulation of the problem on rolling manifolds*, J. Dyn. Control Syst., 18 (2012), pp. 181–214.

[9] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, vol. 80 of Pure and Applied Mathematics, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.
[10] K. Hüper, M. Kleinsteuber, and F. Silva Leite, *Rolling Stiefel manifolds*, Internat. J. Systems Sci., 39 (2008), pp. 881–887.
[11] K. Hüper, K. A. Krakowski, and F. Silva Leite, *Rolling maps and nonlinear data*, in Handbook of variational methods for nonlinear geometric data, Springer, Cham, [2020] ©2020, pp. 577–610.
[12] K. Hüper and F. Silva Leite, *On the geometry of rolling and interpolation curves on $S^n$, $\mathrm{SO}_n$, and Grassmann manifolds*, J. Dyn. Control Syst., 13 (2007), pp. 467–502.
[13] V. Jurdjevic, *The geometry of the plate-ball problem*, Arch. Rational Mech. Anal., 124 (1993), pp. 305–328.
[14] V. Jurdjevic, *Geometric control theory*, vol. 52 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1997.
[15] ——, *Optimal control and geometry: integrable systems*, vol. 154 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2016.
[16] V. Jurdjevic and J. Zimmerman, *Rolling sphere problems on spaces of constant curvature*, Math. Proc. Cambridge Philos. Soc., 144 (2008), pp. 729–747.
[17] A. Korolko and F. S. Leite, *Kinematics for rolling a lorentzian sphere*, in 2011 50th IEEE Conference on Decision and Control and European Control Conference, IEEE, 2011, pp. 6522–6527.
[18] K. A. Krakowski, L. Machado, and F. Silva Leite, *A unifying approach for rolling symmetric spaces*, J. Geom. Mech., 13 (2021), pp. 145–166.
[19] J. M. Lee, *Riemannian manifolds: an introduction to curvature*, vol. 176, Springer Science & Business Media, 2006.
[20] F. S. Leite and F. Louro, *Sphere rolling on sphere: alternative approach to kinematics and constructive proof of controllability*, in Dynamics, games and science, vol. 1 of CIM Ser. Math. Sci., Springer, Cham, 2015, pp. 341–356.
[21] I. Markina and F. Silva Leite, *Introduction to the intrinsic rolling with indefinite metric*, Comm. Anal. Geom., 24 (2016), pp. 1085–1106.
[22] A. Marques and F. S. Leite, *Pure rolling motion of hyperquadrics in pseudo-Euclidean spaces*, J. Geom. Mech., 14 (2022), pp. 105–129.
[23] K. Nomizu, *Kinematics and differential geometry of submanifolds*, Tohoku Math. J. (2), 30 (1978), pp. 623–637.
[24] B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, vol. 103 of Pure and Applied Mathematics, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983.
[25] R. W. Sharpe, *Differential geometry. Cartan's generalization of Klein's Erlangen program*, vol. 166 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1997.
[26] J. A. Zimmerman, *Optimal control of the sphere $S^n$ rolling on $E^n$*, Math. Control Signals Systems, 17 (2005), pp. 14–37.

V. Jurdjevic
Department of Mathematics, University of Toronto, Toronto, Ontario M5S 3G3, Canada
*E-mail address*: jurdj@math.utoronto.ca

I. Markina
Department of Mathematics, University of Bergen, P.O. Box 7803, Bergen N-5020, Norway

*E-mail address*: irina.markina@uib.no

F. Silva Leite
Department of Mathematics, University of Coimbra, Largo D. Dinis, 3001-454 Coimbra, Portugal, and Institute of Systems and Robotics, University of Coimbra - Pólo II, Pinhal de Marrocos, 3030-290 Coimbra, Portugal

*E-mail address*: fleite@mat.uc.pt