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REPRESENTING THE STIRLING POLYNOMIALS $\sigma_n(x)$ IN DEPENDENCE OF n AND AN APPLICATION TO POLYNOMIAL ZERO IDENTITIES

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ABSTRACT: Denote by σ_n the n-th Stirling polynomial in the sense of the much cited book Concrete Mathematics [GKP] by Graham, Knuth and Patashnik. We show that there exist developments $x\sigma_n(x) = \sum_{j=0}^{n}(2^{n-j}(n-j)!)^{-1}p_j(n)x^{n-j}$ with polynomials p_j of degree j. This should have some importance for refined asymptotic analyses of the Stirling numbers of the second kind. We use the result to deduce from it with extra effort the polynomial identities

$$\sum_{a+b+c+d=n}(-1)^d\frac{(x-2a-2b)^{3n-s-a-c}}{a!b!c!d!(3n-s-a-c)!}=0, \quad \text{for} \quad s\in\mathbb{Z}_{\geq 1},$$

or equivalently the identities

$$\sum_{a+b+c+d=n}\frac{(-1)^d(-2)^{t-a-c}}{a!b!c!d!(t-a-c)!}(a+b)^{t-a-c}=0 \quad \text{for} \quad t=0,1,2,...,3n-1,$$

found in an attempt to find a simpler formula for the density function in a 5-dimensional random flight problem. Other, similarly looking identities should be provable by the techniques employed.

KEYWORDS: Stirling polynomials, Polynomial identities, Random flights .
MATH. SUBJECT CLASSIFICATION (2020): 05A19, 11B37.

1. Introduction

In the context of research on a problem of random flights in dimension 5, a little background of which we relate in Section 4, the second author conjectured the identities in the abstract, for whose attack the authors could not find many hints in the literature. The work on its proof led us to a (for us at least) surprising result about the behaviour of the coefficients of sequences of Stirling polynomials. Let $\sigma_n(x)$ be the n-th Stirling polynomial in the sense of [GKP]; the precise definition is given in Section 2, but here we

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present the following table of coefficients of the first few Stirling polynomials. The table tells us for example that $x\sigma_3(x) = 0 + 0x - \frac{1}{48}x^2 + \frac{1}{48}x^3$.

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8
$x\sigma_0(x)$	1								
$x\sigma_1(x)$	0	$\frac{1}{2}$							
$x\sigma_2(x)$	0	$-\frac{1}{24}$	$\frac{1}{8}$						
$x\sigma_3(x)$	0	0	$-\frac{1}{48}$	$\frac{1}{48}$					
$x\sigma_4(x)$	0	$\frac{1}{2880}$	$\frac{1}{1152}$	$-\frac{1}{192}$	$\frac{1}{384}$				
$x\sigma_5(x)$	0	0	$\frac{1}{5760}$	$\frac{1}{2304}$	$-\frac{1}{1152}$	$\frac{1}{3840}$			
$x\sigma_6(x)$	0	$-\frac{1}{181440}$	$-\frac{1}{69120}$	$\frac{13}{414720}$	$\frac{1}{9216}$	$-\frac{1}{9216}$	$\frac{1}{46080}$		
$x\sigma_7(x)$	0	0	$-\frac{1}{362880}$	$-\frac{1}{138240}$	$\frac{1}{829440}$	$\frac{1}{55296}$	$-\frac{1}{92160}$	$\frac{1}{645120}$	
$x\sigma_8(x)$	0	$\frac{1}{9676800}$	$\frac{101}{348364800}$	$-\frac{1}{2580480}$	$-\frac{67}{39813120}$	$-\frac{1}{1658880}$	$\frac{1}{442368}$	$-\frac{1}{1105920}$	$\frac{1}{10321920}$

We convene to begin row and column indices both with 0 and then multiply the entries in column j of this table with $2^j j!$. We get this table:

	0	1	2	3	4	5	6	7	8
0	1								
1	0	1							
2	0	$-\frac{1}{12}$	1						
3	0	0	$-\frac{1}{6}$	1					
4	0	$\frac{1}{1440}$	$\frac{1}{144}$	$-\frac{1}{4}$	1				
5	0	0	$\frac{1}{720}$	$\frac{1}{48}$	$-\frac{1}{3}$	1			
6	0	$-\frac{1}{90720}$	$-\frac{1}{8640}$	$\frac{13}{8640}$	$\frac{1}{24}$	$-\frac{5}{12}$	1		
7	0	0	$-\frac{1}{45360}$	$-\frac{1}{2880}$	$\frac{1}{2160}$	$\frac{5}{72}$	$-\frac{1}{2}$	1	
8	0	$\frac{1}{4838400}$	$\frac{101}{43545600}$	$-\frac{1}{53760}$	$-\frac{67}{103680}$	$-\frac{1}{432}$	$\frac{5}{48}$	$-\frac{7}{12}$	1

The first main result proven in the current paper can be expressed as saying that the j-th diagonal of this table is a polynomial sequence of degree j, $j = 0, 1, 2, \ldots$. Accept this for the moment and denote the sequence by $(q_j(n))_{n\geq 0}$. As is well known, see almost any text on numerical analysis, e.g. [BF, p. 95ff], using $j+1$ interpolation points of distinct abscissae, in our case $n = 0, 1, 2, \ldots$, one can compute a unique polynomial of degree $\leq j$ whose graph passes through these points. From the second table one finds for example

$$q_0(n) = 1, \quad q_1(n) = \frac{-n}{12}, \quad q_2(n) = \frac{(-1+n)n}{288}, \quad q_3(n) = \frac{26n + 15n^2 - 5n^3}{51840}.$$

The sequence of numbers in the j-th diagonal of the original table is given by $(\frac{q_j(\ell)}{2^\ell \ell!})_{\ell\geq 0}$. We can use its row n to determine the polynomial $x\sigma_n(x)$ for symbolic n. The leftmost coefficient is the beginning and hence at position 0 of diagonal n. So it has value $(\frac{q_n(0)}{2^0 0!})$. The coefficient of x^1 pertains to diagonal $n-1$. It is at position 1 of that diagonal so has value $\frac{q_{n-1}(1)}{2^1 1!}$. In general the coefficient pertaining to x^j is at position j of diagonal $n-j$ and therefore has value $\frac{q_{n-j}(j)}{2^j j!}$. Thus we get $x\sigma_n(x) = \sum_{j=0}^{n} \frac{q_{n-j}(j)}{2^j j!} x^j$.

As it happens, the fact $q_j(0) = 0$ for $j \geq 1$ implies $n | q_j(n)$ so that $(n-j) | q_j(n-j)$. This means that putting $\tilde{q}_j(n) := q_j(n)/n$ and using the polynomials q_0, q_1, q_2, q_3 computed above we find

$$\begin{aligned} x\sigma_n(x) &= \sum_{j=0}^{n-4} \frac{q_{n-j}(j)\, x^j}{2^j j!} + \frac{q_3(n-3)x^{n-3}}{2^{n-3}(n-3)!} + \frac{q_2(n-2)x^{n-2}}{2^{n-2}(n-2)!} + \frac{q_1(n-1)x^{n-1}}{2^{n-1}(n-1)!} + \frac{q_0(n)x^n}{2^n n!} \\ &= \sum_{j=0}^{n-4} \frac{\tilde{q}_{n-j}(j)x^j}{2^j (j-1)!} + \frac{\tilde{q}_3(n-3)x^{n-3}}{2^{n-3}(n-4)!} + \frac{\tilde{q}_2(n-2)x^{n-2}}{2^{n-2}(n-3)!} + \frac{\tilde{q}_1(n-1)x^{n-1}}{2^{n-1}(n-2)!} + \frac{x^n}{2^n n!} \\ &= \sum_{j=0}^{n-4} \frac{\tilde{q}_{n-j}(j)x^j}{2^j (j-1)!} + \frac{(64-54n+5n^2)x^{n-3}}{51840 \cdot (2^{n-3}(n-4)!)} + \frac{(-3+n)x^{n-2}}{288(2^{n-2}(n-3)!)} + \frac{-1x^{n-1}}{12(2^{n-1}(n-2)!)} + \frac{x^n}{2^n n!}, \end{aligned}$$

thus illustrating the claim in the abstract. A simple closed expression $f(j,n)$ such that $x\sigma_n(x) = \sum_{j=0}^{n} f(j,n)x^{n-j}$ for all $j, n \in \mathbb{Z}_{\geq 0}$ does probably not exist because it would for example via the identity $B_m = -m\, m!\sigma_m(0)$ imply a simple formula for the Bernoulli numbers.

In the following section we collect a number of results on Stirling numbers and Stirling polynomials. In Section 3 we assume the representation $x\sigma_n(x) = \sum_{k=0}^{n}(-1)^k a_{n,k} x^{n-k}$ and prove that the sequence $\mathbb{Z}_{\geq k} \ni n \mapsto 2^{n-k}(n-k)! a_{n,k}$ is polynomial of degree k; a fact equivalent to the representation claimed for $x\sigma_n(x)$ given in the abstract. For this we have to solve a first order difference equation with polynomial coefficients. As we do so we formulate a certain 'meta-theorem' according to which most of the difference equations of a certain type should have a particular solution which is polynomial. In Section 4 we deduce the identities mentioned in the abstract.

More important than the particular polynomial identity which we derive might be the methods which we employ. They should be applicable in a number of similarly looking identities. But we admit it would be desirable to first simplify our proof significantly. In this vein note also that by introducing the notation $x^{[k]} := x^k/k!$ the identities assume a more convenient form.

2. Stirling numbers, Stirling polynomials, and some known auxiliary facts.

We collect here facts on Stirling numbers and Stirling polynomials. Our sources are a paper by Gessel and Stanley [GS] and the book by Graham, Knuth, Patashnik [GKP, pp 257-272]. Very informative is also the article by Boyadzhiev [B]

Stirling polynomials are born from investigations on Stirling numbers. Stirling numbers, in a notation proposed by Jovan Karamata and promoted by [GKP] are defined for integers $n, k \geq 0$, and come in two kinds. 1st kind Stirling numbers are denoted by $\left[{n \atop k}\right]$ and verbalized by 'n cycle k' . They count the number of partitions of $[n] = \{1, 2, ..., n\}$ into k nonempty cycles. 2nd kind Stirling numbers are denoted by $\left\{{n \atop k}\right\}$ and verbalized by 'n subset k' . They count the number of partitions of $[n] = \{1, 2, ..., n\}$ into k nonempty subsets.

With the supplementary conditions $\left[{n \atop 0}\right] = \left\{{n \atop 0}\right\} = \delta_{n,0}$, there hold for $n > 0$ the following recursions, for whose easy combinatorial explanations see [GKP].

$$\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right], \quad \left\{{n \atop k}\right\} = k\left\{{n-1 \atop k}\right\} + \left\{{n-1 \atop k-1}\right\}.$$

Jekuthiel Ginsberg discovered in 1928 that there is a way to meaningfully define for $n \geq 0$, $\left[{x \atop x-n}\right]$ and $\left\{{x \atop x-n}\right\}$ as polynomials in x of degree $2n$ (so that whenever x is an integer$> n$ there occur the usual Stirling numbers). This is explained in [GKP] where it is observed also that when $x \in \{0, 1, 2, ..., n\}$ then these polynomials are zero and hence we find that with the exception of the case $n = 0$, the expressions

$$\sigma_n(x) = \left[{x \atop x-n}\right] / x(x-1)(x-2)\cdots(x-n)$$

are polynomials, called there Stirling polynomials. The exception is $\sigma_0(x) = 1/x$. We have $\deg \sigma_n(x) = n - 1$.

The authors of [GS] approach the topic of Stirling polynomials differently. They are interested in the sequences $\mathbb{Z}_{n\geq 1} \mapsto f_k(n) := S(n+k, n)$ (where $S(n,k) = \left\{{n \atop k}\right\}$), not so much for its own sake but for giving a combinatorial interpretation to the coefficients of the power series $(1-x)^{2k+1}\sum_{k\geq 0} f_k(n)x^n$. In this context they establish that the functions f_k are polynomial of degree

$2k$ with leading coefficient $(2^k k!)^{-1}$, a fact attributed to C. Jordan' s book on difference equations not available to us. For $k = 0$ the claim is clear since $f_0(n) = S(n,n) = 1$. The general case is done by induction on k. It is observed, with the not completely trivial proof left to the reader, that the recursion for the second kind Stirling numbers can be recast into the equation $(\Delta f_k)(n) = (n+1)f_{k-1}(n+1)$, valid for all $n \geq k \in \mathbb{Z}_{>0}$, where Δ is the forward difference operator. Using the elementary fact that a sequence $\{(\Delta f)(n)\}_{n\geq 0}$ is polynomial of degree d if and only if the sequence $\{f(n)\}_{n\geq 0}$ is polynomial of degree $d+1$; and that then the corresponding leading coefficients stand in the relation $\mathrm{lc}(\Delta f) = \deg f \cdot \mathrm{lc}(f)$ one obtains the claim.

Once one has that the map $n \mapsto f_k(n)$ coincides on the infinitely many points constituting $\mathbb{Z}_{\geq 1}$ with the values of a polynomial of degree $2k$ the authors can define $f_k(x)$ as being this polynomial. Observing that the difference equation $f_k(x+1) - f_k(x) = (x+1)f_{k-1}(x+1)$ holds for all x and supposing $f_{k-1}(0) = f_{k-1}(-1) = \cdots f_{k-1}(1-k) = 0$ and $f_k(0) = 0$ one derives successively $0 = f_k(0) = f_k(-1) = f_k(-2) = \cdots = f_k(-k)$. From this in turn it follows that $f_k(x) = x(x+1)\cdots(x+k)\cdot$(a monic polynomial of degree $k-1$)$\cdot$ $(1/2^k k!)$. In [GS], it is the $f_k(x)$ that are called Stirling polynomials.

The 'Stirling polynomials' of [GKP] and the 'Stirling polynomials' of [GS] are not the same but they are easily transformed to each other. By [GKP, p. 267], for all $k, n \in \mathbb{Z}$, $\left[{n \atop k}\right] = \left\{{-k \atop -n}\right\}$. It follows, first for integer x and then, by the usual polynomial argument on the formal level, that

$$\begin{aligned} f_n(x) &= \left\{{x+n \atop x}\right\} = \left[{-x \atop -x-n}\right] \\ &= \sigma_n(-x)\cdot(-x)(-x-1)\cdots(-x-n) \\ &= \sigma_n(-x)(-1)^{n+1}x(x+1)\cdots(x+n). \end{aligned}$$

(While we are at it we should mention that we know of at least two other notions of Stirling polynomials which do not seem to have a close connection to either of the polynomials $f_k(x)$ or $\sigma_m(x)$.)

We shall need the following recursion formula for the σ_n, mentioned in [GKP, Exercise 6.18].

Lemma 1. *For $n \geq 1$, one has*

$$(x+1)\sigma_n(x+1) = (x-n)\sigma_n(x) + x\sigma_{n-1}(x).$$

Proof: Substituting, for the left- and right hand sides of this equation, respectively, the definitions of the σs, we get

$$\begin{aligned}\text{lhs} &= \begin{bmatrix} x+1 \\ x+1-n \end{bmatrix} / x(x-1)\cdots(x+1-n),\\ \text{rhs} &= \begin{bmatrix} x \\ x-n \end{bmatrix} / x(x-1)\cdots(x+1-n) + \begin{bmatrix} x \\ x+1-n \end{bmatrix} / (x-1)\cdots(x+1-n).\end{aligned}$$

Multiplying everything with $x(x-1)\cdots(x+1-n)$ we get that the claim is equivalent to

$$\begin{bmatrix} x+1 \\ x+1-n \end{bmatrix} = \begin{bmatrix} x \\ x-n \end{bmatrix} + x \begin{bmatrix} x \\ x+1-n \end{bmatrix}$$

Now this is simply an instance of the recursion formula for Stirling polynomials of the first kind. $\square$

In Section 4 we will also use the following known facts.

Proposition 2. *Let $a = a(x) = \sum_{j\geq 0} a_j x^j$ be any polynomial and let $p_n(x) := (-1)^n \cdot -x\sigma_n(-x)$. Then:*
a. One has the following equivalent identities of finite sums

$$\sum_{k\geq 0} \frac{(-1)^k}{k!(m-k)!} a(k) = (-1)^m \sum_{j\geq 0} a_j \begin{Bmatrix} j \\ m \end{Bmatrix}; \quad \sum_{k+l=m} \frac{(-1)^l}{k!l!} a(k) = \sum_{j\geq 0} a_j \begin{Bmatrix} j \\ m \end{Bmatrix}.$$

b. For n, k nonnegative integers, there holds

$$\begin{Bmatrix} n+k \\ k \end{Bmatrix} = \frac{(n+k)!}{k!} p_n(k).$$

Proof. a. The sums are finite because the a_j for $j > \deg a$ are 0 and because for a negative integer s, one has $1/s! = 0$. The left formula is then essentially mentioned for polynomials $a(x)$ that are of the form x^l as [GKP, formula (6.19)]. The formula given follows as any polynomial is a linear combination of monomials. The right formula follows by multiplying both sides with $(-1)^m$ and using that $(-1)^{m+k} = (-1)^{m-k}$.

b. From the relation mentioned before Lemma 1 we see that $\begin{Bmatrix} n+k \\ k \end{Bmatrix} = \sigma_n(-k)(-1)^{n+1} k(k+1)\cdots(k+n)$. Use the definition of p_n to conclude the proof. $\square$

Remarks. In part a of the proposition note that if $\deg a < m$ then in the equalities of sums the ones at the right hand sides and hence at the left hand sides are 0. Also, if $a(x) = x^l$ then the right hand sides reduce to

$(-1)^m\left\{{l \atop m}\right\}$ and $\left\{{l \atop m}\right\}$, respectively. The identities in part a are usually proved by applying the forward difference operator. A multivariate generalization based on completely different reasoning can be found in [SK3].

3. The main result on the diagonals of the modified coefficient table of Stirling polynomials

We transform the recursion of Lemma 2.1 into a matrix equation for the coefficients.

Proposition 1. *Writing*

$$\sigma_n(x) = \sum_{j=0}^{n-1} a_j x^j \quad \text{and} \quad x\sigma_{n-1}(x) = \sum_{j=0}^{n-1} b_j x^j, \quad n = 2, 3, 4...,$$

there holds the following $(n-1)\times(n-1)$ *matrix equation :*

$$\begin{pmatrix} (1+n) & \binom{2}{0} & \binom{3}{0} & \binom{4}{0} & \cdots & \binom{n-1}{0} \\ & 2+n & \binom{3}{1} & \binom{4}{1} & \cdots & \binom{n-1}{1} \\ & & 3+n & \binom{4}{2} & \cdots & \binom{n-1}{2} \\ & & & \ddots & & \vdots \\ & & & & (2n-2) & \binom{n-1}{n-3} \\ & & & & & 2n-1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-2} \end{pmatrix} = \begin{pmatrix} b_0 - \binom{n}{0} a_{n-1} \\ b_1 - \binom{n}{1} a_{n-1} \\ b_2 - \binom{n}{2} a_{n-1} \\ \vdots \\ b_{n-2} - \binom{n}{n-2} a_{n-1} \end{pmatrix}.$$

Proof. With the understanding that $a_{-1} = a_n = 0$ we have

$$\begin{aligned} (x+1)\sigma_n(x+1) &= \sum_{j=0}^{n-1} a_j (1+x)^{j+1} = \sum_{j=0}^{n-1} a_j \sum_{l=0}^{j+1} \binom{j+1}{l} x^l \\ &= \sum_{l=0}^{n-1} \left(\sum_{j=l}^{n-1} a_j \binom{j+1}{l} \right) x^l + \sum_{j=0}^{n-1} a_j x^{j+1} \\ &= \sum_{l=0}^{n} \left(\sum_{j=l}^{n-1} a_j \binom{j+1}{l} + a_{l-1} \right) x^l; \\ (x-n)\sigma_n(x) &= \sum_{l=0}^{n} (a_{l-1} - n a_l) x^l, \end{aligned}$$

and hence, since below the inner expression for $l = n$ is 0,

$$(x+1)\sigma_n(x+1) - (x-n)\sigma_n(x) = \sum_{l=0}^{n-1} \left(\sum_{j=l}^{n-1} a_j \binom{j+1}{l} + n a_l \right) x^l.$$

By the Lemma 2.1, hence we have that $\sum_{j=l}^{n-1} a_j \binom{j+1}{l} + n a_l = b_l$ for $l = 0, 1, ..., n-1$. It is easy to see that these equations can be encoded in the above matrix equation: for example, for $l = n-2$ we get $a_{n-2}\binom{n-1}{n-2} +$

$a_{n-1}\binom{n}{n-2} + na_{n-2} = b_{n-2}$ which is equivalent to the last encoded equation. (The case $l = n-1$ needs not be encoded since it expresses $2na_{n-1} = b_{n-1}$ which is a consequence of the known fact that the leading coefficients for σ_n and σ_{n-1} are, as mentioned earlier, $a_{n-1} = 1/(2^n n!)$, and $\mathrm{lc}(\sigma_{n-1}(x)) = \mathrm{lc}(x \cdot \sigma_{n-1}(x)) = b_{n-1} = 1/2^{n-1}(n-1)!$.) □

In this proposition σ_n was fixed and a_{n-k} is the coefficient of x^{n-k} in σ_n and hence the coefficient of $x^{n+1-k} = x^{n-(k-1)}$ of $x\sigma_n(x)$. Similarly b_{n-k} is the coefficient of x^{n-k} in $x\sigma_{n-1}(x)$. Since in the sequel we have to consider the dependence on n as well, we define $a_{n,k} := (-1)^k\cdot$coefficient of x^{n-k} of $x\sigma_n(x)$. The matrix equation of the previous proposition says that for $k = 2, 3, ..., n$,

$$(2n-k+1)a_{n-k} + \binom{n-k+2}{n-k}a_{n-k+1} + \binom{n-k+3}{n-k}a_{n-k+2} + \cdots + \binom{n-1}{n-k}a_{n-2}$$
$$= b_{n-k} - \binom{n}{n-k}a_{n-1}.$$

Doing the proper replacements according to $a_{n-k} \to (-1)^{k-1}a_{n,k-1}$, and $b_{n-k} \to (-1)^{k-1}a_{n-1,k-1}$ we get after a rearrangement the following equation valid for $k = 2, ..., n$.

$$(2n-k+1)a_{n,k-1} - a_{n-1,k-1} = (-1)^k \sum_{j=0}^{k-2} (-1)^j \binom{n-j}{n-k} a_{n,j}.$$

The last two lines in the following formula being clear, and writing now k for $k-1$ this can also be written as follows.

$$a_{n,k} = \begin{cases} \frac{1}{2n-k}\left(a_{n-1,k} + (-1)^{k+1}\sum_{j=0}^{k-1}(-1)^j\binom{n-j}{n-k-1}a_{n,j}\right) & \text{for } n \geq k > 0 \text{ or } n > k = 0 \\ 0 & \text{for } n < k \text{ or } k < 0 \\ 1 & \text{for } n = k = 0. \end{cases}$$

The main line is valid at first for $k = 1, ..., n-1$ but as it happens it also is valid for $k = 0$; in which case it reproduces that $a_{n,0} = 1/(2^n n!)$. By systematically checking the nine cases $n\varepsilon_1 k$ & $k\varepsilon_2 0$ for $\varepsilon_1, \varepsilon_2 \in \{<, =, >\}$ one finds that any $(n, k) \in \mathbb{Z}^2$ satisfies exactly one of the cases indicated; and beginning with any n, k the base of the recursion will be in the second or third cases. Thus the recursion is well defined.

The recursion serves well if one would desire a rapid computation of the polynomials σ_n or $f_n(x)/(x+1)\cdots(x+n)$. The following MATHEMATICA© code can be used to define $a_{n,k}$ (as `a[n,k]`). Then e.g. `(-1)^3 a[5,3]` gives the coefficient of x^2 in $x\sigma_5(x)$.

```
a[n_,k_]:=(-k+2*n)^(-1)*((-1)^(k+1)*
Sum[(-1)^j*a[n,j]*Binomial[n-j,-k+n-1],{j,0,k-1}]+
  a[n-1,k])/;n>=k>0||n>k==0;
a[0,0]:=1;a[n_,k_]:=0/;n<k||k<0;
```

Alternatively one also may use generating function approaches like the identity [GKP, (6.50)], which reads $((ze^z)/(e^z-1))^x = \sum_n x\sigma_n(x)z^n$ and the `Series[ . . . ]` command to get the polynomials σ_n.

Recall that one of our main goals is to show that the sequence $\mathbb{Z}_{\geq k} \ni n \mapsto f_{n,k} := 2^{n-k}(n-k)!a_{n,k}$ is polynomial of degree k. Motivated by this, one feels it might be simpler to work with a recursion for the $f_{n,k}$ instead of the $a_{n,k}$.

Multiplying the main line of the recursion above with $2^{n-k}(n-k)!$, the replacements $a_{n,j} \to f_{n,j}/(2^{n-j}(n-j)!)$ and simplification yield

$$\begin{aligned} f_{n,k} &= \tfrac{2^{n-k}(n-k)!}{2n-k} \cdot \tfrac{f_{n-1,k}}{2^{n-1-k}(n-1-k)!} + \tfrac{1}{2n-k}\sum_{j=0}^{k-1}(-1)^{1+k+j}\tbinom{n-j}{n-k-1} \cdot \tfrac{2^{n-k}(n-k)!}{2^{n-j}(n-j)!}f_{n,j} \\ &= \tfrac{2(n-k)}{2n-k}f_{n-1,k} + \tfrac{1}{2n-k}\sum_{j=0}^{k-1}(-1)^{1+k+j}\tbinom{n-j}{n-k-1}\tfrac{f_{n,j}}{2^{k-j}(n-k+1)\cdots(n-j)} \end{aligned}$$

One more simplification now yields:

Corollary 2. The numbers $f_{n,k}$ satisfy the following recursion:

$$f_{n,k} = \begin{cases} \frac{(n-k)}{(2n-k)}(2f_{n-1,k} - \sum_{j=0}^{k-1}(-1/2)^{k-j}\frac{f_{n,j}}{(k+1-j)!}) & \text{for } n \geq k > 0 \text{ or } n > k = 0 \\ 0 & \text{for } n < k \text{ or } k < 0 \\ 1 & \text{for } n = k = 0. \end{cases}$$

□

Our guiding principle for proving the theorem below was the following observation.

Observation. Assume p_{11} and p_{12} are two polynomials of degree 1 with the same leading coefficient and assume q is a polynomial of degree k. Then 'in general' there will exist a particular solution $\mathbb{Z}_{\geq 0} \ni n \mapsto (f_n)$ for the difference equation $p_{11}(n)f_n = p_{12}(n)f_{n-1} + q(n)$ which is polynomial of degree k.

The 'proof' for this goes as follows. We make an *ansatz* $f_n = a_0 + a_1 n + \cdots + a_k n^k$. Then the expressions $p_{11}(n)f_n$ and $p_{12}(n)f_{n-1}$ will be polynomials of degree $k+1$ whose coefficients are linear forms in $a_0, a_1, ..., a_k$. However, the coefficient of n^{k+1} of the expression $p_{11}(n)f_n - p_{12}(n)f_{n-1}$ will be $\mathrm{lc}(p_{11})\mathrm{lc}(f_n) -$

$\mathrm{lc}(p_{12})\mathrm{lc}(f_{n-1}) = (\mathrm{lc}(p_{11}) - \mathrm{lc}(p_{12}))a_k = 0$, so that the expression can be expected to be of degree k and it can be written $\sum_{j=0}^{k} l_j(a_0, ..., a_k)n^j$ for certain linear forms l_j. It can then be expected that the linear map

$$\mathbb{R}^{k+1} \ni a_{0:k} \mapsto l_{0:k}(a_{0:k}) \in \mathbb{R}^{k+1}$$

is surjective and hence it will be possible to solve the system of $k+1$ equations $l_j(a_{0:k}) =$ coefficient of x^j of polynomial q, $j = 0, ..., k$, uniquely for $a_{0:k} = (a_0, ..., a_k)$. To be sure, it is well possible that the map is not surjective. For example, if $p_{11}(n) = a + bn, p_{12}(n) = c + bn$ and q is a polynomial of degree 3, then surjectivity holds if and only if $a + ib - c \neq 0$ for $i = 0, 1, 2, 3$. □

Strangely, in various books on difference equations consulted we did not find a hint for a fact like this which should allow various generalizations.

Theorem 3. *Let k be a nonnegative integer. Then the sequence*

$$\mathbb{Z}_{\geq k} \ni n \mapsto f_{n,k}$$

is polynomial of degree k.

Proof. The main line of the recursion for the $f_{n,k}$ can be rewritten as

$$*_1 : (2n-k)f_{n,k} - (2n-2k)f_{n-1,k} = (n-k)\sum_{j=0}^{k-1}((-2)^{1+k-j}(k+1-j)!)^{-1}f_{n,j}.$$

This is a necessary condition which the $f_{n,k}$, uniquely and well defined by the recursion, must satisfy.

We know that $a_{n,0} = (2^n n!)^{-1}$, and so by definitions, $f_{n,0} = 1$ for all n. (This can also be deduced from the recursion which reduces for the case $k = 0$ to $f_{n,0} = f_{n-1,0}$ and uses $f_{0,0} = 1$.) So $f_{n,0}$ is a polynomial of degree 0. We fix now $k > 0$ and assume already proved for $j = 0, 1, 2, \ldots, k-1$, that the sequences $\mathbb{Z}_{\geq j} \ni n \mapsto f_{n,j}$ are polynomial of degree j. The right hand side of the recursion shown is then a polynomial; and it must be of degree k since there exists only one polynomial sequence of degree $k-1$ in the sum, namely $\mathbb{Z}_{\geq k-1} \ni n \mapsto f_{n,k-1}$, all other sequences $f_{n,j}$ occurring have lower degree. We will denote the polynomial (sequence) defining the right hand side by $q(n) = \sum_{j=0}^{k} c_j x^j$ and have $c_k \neq 0$.

Now we make the ansatz

$$f_{n,k} = a_0 + a_1 n + \cdots + a_k n^k$$

and (again) with the understanding that $a_{-1} = a_{k+1} = 0$, find

$$\begin{aligned}
(2n-k)f_{n,k} &= (2n-k)\sum_{i=0}^{k} a_i n^i = \sum_{i=0}^{k+1}(2a_{i-1}-ka_i)n^i;\\
(2n-2k)f_{n-1,k} &= (2n-2k)\sum_{j=0}^{k} a_j (n-1)^j\\
&= (2n-2k)\sum_{j=0}^{k} a_j\left(\sum_{i=0}^{j}\binom{j}{i}n^i(-1)^{j-i}\right)\\
&= (2n-2k)\sum_{i=0}^{k}\left(\sum_{j=i}^{k} a_j\binom{j}{i}(-1)^{j-i}\right)n^i\\
&= \sum_{i=0}^{k}\left(\sum_{j=i}^{k}2a_j\binom{j}{i}(-1)^{j-i}\right)n^{i+1}-\sum_{i=0}^{k}\left(\sum_{j=i}^{k}2ka_j\binom{j}{i}(-1)^{j-i}\right)n^i\\
&= \sum_{i=0}^{k+1}\left(\sum_{j=i-1}^{k}2a_j\binom{j}{i-1}(-1)^{j-i+1}-\sum_{j=i}^{k+1}2ka_j\binom{j}{i}(-1)^{j-i}\right)n^i\\
&= \sum_{i=0}^{k+1}\left(2a_{i-1}-\sum_{j=i}^{k+1}2a_j(-1)^{j-i}\left(\binom{j}{i-1}+k\binom{j}{i}\right)\right)n^i
\end{aligned}$$

Thus the left hand side of the recursion $*_1$ is

$$\begin{aligned}
&(2n-k)f_{n,k}-(2n-2k)f_{n-1,k}\\
&= \sum_{i=0}^{k+1}\left(\sum_{j=i}^{k+1}2a_j(-1)^{j-i}\left(\binom{j}{i-1}+k\binom{j}{i}\right)-ka_i\right)n^i\\
&= \sum_{i=0}^{k}\left(a_i(2i+k)+2\sum_{j=i+1}^{k}(-1)^{j-i}\left(\binom{j}{i-1}+k\binom{j}{i}\right)a_j\right)n^i,
\end{aligned}$$

and so we have to solve, for $a_0, a_1, ..., a_k$, the system

$$l_i(a_{0:k}) := a_i(2i+k)+2\sum_{j=i+1}^{k}(-1)^{j-i}\left(\binom{j}{i-1}+k\binom{j}{i}\right)a_j = c_i, \quad i=0,1,...,k.$$

That this will be possible is evident since the linear form l_i depends actually only on $a_i, ..., a_k$ and the coefficient of a_i is $(2i+k)\neq 0$. So in matrix form the system would be upper triangular $k+1\times k+1$ without zeros on the diagonal.

What we have done till here is to have shown that the equation $*_1$ at the beginning of the proof permits a polynomial solution of degree k. As yet our reasoning did not take into account the hypothesis $n\geq k$ nor any initial values. The general solution to the equation is obtained as the family of all sequences $(f_{n,k} + \dot{f}_n)_{n\in\mathbb{Z}}$, where $f_{n,k}$ is the polynomial sequence obtained above and $(\dot{f}_n)_{n\in\mathbb{Z}}$ is any solution to the homogeneous equation $(2n-k)\dot{f}_n - (2n-2k)\dot{f}_{n-1} = 0$. Now at $n = k$ this equation degenerates to $k\dot{f}_k = 0$ so that $\dot{f}_k = 0$. But then we see, putting successively $n = k+1, k+2, \ldots$ that $\dot{f}_n = 0$. Therefore the *only* solution to the equation $*_1$ possible for $n \geq k$ is the polynomial solution found. By putting in that equation $n = k$ we get $(2k-k)f_{k,k} = 0$, that is $f_{k,k} = 0$. It so happens that the recursion of the Corollary before requires precisely $f_{k,k} = 0$ for the case $k \geq 1$. Therefore the sequence $\mathbb{Z}_{\geq k} \ni n \mapsto f_{n,k}$ coincides indeed with the polynomial sequence of degree k found for $*_1$. □

4. Origin, proof, and impact of the zero identities

Since some time the authors are involved in approaching the odd-dimensional uniform random flight problem in ways differing from those given by García-Pelayo [G-P] and followed up by Borwein and Sinnamon [BS]. For this see the article [SK2]. In particular, in an attempt to find an alternative formula for the probability of a particle subject to a uniform random flight in five dimensional space after n steps being within a ball of radius r of the origin, the second author was led to conjecture that, putting

$$c_{t,l} = (-1)^l \sum_{\mu} (-1)^{\mu} \binom{n}{\mu, l-\mu, t-\mu, \mu+n-l-t}$$

with $t, l \in \mathbb{Z}$, the finite sum

$$\sum_{t,l} \frac{(-1)^t c_{t,l}}{(3n-t-1)!}(x+n-2l)^{3n-t-1},$$

(certainly a polynomial in x of degree $\leq 3n-1$) should be actually 0.

Let us cast the formulation of this proposition into a more manageable form. First note that the sum is actually finite. Assume the expresson for $c_{t,l}$ incorporated in the second sum. Then we can speak of an outer and an inner sum. Recall that $k \in \mathbb{Z}_{<0}$, implies $1/k! = 0$. Assume, say, we choose in the outer sum $t > n$. Then $n - t < 0$ and consequently at least one of $l-\mu, \mu+n-l-t$ is negative. Hence then by the definition of multinomial coefficients, each of the multinomial coefficients associated in the inner sum is 0. Thus since the rôles of l, t are symmetrical we can limit the outer sum and assume it written as $\sum_{0\leq t,l\leq n} \ldots$. It follows that the inner sum also can be limited as $\sum_{\mu=(t+l-n)^+}^{\min\{l,t\}} \cdots$. So the sum is finite and under these conditions the four lower indices of the multinomial coefficients are nonnegative and define a composition of n; that is their sum is n. Now assume four nonnegative integers a, b, c, d define a composition of n. Then let $\mu = a, l = a+b, t = a+c$; we also have $n = a+b+c+d$. Then clearly $t, l \in \{0, 1, ..., n\}$, $0 \leq \mu \leq \min\{t, l\}$, and $t+l-n = 2a+b+c-n = a-d \leq a = \mu$. So $(t+l-n)^+ \leq \mu \leq \min\{t, l\}$. This entails that within the double sum $\sum_{0\leq t,l\leq n} \sum_{\mu=(t+l-n)^+}^{\min\{l,t\}} \ldots$, the quadruple $(\mu, l-\mu, t-\mu, \mu+n-l-t)$ ranges precisely over the compositions (a, b, c, d) of n. What concerns the power $(-1)^{t+l+\mu} = (-1)^{3a+b+c} = (-1)^{n-d}$ occurring in the double sum, in the context what we wish to prove, it can evidently be replaced by $(-1)^d$. Finally we may also replace $x + n$ by x in the proposition above and after dividing by $n!$ we see that the conjecture can be rewritten as claiming that

$$\sum_{a+b+c+d=n} (-1)^d \frac{(x-2a-2b)^{3n-a-c-1}}{a!b!c!d!(3n-a-c-1)!} = 0.$$

Clearly this expression is a polynomial in x of degree at most $3n - 1$. It is 0 as claimed if and only if all its coefficients are 0. The claim in the abstract is obtained simply by taking the $(s-1)$-st derivative of the identity shown. So we focus on the shown here which corresponds to the case $s = 1$. Now for any positive integer λ we have that

$$\text{coefficient of } x^t \text{ of } (x-2a-2b)^\lambda \ = \frac{\lambda!(-2)^{\lambda-t}}{t!(\lambda-t)!}(a+b)^{\lambda-t}.$$

We use this for $\lambda = 3n - a - c - 1$. So we see that to prove the conjecture is to show the following proposition.

Proposition 1. *There holds for* $t = 0, 1, 2, ..., 3n - 1,$ *that*

$$S = S(t, n) = \sum_{a+b+c+d=n} \frac{(-1)^d(-2)^{t-a-c}}{a!b!c!d!(t-a-c)!}(a+b)^{t-a-c} = 0.$$

Here we simplified notation by replacing t by $t_{\text{new}} = 3n - 1 - t$ (and then renaming t_{new} to t). The previous identity for general s similarly translated leads exactly to this same identity. The rest of this section is dedicated to the task of proving Proposition 1.

Recall that in Proposition 2.2 we introduced the polynomials $p_n(x) = (-1)^n \cdot -x \cdot \sigma_n(-x)$ and after Proposition 3.1 we introduced $a_{n,k} = (-1)^k\cdot$ c (coefficient of x^{n-k} in $x\sigma_n(x)$).

Lemma 2. *Let m, n, μ, ν be integers for which $0 < n \leq m \leq 2n-1$ and $0 \leq \mu \leq \nu \leq m-n$. Then*

$$\sum_{i=0}^{n}(-1)^i \frac{p_{m-n-\nu}(i)p_\mu(n-i)}{i!(n-i)!}2^{-i}a_{n-i+\nu,\nu-\mu}(n-i+\mu)! = 0.$$

Proof. The sequence $\{0,1,2,...,n\} \ni i \mapsto 2^{-i}a_{n-i+\nu,\nu-\mu}(n-i+\mu)!$ is well defined and nontrivial in the sense that for the i used, the subindices of a occurring are nonnegative and the first one is larger than or equal to the second one and the factorial occurring is also nonnegative. By Theorem 3.3 the sequence $\mathbb{Z}_{\geq \nu-\mu} \ni n \mapsto f_{n,\nu-\mu} = 2^{n-\nu+\mu}(n-\nu+\mu)!a_{n,\nu-\mu}$ is polynomial of degree $\nu-\mu$. If we replace in a polynomial $p = p(n)$ with coefficients in $\mathbb{R}$ the n by $n+\nu-i$ we get a polynomial expression $p(n+\nu-i)$ which we may view as a polynomial in i. Its leading coefficient as a polynomial in i will be real number equal to $\pm$ its leading coefficient as a polynomial in n. In particular its degree in i will be equal its degree in n. In particular thus the sequence $\{0,1,...,n+\nu\} \ni i \mapsto f_{n-i+\nu,\nu-\mu}$ and for that matter the sequence on $\{0,...,n\}$ above defined at the beginning of the proof will be polynomial of degree $\nu-\mu$. The sum of the lemma is of the form $\sum_{i=0}^{n}\frac{(-1)^i}{i!(n-i)!}q(i)$ where q is a polynomial of degree $\leq (m-n-\nu)+\mu+(\nu-\mu) = m-n < n$ and therefore 0 as follows from the remark to Proposition 2.2. $\square$

Lemma 3. *If $m \in \{n, n+1, ..., 2n-1\}$ and $\nu \in \{0,1,...,m-n\}$, then*

$$\sum_{i=0}^{n}2^{-i}\frac{p_{m-n-\nu}(i)}{i!}\sum_{k=0}^{n-i}(-1)^k\frac{p_{n-i+\nu}(k)}{k!(n-i-k)!} = 0.$$

Proof. It is sufficient to establish the claim substituting the polynomial $p_{n-i+\nu}(k)$ of degree $\leq n-i+\nu$ in the sum by any term of this polynomial. Such a term is given by $a_{n-i+\nu,\nu-\mu}k^{n-i+\mu}$ with $\mu \leq \nu$. By Proposition 2.2 and the remarks following it, we get

$$\sum_{k=0}^{n-i}(-1)^k\tfrac{k^{n-i+\mu}}{k!(n-i-k)!} = (-1)^{n-i}\tfrac{(n-i+\mu)!}{(n-i)!}p_\mu(n-i).$$

If we substitute as indicated , the sum occurring is precisely of the form of the previous lemma and hence it is 0. So the current lemma follows. $\square$

A trivial, but at the end important corollary is:

Theorem 4. *If* $m \in \{n, n+1, ..., 2n-1\}$, *then*

$$\sum_{i=0}^{n} 2^{-i} \sum_{\nu=0}^{m-n} \frac{p_{m-n-\nu}(i)}{i!} \sum_{k=0}^{n-i} (-1)^k \frac{p_{n-i+\nu}(k)}{k!(n-i-k)!} = 0$$

Proof. Take the sum over all $\nu \in \{0, 1, ..., m-n\}$, of the expression above and interchange the two outer sums obtained. $\square$

We shall show that Proposition 1 follows from Theorem 4. We need two further lemmas.

Lemma 5. *If* $0 \le \kappa \le n$ *and* $l \ge 0$ *are integers, then*

$$\sum_{\substack{b+d=n-\kappa \\ a+c=\kappa}} \frac{(-1)^d}{a!b!c!d!}(a+b)^l = \sum_{a+c=\kappa} \frac{1}{a!c!} \sum_{\nu=0}^{l} \binom{l}{\nu} a^{l-\nu} \left\{ \begin{matrix} \nu \\ n-\kappa \end{matrix} \right\}.$$

Proof. Using Proposition 2.2 b we find

$$\begin{aligned} \text{lhs} &= \sum_{a+c=\kappa} \frac{1}{a!c!} \sum_{b+d=n-\kappa} \frac{(-1)^d}{b!d!} \sum_{\nu=0}^{l} \binom{l}{\nu} a^{l-\nu} b^{\nu} \\ &= \sum_{a+c=\kappa} \frac{1}{a!c!} \sum_{\nu=0}^{l} \binom{l}{\nu} a^{l-\nu} \sum_{b+d=n-\kappa} \frac{(-1)^d}{b!d!} b^{\nu} = \text{rhs}. \end{aligned}$$

$\square$

In accordance with [GKP], In the following lemma we use for integer $i \ge 0$ the notation $x^{\underline{i}} = x(x-1)\cdots(x-i+1)$ for falling factorials.

Lemma 6. *If* n *and* k *are nonnegative integers, then*

$$\sum_{l=0}^{n} \binom{n}{l} l^k = 2^n \sum_{i=0}^{k} \left\{ \begin{matrix} k \\ i \end{matrix} \right\} n^{\underline{i}} 2^{-i}, \text{ or, equivalently, } \sum_{l+h=n} \frac{1}{l!h!} l^k = \sum_{i=0}^{k} \left\{ \begin{matrix} k \\ i \end{matrix} \right\} \frac{2^{n-i}}{(n-i)!}.$$

Proof. It is known that $l^k = \sum_i \left\{ \begin{smallmatrix} k \\ i \end{smallmatrix} \right\} l^{\underline{i}}$, see [GKP, equation (6.10)]. Hence the left equality can be deduced as follows:

$$\text{lhs} = \sum_{i=0}^{k} \left\{ \begin{matrix} k \\ i \end{matrix} \right\} \sum_{l=0}^{n} \binom{n}{l} l^{\underline{i}} = \sum_{i=0}^{k} \left\{ \begin{matrix} k \\ i \end{matrix} \right\} \sum_{l=0}^{n} n^{\underline{i}} \binom{n-i}{l-i} = \sum_{i=0}^{k} \left\{ \begin{matrix} k \\ i \end{matrix} \right\} n^{\underline{i}} 2^{n-i} = \text{rhs}.$$

The right equality follows by dividing by $n!$. $\square$

Proof of Proposition 1. This proof follows from the following chain of equalities. Where it eases understanding we give immediately after a step its justification. Have always in mind that $S = S(t,n)$ and t,n are the only fixed quantities of the integers occurring.

$$\begin{aligned}
S &= \sum_{a+b+c+d=n} \frac{(-1)^d(-2)^{t-a-c}}{a!b!c!d!(t-a-c)!}(a+b)^{t-a-c} \\
&= \sum_{\kappa=0}^{n} \sum_{\substack{a+c=\kappa \\ b+d=n-\kappa}} \frac{(-1)^d(-2)^{t-\kappa}}{a!b!c!d!(t-\kappa)!}(a+b)^{t-\kappa} \\
&\quad \text{since for } \kappa > n \text{ the inner sum is empty, and hence } 0 \\
&= \sum_{\kappa=0}^{n} \frac{(-2)^{t-\kappa}}{(t-\kappa)!} \sum_{\substack{a+c=\kappa \\ b+d=n-\kappa}} \frac{(-1)^d}{a!b!c!d!}(a+b)^{t-\kappa} \\
&= \sum_{\kappa=0}^{n} \frac{(-2)^{t-\kappa}}{(t-\kappa)!} \sum_{a+c=\kappa} \frac{1}{a!c!} \sum_{\nu=0}^{t-\kappa} \binom{t-\kappa}{\nu} a^{t-\kappa-\nu} \left\{ {\nu \atop n-\kappa} \right\} \\
&\quad \text{here we used Lemma 4.5 with } l = t-\kappa \\
&= \sum_{\kappa=0}^{n} (-2)^{t-\kappa} \sum_{a+c=\kappa} \frac{1}{a!c!} \sum_{\nu=0}^{t-\kappa} \frac{a^{t-\kappa-\nu}}{\nu!(t-\kappa-\nu)!} \left\{ {\nu \atop n-\kappa} \right\} \\
&\quad \text{use the definition of binomial coefficients; cancel } (t-\kappa)! \\
&= \sum_{\kappa=0}^{n} (-2)^{t-\kappa} \sum_{\nu=n-\kappa}^{t-\kappa} \frac{\left\{ {\nu \atop n-\kappa} \right\}}{\nu!(t-\kappa-\nu)!} \sum_{a+c=\kappa} \frac{a^{t-\kappa-\nu}}{a!c!} \\
&\quad \text{permute inner sums; use that } \left\{ {\nu \atop n-\kappa} \right\} = 0 \text{ for } \nu < n-\kappa \\
&= 2^t \sum_{\kappa=0}^{n} (-1)^{t-\kappa} \sum_{\nu=n-\kappa}^{t-\kappa} \frac{\left\{ {\nu \atop n-\kappa} \right\}}{\nu!(t-\kappa-\nu)!} \sum_{i=0}^{t-\kappa-\nu} \frac{\left\{ {t-\kappa-\nu \atop i} \right\}}{2^i(\kappa-i)!} \\
&\quad \text{use Lemma 4.6; replace there } n \text{ by } \kappa,\ h \text{ by } c,\ l \text{ by } a,\ k \text{ by } t-\kappa-\nu. \\
&\quad \text{Note } (-2)^{t-\kappa}2^{\kappa-i} = (-1)^{t-\kappa}2^{t-i}.
\end{aligned}$$

$$= 2^t \sum_{\kappa=0}^{n} (-1)^{t-\kappa} \sum_{\nu=n-\kappa}^{t-\kappa} \frac{p_{\nu-n+\kappa}(n-\kappa)}{(n-\kappa)!} \sum_{i=0}^{t-\kappa-\nu} \frac{1}{2^i(\kappa-i)!} \frac{p_{t-\kappa-\nu-i}(i)}{i!}$$

use $\left\{ \begin{matrix} a \\ b \end{matrix} \right\} / a! = p_{a-b}(b)/b!$ whenever $a \geq b$ are nonnegative integers; see Proposition 2.2b

$$= 2^t \sum_{\kappa=0}^{n} (-1)^{t-n+\kappa} \sum_{\nu=\kappa}^{t-n+\kappa} \frac{p_{\nu-\kappa}(\kappa)}{\kappa!} \sum_{i=0}^{t-n+\kappa-\nu} \frac{1}{2^i(n-\kappa-i)!} \frac{p_{t-n+\kappa-\nu-i}(i)}{i!}$$

replace summation index κ by $\kappa_{\text{new}} = n - \kappa$

$$= 2^t \sum_{\kappa=0}^{n} (-1)^{m+\kappa} \sum_{\nu=\kappa}^{m+\kappa} \frac{p_{\nu-\kappa}(\kappa)}{\kappa!} \sum_{i=0}^{m+\kappa-\nu} \frac{1}{2^i(n-\kappa-i)!} \frac{p_{m+\kappa-\nu-i}(i)}{i!}$$

put $m = t - n$

$$= 2^t(-1)^m \sum_{\kappa=0}^{n} (-1)^{\kappa} \sum_{\nu=0}^{m} \frac{p_{\nu}(\kappa)}{\kappa!} \sum_{i=0}^{m-\nu} \frac{1}{2^i(n-\kappa-i)!} \frac{p_{m-\nu-i}(i)}{i!}$$

introduce $\nu_{\text{new}} = \nu - \kappa$; revert name to ν.

$$= 2^t(-1)^m \sum_{\nu=0}^{m} \sum_{i=0}^{m-\nu} 2^{-i} \frac{p_{m-\nu-i}(i)}{i!} \sum_{\kappa=0}^{n-i} (-1)^{\kappa} \frac{p_{\nu}(\kappa)}{\kappa!(n-i-\kappa)!}$$

rearrange summations; push $\sum_{\kappa=0}^{n}$ into interior. Note that replacing upper limit n by $n-i$ is justified. Also if $i > n$, then $S = 0$ is clear. So henceforth, $0 \leq i \leq n$ is assumed.

$$= 2^t(-1)^m \sum_{i=0}^{n} 2^{-i} \sum_{\nu=0}^{m-i} \frac{p_{m-\nu-i}(i)}{i!} \sum_{\kappa=0}^{n-i} (-1)^{\kappa} \frac{p_{\nu}(\kappa)}{\kappa!(n-i-\kappa)!}$$

note that summation index inequalities $0 \leq \nu \leq m,\ 0 \leq i \leq m - \nu,\ 0 \leq i \leq n,\ 0 \leq \kappa \leq n-i$ are equivalent to $0 \leq i \leq n,\ 0 \leq \nu \leq m-i,\ 0 \leq \kappa \leq n-i$.

$$= 2^t(-1)^m \sum_{i=0}^{n} 2^{-i} \sum_{\nu=i-n}^{m-n} \frac{p_{m-\nu-n}(i)}{i!} \sum_{\kappa=0}^{n-i} (-1)^{\kappa} \frac{p_{n-i+\nu}(\kappa)}{\kappa!(n-i-\kappa)!}$$

introduce $\nu_{\text{new}} = \nu + i - n$.

$$= 2^t(-1)^m \sum_{i=0}^{n} 2^{-i} \sum_{\nu=0}^{m-n} \frac{p_{m-\nu-n}(i)}{i!} \sum_{\kappa=0}^{n-i} (-1)^{\kappa} \frac{p_{n-i+\nu}(\kappa)}{\kappa!(n-i-\kappa)!}$$

If $\nu < 0$ then $p_{n-i+\nu}$ has degree $< n-i$ so that the third sum is 0 by Proposition 2.2. So symbol $\sum_{\nu=i-n}^{m-n}$ can be replaced by $\sum_{\nu=0}^{m-n}$.

Looking at this last expression found for S we can argue why $S = 0$ if $t = 0, 1, ..., 3n-1$. If $m < n$ then the middle sum $\sum_{\nu=0}^{m-n}$ ... is empty, and hence $S = 0$. (This is actually already visible in the sixth equation.) For the cases $m = n, n+1, ..., 2n-1$, Theorem 4 yields that $S = 0$. These m correspond by the 10th equation to $t = 2n, ..., 3n-1$. So we have proved $S = 0$ for all $t = 0, 1, ..., 3n-1$. This shows Proposition 1. □

We conclude by mentioning that the zero identity presented is very probably not an isolated phenomenon. We have experimental reasons to believe that, for example, the following is true as well

Put

$$c_{n,t,l} = \sum_{0\le\mu,\nu,\rho\le t}(-1)^{\mu+t}3^{n-\nu-\rho}\binom{n}{\mu,\nu,l-\mu-\nu,-\mu-2\nu-2\rho+t,\rho,-l+\mu+2\nu+n+\rho-t}.$$

Then

$$\sum_{l=0}^{n}\sum_{t=0}^{2n}\frac{c_{n,t,l}}{(5n-t-1)!}(x+n-2l)^{5n-t-1} = 0.$$

Establishing this would probably be useful to give an alternative the Borwein-Sinnamon formula for dimension 7 just as the zero identity mentioned in the first paragraph of the current section (and proved via Proposition 1) allows to prove an alternative to the Borwein Sinnamon formula for dimension 5.

Possibly by a profound re-inspection of the proofs given in the current section, the proof of Proposition 1 can be abbreviated somewhat; or a completely different less computational proof can be found. But the authors are currently unable to perform any significant cuts without compromising readability. After a significantly more transparent proof of what we have is available, it may be worth to attack the further conjectured identities.

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