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A COUNTEREXAMPLE TO A CONJECTURE OF M. ISMAIL

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Abstract: In an earlier work [K. Castillo et al., J. Math. Anal. Appl. **514** (2022) 126358], we give positive answer to the first part of a conjecture posed by M. Ismail concerning the characterization of the continuous q-Jacobi polynomials, Al-Salam-Chihara polynomials or special or limiting cases of them. In this note we present a counterexample to the second part of such a conjecture, and so this issue is definitively closed.

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1. Introduction

The Askey-Wilson divided difference operator is defined by

$$\mathcal{D}_q f(x) = \frac{\breve{f}\big(q^{1/2}e^{i\theta}\big) - \breve{f}\big(q^{-1/2}e^{i\theta}\big)}{\breve{e}\big(q^{1/2}e^{i\theta}\big) - \breve{e}\big(q^{-1/2}e^{i\theta}\big)}, \tag{1}$$

where, for each polynomial f, $\breve{f}(e^{i\theta}) = f(\cos\theta)$ and $e(x) = x$ (see [3, Section 12.1]). In [1], we give positive answer to the first part of the following Ismail's conjecture (see [3, Conjecture 24.7.8]):

Conjecture 1. *Let $(p_n)_{n\geq 0}$ be a sequence of orthogonal polynomials and let π be a polynomial which does not depend on n. If*

$$\pi(x)\mathcal{D}_q\, p_n(x) = \sum_{k=-1}^{1} c_{n,k} p_{n+k}(x),$$

then $(p_n)_{n\geq 0}$ are continuous q-Jacobi polynomials or Al-Salam-Chihara polynomials, or special or limiting cases of them. The same conclusion follows

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if

$$\pi(x)\mathcal{D}_q\, p_n(x) = \sum_{k=-r}^{s} c_{n,k} p_{n+k}(x), \tag{2}$$

for positive integers r and s.

The second part of this conjecture is certainly a much more complex problem than the first one. However, after much manipulation of similar structural relations in a number of recent works, we found the second part of the conjecture less and less convincing. Now in the next section we propose a counterexample.

2. Counterexample

Throughout this section we assume that $0 < q < 1$. Set $x(s) = (q^s + q^{-s})/2$. Taking $e^{i\theta} = q^s$ in (1), $\mathcal{D}_q$ reads

$$\mathcal{D}_q f(x(s)) = \frac{f\big(x(s+\frac{1}{2})\big) - f\big(x(s-\frac{1}{2})\big)}{x(s+\frac{1}{2}) - x(s-\frac{1}{2})}.$$

Define

$$\mathcal{S}_q f(x(s)) = \frac{f\big(x(s+\frac{1}{2})\big) + f\big(x(s-\frac{1}{2})\big)}{2},$$

and

$$\alpha_n = \frac{q^{n/2} + q^{-n/2}}{2}, \quad \gamma_n = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} \quad (n = 0, 1, \dots),$$

and $\alpha = \alpha_1$. Recall that

$$\mathcal{D}_q\big(fg\big) = \big(\mathcal{D}_q f\big)\big(\mathcal{S}_q g\big) + \big(\mathcal{S}_q f\big)\big(\mathcal{D}_q g\big), \tag{3}$$

$$\mathcal{S}_q\big(fg\big) = \big(\mathcal{D}_q f\big)\big(\mathcal{D}_q g\big)\mathtt{U}_2 + \big(\mathcal{S}_q f\big)\big(\mathcal{S}_q g\big), \tag{4}$$

where $\mathtt{U}_2(x) = (\alpha^2 - 1)(x^2 - 1)$. All these properties and definitions, even the notation, can be found, for instance, in [2]. The Askey-Wilson polynomials are defined by

$$p_n(x; a, b, c, d \,|\, q) = \frac{(ab, ac, ad; q)_n}{a^n}\, {}_4\phi_3\left(\left.\begin{matrix} q^{-n},\ abcdq^{n-1},\ ae^{i\theta},\ ae^{-i\theta} \\ ab,\ ac,\ ad \end{matrix}\right| q,\ q\right),$$

where $x = \cos\theta$. If we take $a = q^{1/2\alpha+1/4}$, $b = q^{1/2\alpha+3/4}$, $c = -a$, and $d = -b$, we get the continuous q-Jacobi polynomials. If we take $d = 0$, we get the

where

$$d_{k,1} = (\alpha^2-1)\alpha_k + \alpha c_{k,1},$$

$$d_{k,2} = (\alpha^2-1)(c_k + \alpha_k(B_k + B_{k+1})) + \alpha c_{k,2} - (B_k - \alpha B_{k+1})c_{k,1},$$

$$d_{k,3} = (\alpha^2-1)((B_k + B_{k-1})c_k + \alpha_k(B_k^2 + C_k + C_{k+1} - 1)) + \alpha c_{k,1}C_{k+1} - c_{k-1,1}C_k + (\alpha-1)c_{k,2}B_n + \alpha c_{k,3},$$

$$d_{k,4} = (\alpha^2-1)((B_k + B_{k-1})\alpha_k C_k + (C_k + B_{k-1}^2 + C_{k-1} - 1)c_k) - (c_{k-1,2} - \alpha c_{k,2})C_k - (B_k - \alpha B_{k-1})c_{k,3} + \alpha c_{k,4},$$

$$d_{k,5} = (\alpha^2-1)C_{k-1}(\alpha_k C_k + c_k(B_{k-1} + B_{k-2})) + \alpha c_{k,3}C_{k-1} - c_{k-1,3}C_k - (B_k - \alpha B_{k-2})c_{k,4},$$

$$d_{k,6} = (\alpha^2-1)c_k C_{k-1}C_{k-2} + \alpha c_{k,4}C_{k-2} - c_{k-1,4}C_k.$$

Finally, the reader should satisfy himself that $d_{k,1} = c_{k+1,1}$, $d_{k,2} = c_{k+1,2}$, $d_{k,3} = c_{k+1,3}$, $d_{k,4} = c_{k+1,4}$, $d_{k,5} = 0$, and $d_{k,6} = 0$, and (7) holds for $n = k+1$. That completes the inductive step, and hence the proof. ∎

In view of Proposition 2.1, an interesting open problem is to characterize the sequence of orthogonal polynomials such that $\mathcal{S}_q P_n$ can be writen as a linear combination of P_n and P_{n-1}.

References

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continuous dual q-Hahn polynomials. If we take $c = d = 0$, we get the Al-Salam-Chihara polynomials. The sequence of monic continuous dual q-Hahn polynomials, $(H_n(x; a, b|q))_{n\geq 0}$, satisfies

$$xH_n(x; a, b, c|q) = H_{n+1}(x; a, b, c|q) + a_n H_n(x; a, b, c|q) + b_n H_{n-1}(x; a, b, c|q), \tag{5}$$

where $H_{-n-1}(\cdot; a, b, c|q) = 0$ and

$$a_n = \frac{1}{2}(a + a^{-1} - a(1-q^n)(1-bcq^{n-1}) - a^{-1}(1-abq^n)(1-acq^n)),$$

$$b_n = \frac{1}{4}(1-abq^{n-1})(1-acq^{n-1})(1-bcq^{n-1})(1-q^n).$$

Define $P_n = H_n(\cdot; 1, -1, q^{1/4}|q^{1/2})$. Clearly, P_n is not a continuous q-Jacobi polynomial or Al-Salam-Chihara polynomial or, much less, special or limiting cases of them. These polynomials satisfy, among other relations, a relation of type (2) with $r = 2$ and $s = 1$.

Proposition 2.1. *Let $P_n(x) = H_n(x; 1, -1, q^{1/4}|q^{1/2})$. The sequence $(P_n)_{n\geq 0}$ satisfies the following relations:*

$$\mathcal{S}_q P_n(x) = \alpha_n P_n(x) + c_n P_{n-1}(x), \tag{6}$$

$$\begin{aligned}\mathcal{U}_2(x)\mathcal{D}_q P_n(x) &= (\alpha^2-1)\gamma_n P_{n+1}(x) + \big(c_{n+1} - \alpha c_n + (1-\alpha)\alpha_n B_n\big)P_n(x)\\ &\quad + \big((B_n - \alpha B_{n-1})c_n + (1-\alpha^2)\gamma_n C_n\big)P_{n-1}(x)\\ &\quad + (c_{n-1}C_n - \alpha c_n C_{n-1})P_{n-2}(x),\end{aligned} \tag{7}$$

where

$$B_n = \frac{1}{2}\Big((1+q^{-1/2})q^{n/2} + 1 - q^{-1/2}\Big)q^{(2n+1)/4},$$

$$C_n = \frac{1}{4}(1+q^{(n-1)/2})(1-q^{n/2})(1-q^{n-1/2}),$$

$$c_n = \frac{1}{4}(1-q^{n/2})(1-q^{n-1/2})(1+q^{(n-1)/2})q^{-(2n-1)/4}.$$

Proof: (5) makes it obvious that

$$xP_n(x) = P_{n+1}(x) + B_n P_n(x) + C_n P_{n-1}(x) \quad (n = 0, 1, \dots). \tag{8}$$

The proof is by complete mathematical induction on n. Note that $c_0 = 0$ and $\alpha_0 = 1$, and so

$$c_1 - \alpha c_0 + (1-\alpha)\alpha_0 B_0 = 0.$$

Hence, for $n = 0$ we have

$$\mathcal{S}_q P_0 = 1 = \alpha_0 P_0 + c_0 P_{-1} = 1,$$

$$\mathtt{U}_2(x)\mathcal{D}_q P_0(x) = 0 = (\alpha^2 - 1)\gamma_0 P_1 + 0P_0 = 0.$$

Assuming (6) and (7) hold for all $n \leq k$, we will prove it for $n = k+1$. Set

$$\begin{aligned} c_{n,1} &= (\alpha^2 - 1)\gamma_n, \\ c_{n,2} &= c_{n+1} - \alpha c_n + (1-\alpha)\alpha_n B_n, \\ c_{n,3} &= (B_n - \alpha B_{n-1})c_n + (1-\alpha^2)\gamma_n C_n, \\ c_{n,4} &= c_{n-1}C_n - \alpha c_n C_{n-1}. \end{aligned}$$

Now (7) reads as

$$\mathtt{U}_2(x)\mathcal{D}_q P_n(x) = c_{n,1}P_{n+1}(x) + c_{n,2}P_n(x) + c_{n,3}P_{n-1}(x) + c_{n,4}P_{n-2}(x). \tag{9}$$

Applying $\mathcal{S}_q$ to (8), and using (4), we get

$$\begin{aligned} \mathcal{S}_q P_{n+1}(x) &= \mathtt{U}_2(x)\mathcal{D}_q\, x\mathcal{D}_q P_n(x) + \mathcal{S}_q\, x\mathcal{S}_q P_n(x) - B_n\mathcal{S}_q P_n(x) + C_n\mathcal{S}_q P_{n-1}(x) \\ &= \mathtt{U}_2(x)\mathcal{D}_q P_n(x) - B_n\mathcal{S}_q P_n(x) - C_n\mathcal{S}_q P_{n-1}(x) + \alpha x\mathcal{S}_q P_n(x). \end{aligned} \tag{10}$$

From (10), and using (9) for $n = k$ and (6) for $n = k-1$ and $n = k$, we obtain

$$\begin{aligned} \mathcal{S}_q P_{k+1}(x) &= c_{k,1}P_{k+1}(x) + c_{k,2}P_k(x) + c_{k,3}P_{k-1}(x) + c_{k,4}P_{k-2}(x) \\ &\quad - B_k(\alpha_k P_k(x) + c_k P_{k-1}(x)) - C_k(\alpha_{k-1}P_{k-1}(x) + c_{k-1}P_{k-2}(x)) \\ &\quad + \alpha x(\alpha_k P_k(x) + c_k P_{k-1}(x)). \end{aligned}$$

From (8) we have

$$\begin{aligned} \alpha x(\alpha_k P_k(x) + c_k P_{k-1}(x)) &= \alpha\alpha_k(P_{k+1}(x) + B_k P_k(x) + C_k P_{k-1}(x)) \\ &\quad + \alpha c_k(P_k(x) + B_{k-1}P_{k-1}(x) + C_{k-1}P_{k-2}(x)). \end{aligned}$$

We leave to the reader the verification that

$$c_{k,3} + (\alpha\alpha_k - \alpha_{k-1})C_k + (\alpha B_{k-1} - B_k)c_k = c_{k,4} + \alpha c_k C_{k-1} - c_{k-1}C_k = 0,$$

and

$$\alpha_{k+1} = c_{k,1} + \alpha\,\alpha_k, \quad c_{k+1} = c_{k,2} + \alpha c_k + (\alpha - 1)\alpha_k B_k.$$

We thus get

$$\begin{aligned} \mathcal{S}_q P_{k+1}(x) &= (c_{k,1} + \alpha\,\alpha_k)P_{k+1}(x) + (c_{k,2} + \alpha c_k + (\alpha-1)\alpha_k B_k)P_k(x) \\ &\quad + (c_{k,3} + (\alpha\alpha_k - \alpha_{k-1})C_k + (\alpha B_{k-1} - B_k)c_k)P_{k-1}(x) \\ &\quad + (c_{k,4} + \alpha c_k C_{k-1} - c_{k-1}C_k)P_{k-2}(x) \\ &= \alpha_{k+1}P_{k+1}(x) + c_{k+1}P_k(x), \end{aligned}$$

and (6) holds for $n = k+1$.

Applying now $\mathcal{D}_q$ to (8), and using (3), we get

$$\mathcal{D}_q P_{k+1}(x) = \mathcal{S}_q P_k(x) + (\alpha x - B_k)\mathcal{D}_q P_k(x) - C_k\mathcal{D}_q P_{k-1}(x). \tag{11}$$

From (8) we have

$$\begin{aligned} \mathtt{U}_2(x)P_k(x) &= (\alpha^2-1)P_{k+2}(x) + (\alpha^2-1)(B_k + B_{k+1})P_{k+1}(x) \\ &\quad + (\alpha^2-1)(B_k^2 + C_{k+1} + C_k - 1)P_k(x) \\ &\quad + (\alpha^2-1)C_k(B_k + B_{k-1})P_{k-1}(x) + (\alpha^2-1)C_k C_{k-1}P_{k-2}(x). \end{aligned}$$

Hence, multiplying (11) by $\mathtt{U}_2$ and using (6) for $n = k$ and (9) for $n = k-1$ and $n = k$, we get

$$\begin{aligned} \mathtt{U}_2(x)\mathcal{D}_q P_{k+1}(x) &= d_{k,1}P_{k+2}(x) + d_{k,2}P_{k+1}(x) + d_{k,3}P_k(x) + d_{k,4}P_{k-1}(x) \\ &\quad + d_{k,5}P_{k-2}(x) + d_{k,6}P_{k-3}(x), \end{aligned}$$