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Francis Borceux
Université catholique de Louvain, Institut de Recherche em Mathématique et Physique, 1348 Louvain-la-Neuve, Belgium
E-mail address: francis.borceux@uclouvain.be

Maria Manuel Clementino
University of Coimbra, CMUC, Department of Mathematics, 3000-143 Coimbra, Portugal
E-mail address: mmc@mat.uc.pt

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ON COHERENT SYSTEMS OF SUBOBJECTS WITH APPLICATION TO TORSION THEORY

FRANCIS BORCEUX AND MARIA MANUEL CLEMENTINO

To Marta Bunge, a friendly colleague and an inspiring mathematician

Abstract: In a coherent category, the posets of subobjects have very strong properties. We emphasize the validity of these properties, in general categories, for well-behaved classes of subobjects. As an example of application, we investigate the problem of the various torsion theories which can be universally associated with a pretorsion one.

Keywords: coherent system of subobjects, (pre)torsion theory, torsion objects, torsion free objects, stable category, category of fractions.
Math. Subject Classification (2020): 18E40, 18B50, 18A20, 18E35.

1. Introduction

A coherent category (see [14]) has in particular distributive lattices of subobjects whose operations are preserved by pullbacks. In this paper, we want to draw attention to the case of arbitrary categories in which there exist distributive lattices of particular subobjects – we call them *distinguished* – whose operations are preserved by pullbacks. We call this a *coherent system of subobjects*. A well-known example is that of complemented subobjects in a lextensive category. But there are many other examples of interest.

In the case of small categories, we call a subcategory $\mathcal{S}$ of $\mathcal{A}$ *saturated* when an arrow of $\mathcal{A}$ lies in $\mathcal{S}$ as soon as its domain or its codomain is in $\mathcal{S}$. The saturated subcategories constitute a distributive lattice whose operations are preserved by pullbacks. This result generalizes to the case of internal categories in an arbitrary coherent category $\mathcal{C}$, yielding an interesting bunch of examples of coherent systems of subobjects. This particularizes further to preordered objects in a coherent category.

We apply these considerations to the study of the stable category associated with a pretorsion theory. A *pretorsion theory* (see [12]) in a category $\mathcal{C}$

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consists of giving two classes of objects, respectively called the *torsion objects* and the *torsion free objects*, together with adequate axioms. The objects which are both torsion and torsion free are called *trivial*. When all trivial objects are zero objects, the pretorsion theory is called a *torsion theory*. The problem of the *stable category* associated with a torsion theory is that of constructing naturally a torsion theory from a given pretorsion theory (see [5]).

Of course, to construct a torsion theory from a pretorsion one, one must identify all trivial objects to a zero object. In [11, 5], this is done with the extra requirement that, when $A = S \amalg T$ is the disjoint union of two complemented subobjects, that union is preserved; in particular, if T is trivial, A is identified with S in the stable category. In our paper, we consider the more flexible requirement that when $A = S \cup T$ is the union of two distinguished subobjects with T trivial, then T is identified with $\mathbf{0}$ and A is identified with S.

We prove a general theorem on the existence and the universality of the stable category, which applies in particular to the case of preordered objects in an exact coherent category (see [11, 3]), and to the case of internal categories in a Grothendieck topos (see [6, 7]). But the flexibility of our approach allows also choosing only trivial subobjects (i.e. only $\mathbf{0}$ and the object itself) as distinguished subobjects. In that case, our theorem yields the torsion theory universally associated with the given pretorsion theory. And, under a very mild assumption, we prove that the stable category can then be obtained as the category of fractions which inverts all morphisms $\mathbf{0} \longrightarrow A$, with A a trivial object.

2. Coherent systems of subobjects

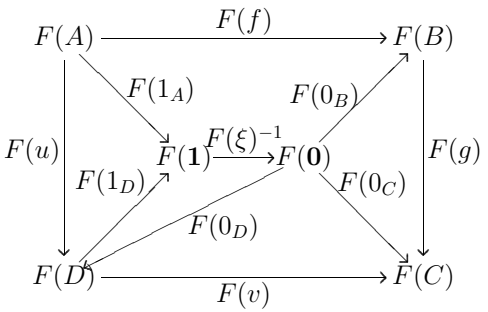
The structure that we want to promote in this paper is the following one*.

Definition 2.1. *Let $\mathcal{C}$ be a category with a strict initial object $\mathbf{0}$. By a* coherent system of subobjects *in $\mathcal{C}$ is meant, for each object $C \in \mathcal{C}$, the choice of a class of so-called* distinguished subobjects of C*, in such a way that:*

(CS1): $\mathbf{0}$ *and A are distinguished in A;*

(CS2): *the union of two distinguished subobjects exists and is distinguished;*

*Since this cannot hurt in this paper, we choose to use freely the common abuse of language which does not distinguish subobjects and monomorphisms.



Since f and g are non trivial, we have

$$G(\sigma(g)) \circ G(\sigma(f)) = F(g) \circ F(f) = F(gf) = F(vu) = F(v) \circ F(u).$$

Since D is trivial, so are u and v and thus we know already that

$$F(u) = F(0_B) \circ F(\xi)^{-1} \circ F(1_A),\ F(v) = F(0_C) \circ F(\xi)^{-1} \circ F(1_B).$$

The composite of these two morphisms reduces to $F(0_C) \circ F(\xi)^{-1} \circ F(1_A)$ since $1_B \circ 0_B = \mathsf{id}_{\mathbf{0}}$. But this is precisely $G(0_{AC})$, that is $G(\sigma(gf))$ since gf is trivial. ∎

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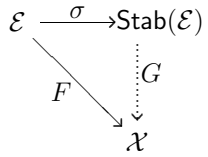
$\mathbf{0} \rightarrowtail A$, *with* A *trivial. Equivalently,* Σ *can be chosen as just the singleton* $\{\mathbf{0} \rightarrowtail \mathbf{1}\}$.

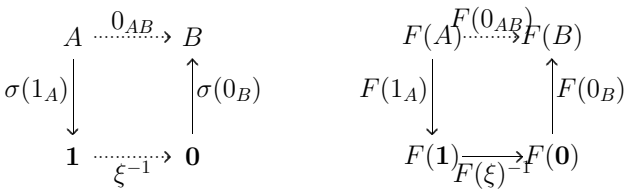
Proof: When A is trivial, $0_A \colon \mathbf{0} \longrightarrow A$ becomes an isomorphism in $\mathsf{Stab}(\mathcal{E})$, by Proposition 5.5. This is in particular the case for the unique morphism $\xi \colon \mathbf{0} \longrightarrow \mathbf{1}$.

Choose now a functor $F \colon \mathcal{E} \longrightarrow \mathcal{X}$ which transforms the morphism ξ in an isomorphism.



We must prove the existence of a unique factorization G. Since σ is bijective on objects, we must put $G(A) = F(A)$ for every object A. And of course by functoriality, we must put $F(\xi^{-1}) = F(\xi)^{-1}$.

Let us use the description of $\mathsf{Stab}(\mathcal{E})$ as in Corollary 7.2, writing 1_A for the unique morphism from A to $\mathbf{1}$ in $\mathcal{E}$. If f is a non trivial morphism in $\mathcal{E}$, we must again put $G(f) = F(f)$. And when it goes about the zero morphism 0_{AB} in $\mathsf{Stab}(\mathcal{E})$, we have the situation



where the left hand square is commutative in $\mathsf{Stab}(\mathcal{E})$. We are thus forced to define

$$G(0_{AB}) = F(0_B) \circ F(\xi)^{-1} \circ F(1_A).$$

This shows in particular that $F(u) = F(0_B) \circ F(\xi)^{-1} \circ F(1_A)$ for every trivial morphism $u \colon A \longrightarrow B$.

Checking the functoriality of G is then routine. For example consider the most involved case: two non trivial morphisms f, g in $\mathcal{E}$ whose composite is trivial. We have thus a factorization $fg = uv$ through a trivial object D, yielding the following situation in $\mathcal{X}$

(CS3): *the pullback of a distinguished subobject along an arbitrary morphism exists and is distinguished;*

(CS4): *pulling back distinguished subobjects preserves their union;*

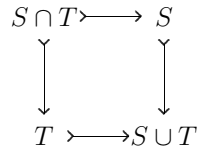
(CS5): *given two monomorphisms*

$$A \rightarrowtail B \rightarrowtail C$$

if A *is distinguished in* B *and* B *is distinguished in* C*, then* A *is distinguished in* C.

A coherent system of subobjects is called effective *when moreover (see* [2]*)*

(CS6): *the union of distinguished objects is effective, that is, given two distinguished subobjects* S *and* T *of* A*, the following pullback is also a pushout:*



Let us recall that, given a strict initial object $\mathbf{0}$, every morphism with domain $\mathbf{0}$ is a monomorphism. Indeed if there exists a morphism $X \longrightarrow \mathbf{0}$, that morphism is necessarily unique since it must be an isomorphism, with the unique morphism $\mathbf{0} \longrightarrow X$ as inverse.

Proposition 2.2. *Let* $\mathcal{C}$ *be a category with a strict initial object, provided with a coherent system of subobjects. The following properties hold:*

(CS7): *the intersection of two distinguished subobjects exists and is distinguished;*

(CS8): *given three distinguished subobjects* R*,* S*,* T *of* A

$$R \cap (S \cup T) = (R \cap S) \cup (R \cap T);$$

(CS9): *if* S *is a distinguished subobject of* A*,* S *is distinguished in every subobject* $S \subseteq T \subseteq A$.

Proof: The case of the intersection follows from (CS3) and (CS5). The distributivity law is a special instance of (CS4). The last assertion is obtained via (CS3) when pulling back $S \subseteq A$ along $T \subseteq A$. ∎

We shall also meet the following related notion, which is reminiscent of the notion of extremal epimorphism:

Definition 2.3. *Let $\mathcal{C}$ be a category with a strict initial object, provided with a coherent system of subobjects. An epimorphism $f\colon A \longrightarrow B$ is called* distinguished *when it does not factor through any proper distinguished subobject of B.*

Proposition 2.4. *Let $\mathcal{C}$ be a category with a strict initial object, provided with a coherent system of subobjects. An epimorphism is distinguished if and only if it is left orthogonal to every distinguished monomorphism.*

Proof: The classical proof for extremal versus strong epimorphism applies as such. ∎

Proposition 2.5. *Let $\mathcal{C}$ be a category with a strict initial object, provided with a coherent system of subobjects.*

(1) *Distinguished epimorphisms are stable under composition.*
(2) *If gf is a distinguished epimorphism, so is g.*

Proof: Once more the standard proof for extremal epimorphisms applies as such. ∎

Examples 2.6. *Here is a first list of rather immediate examples of coherent systems of subobjects; the last three are effective.*

(1) *All subobjects in a coherent category.*
(2) *The complemented subobjects in a lextensive category (see* [8]*).*
(3) *The open (respectively closed, clopen) subspaces in the category of topological spaces.*
(4) *Only* $\mathbf{0}$ *and the object itself in a category with a strict initial object. We call this the* indiscrete *coherent system of subobjects.*

Proof: In the lextensive case, given two complemented subobjects S, T of A, their union is the disjoint coproduct

$$(S \cap \complement T) \amalg (S \cap T) \amalg (\complement S \cap T).$$

The rest is obvious. ∎

Despite its triviality, the last of these examples will play a significant role in Section 7. But we want to focus now on the following example, which suggested to us the notion of coherent system of subobjects.

Example 2.7. *A sub-preordered set $(S, \leq)$ of $(A, \leq)$ with the induced preorder is called* open *(see* [11]*) when*

$$a \leq b \text{ and } b \in S \implies a \in S$$

a category provided with a torsion theory, universally associated with those data.

Proof: The universal solution is the stable category corresponding to the choice of the indiscrete coherent system of subobobjects (see Example 4.13). Indeed, given two distinguished subobjects of A, the only possible bicartesion squares as in Theorem 6.4 are

$$\begin{array}{ccc} \mathbf{0} & = & \mathbf{0} \\ \| & & \| \\ \mathbf{0} & = & \mathbf{0} \end{array} \qquad \begin{array}{ccc} A & = & A \\ \| & & \| \\ A & = & A \end{array} \qquad \begin{array}{ccc} \mathbf{0} & \rightarrowtail & A \\ \| & & \| \\ \mathbf{0} & \rightarrowtail & A \end{array}$$

and these are trivially sent to actual pushouts by every torsion functor. ∎

Corollary 7.2. *In the conditions of Theorem 7.1, the universal stable category* $\mathsf{Stab}(\mathcal{E})$ *admits the following description:*

- *the objects are those of $\mathcal{E}$;*
- *the arrows $A \longrightarrow B$ in* $\mathsf{Stab}(\mathcal{E})$ *are the non-trivial arrows from A to B in $\mathcal{E}$, together with a formally added arrow 0_{AB}.*

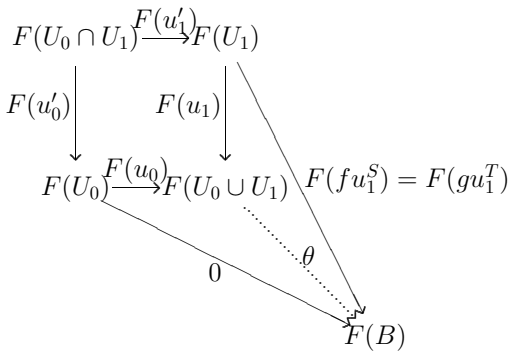
The composition in $\mathsf{Stab}(\mathcal{E})$ *of two non-trivial morphisms of $\mathcal{E}$ is their composition in $\mathcal{E}$ when this composite is non-trivial, and the corresponding zero morphism otherwise. Composing whatever arrow with a zero arrow is the corresponding zero arrow.*

Proof: The morphisms from A to B in $\mathsf{DisPar}(\mathcal{E})$ are

- $(\mathbf{0}, A, f)$ for every morphism $f\colon A \longrightarrow B$;
- $(A, \mathbf{0}, 0_B)$;
- (A, A, f) for every trivial morphism $f\colon A \longrightarrow B$.

By Lemma 5.3, all cases where f is trivial are identified with $(A, \mathbf{0}, 0_B)$ in $\mathsf{Stab}(\mathcal{E})$. And when f, g are non-trivial, the only possible choice for U_0 in a congruence diagram is $U_0 = \mathbf{0}$, thus $U_1 = A$, which yields $f = g$. The conclusion follows at once, when putting $0_{AB} = \sigma(A, \mathbf{0}, B)$ ∎

Proposition 7.3. *Consider a category $\mathcal{E}$ provided with a torsion theory, a strict initial object and a terminal object. Suppose that the initial and the terminal object are both trivial. The stable category* $\mathsf{Stab}(\mathcal{E})$ *of Theorem 7.1 is then the category of fractions $\mathcal{E}[\Sigma^{-1}]$, where Σ is the class of morphisms*



yielding the unique factorization θ through the zero-pushout. But $U_0 \cup U_1 = A$ so that θ is a morphism from $F(A)$ to $F(B)$. Let us prove that both morphisms $\varphi = G(U_0, U_1, f)$ and $\psi = G(T_0, T_1, g)$ are equal to θ. Of course it suffices to do the job for one of them: we do it for φ.

We have first

$$\varphi F(u_1) = \varphi F(s_1) F(u_1^S) = F(f) F(u_1^S).$$

To prove that $\varphi \circ F(u_0) = 0$, observe that

$$U_0 = U_0 \cap A = U_0 \cap (S_0 \cup S_1) = (U_0 \cap S_0) \cup (U_0 \cap S_1).$$

Using the zero-pushout in $\mathcal{X}$ constructed from these last two distinguished subobjects in $\mathcal{E}$, it suffices to prove that $\varphi \circ F(u_0)$ is zero on $F(U_0 \cap S_0)$ and $F(U_0 \cap S_1)$. In the first case it is because φ is already zero at the level $F(S_0)$ and in the second case, because $F(f)$ is zero at the level $F(U_0 \cap S_1)$.

Checking the functoriality of G is just routine verification. Its uniqueness is attested by that of φ at the beginning of the proof. And by definition of the torsion theory on $\mathsf{Stab}(\mathcal{E})$ and assumption on F, G is a torsion functor. ∎

All the results of this section apply again to Examples 4.13, 4.14, 4.15 and 4.16.

7. The associated torsion theory

In this section, we want to focus on the torsion theory universally associated with a pretorsion theory, without any further requirement of compatibility with unions, coproducts, or whatever.

Theorem 7.1. *Let $\mathcal{E}$ be a category provided with a pretorsion theory and a strict initial object. Suppose that the initial object is trivial. There exists*

and closed *when*

$$a \leq b \text{ and } a \in S \implies b \in S.$$

Clopen means, as usual, open and closed. The open (respectively closed, clopen) sub-preordered sets in the sense of [11] *constitute effective coherent systems of subobjects.*

Proof: The proof of having effective coherent systems of subobjects is a special case of our next example. Indeed, a preordered set is a small category with at most one arrow between every two objects. ∎

We generalize now the notions of open, closed and clopen subobjects to the case of subcategories. To avoid any ambiguity of terminology in the closed case, we prefer to use *left saturated*, *right saturated* and *saturated* instead of open, closed and clopen.

Example 2.8. *A subcategory $\mathcal{S} \subseteq \mathcal{C}$ is* left saturated *when*

$$f\colon A \to B \text{ and } B \in \mathcal{S} \implies f \in \mathcal{S}$$

and right saturated when

$$f\colon A \to B \text{ and } A \in \mathcal{S} \implies f \in \mathcal{S}.$$

$\mathcal{S}$ is saturated *in $\mathcal{C}$ when it is both left and right saturated. Left saturated, right saturated and saturated subcategories constitute effective coherent systems of subobjects in* Cat.

Proof: Only the case of the union requires a comment. To construct the union of two subcategories $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{C}$, one considers first the graph constructed on the set theoretical union of both sets of objects, and the set theoretical union of both sets of arrows. One adds further all the possible composites in $\mathcal{C}$ of chains of consecutive arrows in this last union. When both $\mathcal{S}$ and $\mathcal{T}$ are left saturated, the last arrow of a chain is thus in $\mathcal{S}$ or in $\mathcal{T}$, and therefore by left saturation, so do all the arrows of the chain and thus also their composite. Analogously for right saturation, starting with the first arrow of the chain. Thus the set of arrows of $\mathcal{S} \cup \mathcal{T}$ is the set theoretical union of the sets of arrows of $\mathcal{S}$ and $\mathcal{T}$. Since Set is coherent with effective unions, this forces at once all the properties for having effective coherent systems of subobjects. ∎

In particular, trivially:

Proposition 2.9. *A left saturated, right saturated or saturated subcategory is full.* ∎

Definition 2.10. *In a category $\mathcal{C}$ with finite limits, a left saturated internal subcategory $\mathbb{S} \subseteq \mathbb{A}$ is one which satisfies the axiom*

$$\textit{If } \Big[f \in A_1 \wedge d_1(f) \in S_0 \Big] \textit{ then } \Big[f \in S_1 \Big]$$

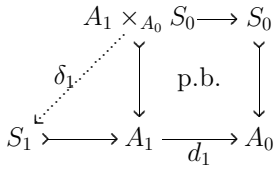
in the cartesian internal logic of $\mathcal{C}$. The right saturated case is obtained when using instead d_0.

We shall use the notation

$$A_2 \xrightarrow{\ m\ } A_1 \underset{d_1}{\overset{d_0}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longrightarrow}}}} A_0 \quad \text{with } n$$

to indicate an internal category $\mathbb{A}$. The internal category is an internal preordered object when the pair (d_0, d_1) turns A_1 in a subobject of $A_0 \times A_0$.

Given an internal subcategory $\mathbb{S}$ of $\mathbb{A}$, the left saturated notion of Example 2.7 (which is a cartesian notion; see [14]) translates as the existence of a (unique) factorization δ_1 in the following pullback diagram



An analogous conclusion holds, using instead d_0 and a corresponding factorization δ_0, in the right saturated case.

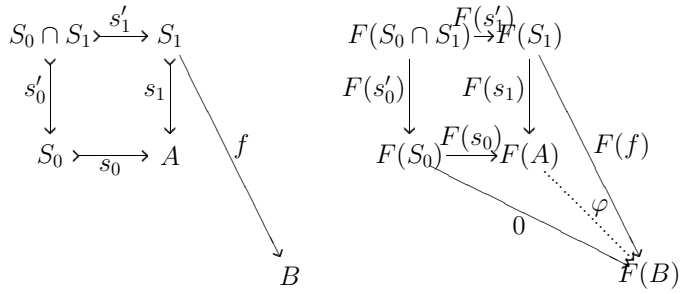
In the situation of Definition 2.10, one could of course be tempted to define a saturated internal subcategory as one which is both left and right saturated. But in the case of a coherent category $\mathcal{C}$ (see [14]), thus in particular in a topos, it sounds more sensible to define:

Definition 2.11. *In a coherent category $\mathcal{C}$, a saturated internal subcategory $\mathbb{S} \subseteq \mathbb{A}$ is one which satisfies the axiom*

$$\textit{If } \Big[f \in A_1 \wedge \big(d_0(f) \in S_0 \vee d_1(f) \in S_0 \big) \Big] \textit{ then } \Big[f \in S_1 \Big]$$

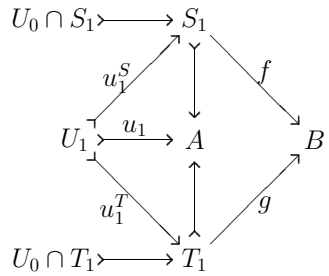
in the internal coherent logic of $\mathcal{C}$.

This can be translated as the existence of a (unique) factorization δ in the following diagram



yielding the corresponding right hand diagram in $\mathcal{X}$, with the square a zero-pushout. By definition of a morphism in $\mathsf{DisPar}(\mathcal{E})$, fs_1' is trivial, thus $F(f)F(s_1') = 0$ because F is a torsion functor. By the zero-pushout property, we get a unique factorization φ which we choose as $G(S_0, S_1, f)$.

We must of course verify that this definition is independent of the choice of the distinguished partial morphism representing the morphism in $\mathsf{Stab}(\mathcal{E})$. Using analogous notation, let the partial morphism (T_0, T_1, g) be equivalent to (U_0, U_1, f) in $\mathsf{DisPar}(\mathcal{E})$ and call ψ the corresponding factorization from $F(A)$ to $F(B)$. We have thus a congruence diagram with $U_0 \cup U_1 = A$



Still with analogous notation, we consider further the following diagram

is a covering of $S \cup T$, while f and g coincide already on W_1. To conclude the proof, it remains to see that f and g are trivial when restricted respectively to $(W_0 \cup T) \cap U_1$ and $(W_0 \cup T) \cap V_1$. We have

$$(W_0 \cup T) \cap U_1 = (W_0 \cap U_1) \cup (T \cap U_1)$$

and we know already that f is trivial on both pieces of this union; condition (CC2) yields the conclusion for f. In the same way

$$(W_0 \cup T) \cap V_1 = (W_0 \cap V_1) \cup (T \cap V_1)$$

and we know already that g is trivial on $T \cap V_1$ and $W_0 \cap S \cap V_1$. This takes already care of the term $T \cap V_1$; for the other term, simply split it in two parts and apply again condition (CC2)

$$W_0 \cap V_1 = (W_0 \cap V_1) \cap (S \cup T) = (W_0 \cap V_1 \cap S) \cup (W_0 \cap V_1 \cap T).$$

The morphism g is already known to be trivial on each part. ∎

Theorem 6.4. *The functor $\sigma \colon \mathcal{E} \longrightarrow \mathsf{Stab}(\mathcal{E})$ is universal among all the functors $F \colon \mathcal{E} \longrightarrow \mathcal{X}$ where*

(1) *$\mathcal{X}$ is provided with a torsion theory;*
(2) *F is a torsion functor;*
(3) *given two distinguished subobjects S, T of A in $\mathcal{E}$, F transforms the square*

$$\begin{array}{ccc} S \cap T & \rightarrowtail & S \\ \downarrow & & \downarrow \\ T & \rightarrowtail & S \cup T \end{array}$$

in a zero-pushout.

Proof: Call $G \colon \mathsf{Stab}(\mathcal{E}) \longrightarrow \mathcal{X}$ the expected unique factorization of F through σ. Since σ is the identity on objects, we must put $G(A) = F(A)$ for every object $A \in \mathcal{E}$. In the same way for every arrow $f \in \mathcal{E}$, the factorization requirement imposes $G\big(\sigma(f)\big) = F(f)$.

Next consider a morphism from A to B in $\mathsf{Stab}(\mathcal{E})$ represented by (S_0, S_1, f) in $\mathsf{DisPar}(\mathcal{E})$. We have in particular the following left hand diagram in $\mathcal{E}$, with $S_0 \cup S_1 = A$.

$$\begin{array}{ccc} A_1' & \longrightarrow & (S_0 \times A_0) \cup (A_0 \times S_0) \\ {\scriptstyle\delta}\,\vdots & \text{p.b.} & \downarrow \\ S_1 & \rightarrowtail & A_1 \xrightarrow[(d_0,d_1)]{} A_0 \times A_0 \end{array}$$

For example in the topos of sheaves on a locale L, this means that if $f \in A_1(u)$ while $u \in L$ is covered by all the levels $v \leq u$ where the domain or the codomain of the restriction of f is in $S_0(v)$, then $f \in S_1(u)$.

Lemma 2.12. *When $\mathcal{C}$ is a coherent category, given an internal subcategory $\mathbb{S} \subseteq \mathbb{A}$, if $\mathbb{S}$ is saturated in $\mathbb{A}$, it is both left and right saturated. The converse holds when $\mathcal{C}$ has effective unions.*

Proof: With the notation of Definitions 2.10 and 2.11, compose (d_0, d_1) with both projections of the product. This yields first

$$p_1^{-1}(S_0) = S_0 \times A_0,\ p_2^{-1}(S_0) = A_0 \times S_0.$$

By axiom (CS4) this implies

$$A_1' = (A_1 \times_{A_0} S_1) \cup (S_1 \times_{A_0} A_1).$$

Since $S_1 \rightarrowtail A_1$ is a monomorphism, the existence of δ implies that of δ_0 and δ_1. The converse holds when $\mathcal{C}$ has effective unions. ∎

Example 2.13. *Let $\mathcal{C}$ be a coherent category. The notions of left saturated, right saturated and saturated internal subcategories yield coherent systems of subobjects in $\mathsf{Cat}(\mathcal{C})$. Restricting one's attention to internal preordered objects, one obtains corresponding coherent systems of subobjects in the category of preordered objects in $\mathcal{C}$.*

Proof: Given two saturated (respectively, left saturated, right saturated) internal subcategories $\mathbb{S}$, $\mathbb{T}$ of $\mathbb{A}$, let us first observe that their union as internal subcategories admits $S_1 \cup T_1$ in $\mathcal{C}$ as object of arrows, and of course $S_0 \cup T_0$ as object of objects. Indeed the pullback defining the object of composable pairs of morphisms in $S_1 \cup T_1$

$$(S_1 \cup T_1) \times_{A_0} (S_1 \cup T_1)$$

can be split in four pieces by coherence of $\mathcal{C}$. The two pieces $S_1 \times_{A_0} S_1$ and $T_1 \times_{A_0} T_1$ yield composites lying respectively in S_1 and T_1 by the category axioms. The other two pieces $S_1 \times_{A_0} T_1$ and $T_1 \times_{A_0} S_1$ yield composites lying

in $S_1 \cup T_1$ in the saturated case, or already in S_1 or T_1 in the left or right saturated case. The rest follows at once from the coherence of the category $\mathcal{C}$. ∎

It is immediate that, in Cat, every saturated subcategory $\mathcal{S}$ is complemented: its complement is the full subcategory generated by those objects which are not connected by a chain of arrows to any object of $\mathcal{S}$. This is not the case for left or right saturated subcategories: here is an example of a right saturated subcategory which is not complemented.

$$\{A \to B\} \rightarrowtail \{A \to B \leftarrow C\}$$

In an arbitrary category with finite limits, even in a topos, it is no longer the case that saturated subcategories are complemented. For example in the topos of sheaves on the Sierpinski space, consider the following preordered sheaf, where the restriction applies a_i on a and b_i on b:

$$\{a_1 < b_1, a_2 < b_2\} \longrightarrow \{a < b\}$$

The two subsheaves

$$\{a_i < b_i\} \longrightarrow \{a < b\}$$

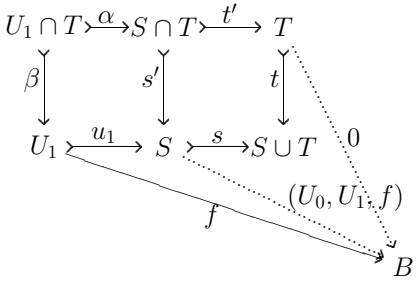
are saturated (or clopen) in the sense of Example 2.13, but are not complemented. Their union is of course the whole sheaf. This shows that our notion of saturated (clopen) sub-preordered object differs from that in [3], where complementarity of the subobject is forced. Of course, both notions coincide in the Set case.

3. Variations on torsion theories

Let us recall that an ideal $\mathcal{Z}$ in a category $\mathcal{C}$ is a class of arrows such that, for every arrow $f \in \mathcal{Z}$, one has $fu \in \mathcal{Z}$ and $vf \in \mathcal{Z}$, for all arrows u, v composable with f (see [10]). When $\mathcal{C}$ has a zero object, the zero morphisms constitute an ideal.

Given an ideal $\mathcal{Z}$ in a category $\mathcal{C}$, an arrow k is the $\mathcal{Z}$-kernel of an arrow f when $fk \in \mathcal{Z}$ and, if $fm \in \mathcal{Z}$ for some arrow m, then m factors uniquely through k. The uniqueness condition forces k to be a monomorphism. When $\mathcal{Z}$ is the ideal of zero morphisms, we recapture the usual notion of kernel. There is of course a dual notion of $\mathcal{Z}$-cokernel. A pair of composable morphisms

$$K \xrightarrow{\ k\ } A \xrightarrow{\ q\ } Q$$



In $\mathsf{Stab}(\mathcal{E})$, we assume thus $(U_0, U_1, f) \circ s' = 0 \circ t'$. By Lemma 5.3, $f\beta$ is trivial, thus f is trivial on $U_1 \cap T$. But by definition of the arrows in $\mathsf{DisPar}(\mathcal{E})$, f is also trivial on $U_1 \cap U_0$. By condition (CC2), f is thus trivial on

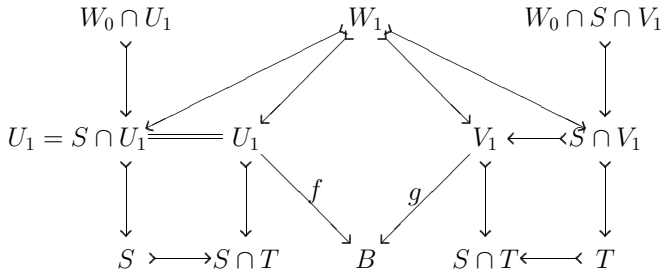
$$(U_1 \cap T) \cup (U_1 \cap U_0) = U_1 \cap (T \cup U_0).$$

Since $U_1 \cup (T \cup U_0) = S \cup T$, this proves that we have a morphism in $\mathsf{Stab}(\mathcal{E})$

$$(T \cup U_0, U_1, f) \colon S \cup T \dashrightarrow B$$

and it follows at once that this is the expected factorization.

It remains to prove the uniqueness of that factorization. Let (V_0, V_1, g) be another factorization, with thus in particular $V_0 \cup V_1 = S \cup T$. On one hand the equality $(V_0, V_1, g) \circ s = (U_0, U_1, f)$ implies the existence of a distinguished covering $S = W_0 \cup W_1$ and a corresponding congruence diagram



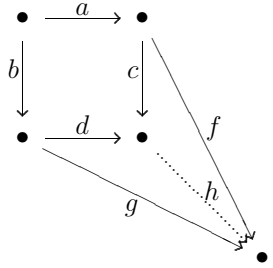
By this congruence diagram, f and g coincide on W_1, while f is trivial on $W_0 \cap U_1$ and g is trivial on $W_0 \cap S \cap V_1$. On the other hand the equality $(V_0, V_1, g) \circ y = 0$ means that g is trivial on $T \cap V_1$. And we know already that f is trivial on $T \cap U_1$. Observe now that

$$S \cup T = W_0 \cup W_1 \cup = W_1 \cup (W_0 \cup T)$$

Proof: Write $s\colon S \rightarrowtail A$ for the canonical inclusion. The two distinguished partial morphisms (T, S, s) from A to S and $(\mathbf{0}, S, \mathsf{id}_S)$ from S to A exhibit the isomorphism in $\mathsf{Stab}(\mathcal{E})$. ∎

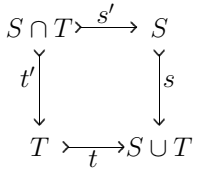
This property somehow suggests the following definition.

Definition 6.2. *Let $\mathcal{X}$ be a category with a zero object. A commutative square $ca = db$ is called a* zero-pushout



when given arrows f, g, with $fa = gb$, there exists a unique morphism h such that $hc = f$ and $hd = g$, provided that at least one of the two morphisms f, g is zero.

Proposition 6.3. *Given two distinguished subobjects S, T of A in $\mathcal{E}$, the functor $\sigma\colon \mathcal{E} \longrightarrow \mathsf{Stab}(\mathcal{E})$ transforms the bicartesian square*



in a zero-pushout.

Proof: Let us consider the following diagram, where the plain arrows are in $\mathcal{E}$ and the dotted arrows in $\mathsf{Stab}(\mathcal{E})$.

is a short $\mathcal{Z}$-exact sequence when k is the $\mathcal{Z}$-kernel of q and q is the $\mathcal{Z}$-cokernel of k.

The following definition was introduced in [11] and then thoroughly investigated in [12]:

Definition 3.1. *A pretorsion theory in a category $\mathcal{C}$ consists of a pair $(\mathcal{T}, \mathcal{F})$ of classes of objects, both of them closed under isomorphisms, whose elements are called the* torsion *and the* torsion-free *objects of the pretorsion theory, respectively. The objects in $\mathcal{T} \cap \mathcal{F}$ are called* trivial*, and the ideal $\mathcal{Z}$ of* trivial morphisms *is that of those arrows factoring through a trivial object.*
These data must satisfy the following two axioms:

(PT1): *every arrow $f\colon A \longrightarrow B$ with $A \in \mathcal{T}$ and $B \in \mathcal{F}$ is trivial;*
(PT2): *for every object $A \in \mathcal{C}$, there exists a short $\mathcal{Z}$-exact sequence*

$$\tau(A) \xrightarrow{\varepsilon_A} A \xrightarrow{\eta_A} \phi(A)$$

with $\tau(A) \in \mathcal{T}$ and $\phi(A) \in \mathcal{F}$.

When the trivial objects are zero objects, the pretorsion theory is called a torsion *theory.*[†]
A torsion functor *is a functor, between two categories provided with a pretorsion theory, which respects torsion objects, torsion free objects and the canonical $\mathcal{Z}$-exact sequences of axiom (PT2).*

Proposition 3.2. *In the conditions of Definition 3.1, the objects $\tau(A)$ and $\phi(A)$ are defined uniquely up to an isomorphism. This extends in functors τ and ϕ which present respectively the full category of torsion objects as a coreflexive category of $\mathcal{C}$, and the full subcategory of torsion free objects as a reflexive subcategory of $\mathcal{C}$.*

Proof: See [12]. ∎

It is trivial to observe that τ and ϕ extend as functors, which turn respectively the full subcategories of torsion/torsion free objects in a coreflective/reflective full subcategory of $\mathcal{C}$ (see [12]).

Example 3.3. *One gets a pretorsion theory on the category of preordered sets when choosing (see* [11]*)*

- *the equivalence relations as torsion objects;*
- *the partial orders as torsion free objects.*

[†]The terminology *pretorsion* is thus somehow unfortunate, since the axioms are the same as for a torsion theory, but with respect to a more flexible choice of ideal.

These two notions are cartesian ones (see [14]*), thus make sense in every category with finite limits. They yield a pretorsion theory in every Barr exact category (see* [1, 3]*).*

Proof: Given a preordered set (A, R) and writing R° for the opposite relation, $\tau(A, R) = (A, R \cap R^\circ)$ while $\phi(A, R)$ is the quotient of A by the equivalence relation $R \cap R^\circ$, this quotient being provided with the image-preorder of R. ■

Example 3.4. *One gets a pretorsion theory on the category of small categories when choosing (see* [6]*)*

- *the groupoids as torsion objects (every arrow is an isomorphism);*
- *the skeletal categories as torsion free objects (every isomorphism is an automorphism).*

These two notions are cartesian ones (see [14]*), thus make sense in every category with finite limits. They yield a pretorsion theory in every Grothendieck topos (see* [7]*).*

Proof: Given a small category $\mathcal{C}$, $\tau(\mathcal{C})$ is the groupoid of isomorphisms of $\mathcal{C}$ while $\phi(\mathcal{C})$ is the quotient of $\mathcal{C}$ which identifies the domain and the codomain of every isomorphism (see [6, 7]). ■

More examples of pretorsion theories can be found in [12, 15].

The notion of *stable category* for a pretorsion theory has been introduced in [11] and further investigated in [3, 4, 5]. The question is, given a category $\mathcal{X}$ provided with a pretorsion theory, to construct "in the best possible way" a category $\mathcal{Y}$ provided with a torsion theory, and a morphism of pretorsion theories from $\mathcal{X}$ to $\mathcal{Y}$. The first way to interpret "in the best possible way" is to look for the universal solution to the problem: this is what we do in Section 7. The idea is of course to identify all the trivial objects to $\mathbf{0}$ ... and check when this yields a solution to the problem.

But one can be interested in a more involved problem. For example, when an object A can be written as a union $A = S \cup T$ of two subobjects, with T a trivial object, one could want A to be identified with S in the stable category ... since T will be identified with $\mathbf{0}$. This yields of course a different universal problem, in which some compatibility with the union of subobjects is requested. In [11, 3, 5], a universal solution of that type is produced in the special case where S and T are complemented subobjects and the union is a disjoint one.

The composite $V_1 w_1^V \varepsilon_{W_1}$ is trivial by axiom (PT1), from which the factorization α_V. The same argument can be repeated with S_1 and w_1^S, yielding α_S such that $\alpha_S \eta_{W_1} = \eta_{S_1} w_1^S$. Then

$$z\alpha_V \eta_{W_1} = z\eta_{V_1} w_1^V = y\eta_{S_1} w_1^S = y\alpha_S \eta_{W_1}.$$

Since η_{W_1} is an isomorphism, this implies $z\alpha_V = y\alpha_S$. This is the first condition for a congruence diagram based on $\phi(W_0) \cup \phi(W_1)$.

It remains to check the triviality of z and y when restricted to $\phi(W_0)$. For this, having in mind Corollaries **??** and 4.11, we consider the diagram

$$\begin{array}{ccccccc} & & \overset{\varepsilon_{W_0\cap W_1}}{\rightarrowtail} & & \overset{\eta_{W_0\cap W_1}}{\twoheadrightarrow} & & \overset{\phi(i)}{\rightarrowtail} \\ \tau(W_0)\cap\tau(W_1) & & W_0\cap W_1 & & \phi(W_0)\cap\phi(W_1) & & \phi(W_1) \\ & \searrow^{\beta} & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle y} & & \\ & & Z & \xrightarrow[\gamma]{} & X & & \end{array}$$

where i is the inclusion of $W_0 \cap W_1$ in W_1. From the congruence diagram built on the covering $A = W_0 \cup W_1$, we know that $y\phi(i)\eta_{W_0\cap W_1}$ is trivial. Thus this morphism factors as $\gamma\beta$ through some trivial object D. But then β, and thus $\beta\varepsilon_{W_0\cap W_1}$ are trivial. This implies that β factors via a unique morphism δ through $\eta_{W_0\cap W_1}$. Since $\eta_{W_0\cap W_1}$ is an epimorphism, the various commutativities imply $\gamma\delta = y\phi(i)$. Thus $y\phi(i)$ factors through the trivial object D and is trivial. The case of z is perfectly analogous. ■

All the results of this section apply thus to Examples 4.13, 4.14, 4.15 and 4.16.

6. The universal property

Again we put:

Blanket assumption for this whole section
Let $\mathcal{E}$ be a category with a strict initial object, provided with a pretorsion theory and a coherent system of distinguished subobjects which are compatible.

Let us begin with the motivating property that we emphasized in our discussion on stable categories at the end of Section 3.

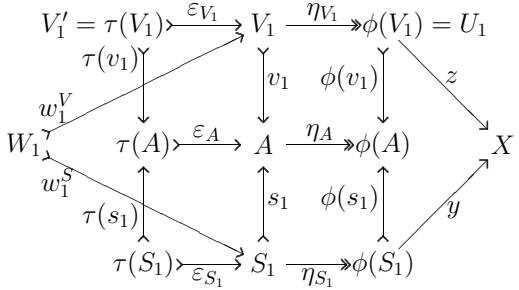
Proposition 6.1. *Consider an object A in $\mathcal{E}$ which is the union $A = S \cup T$ of two distinguished subobjects. When T is trivial, A and S become isomorphic in* $\mathsf{Stab}(\mathcal{E})$.

where, by Property 4.7, both squares are pullbacks and the vertical arrows are distinguished subobjects. In particular,

$$(S_0, S_1, x) \circ \varepsilon_A = \big(\varepsilon_A^{-1}(S_0), \tau(S_1), x\varepsilon_{S_1}\big).$$

Since this composite is zero, $x\varepsilon_{S_1}$ is trivial, by Lemma 5.3. This implies the existence of a unique factorization y of x through η_{S_1}. Then $\big(\phi(S_0), \phi(S_1), y\big)$ yields the expected factorization of (S_0, S_1, x) through η_A in $\mathsf{Stab}(\mathcal{E})$.

It remains to prove the uniqueness of the factorization. Let thus (U_0, U_1, z) be another factorization of (S_0, S_1, x) through η_A in $\mathsf{Stab}(\mathcal{E})$. Consider the following diagram

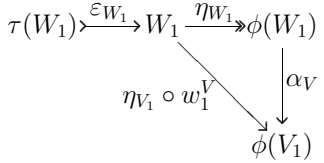


where V_1 and V_1' are obtained from U_1 by pullbacks. By Proposition 4.12, the upper line is a $\mathcal{Z}$-exact sequence, allowing as in Proposition 4.7 to rewrite it in terms of V_1, and the vertical morphisms in terms of v_1. The equalities

$$(V_0, V_1, z\eta_{V_1}) = (U_0, U_1, z) \circ \eta_A = (S_0, S_1, x)$$
$$= \big(\phi(S_0), \phi(S_1), y\big) \circ \eta_A = \big(S_0, S_1, y\eta_{S_1}\big)$$

mean the existence of a congruence diagram built from a distinguished covering $A = W_0 \cup W_1$ of A.

We know by Corollary 4.11 that $\phi(A) = \phi(W_0) \cup \phi(W_1)$ and we shall now prove that this covering allows constructing a congruence diagram proving the expected uniqueness of the factorization. Consider first the canonical $\mathcal{Z}$-exact sequence of W_1.



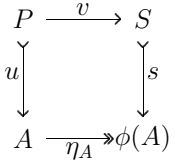
In this paper, we shall give evidence that complemented subobjects do not play any canonical role: a stable category compatible with the union of distinguished subobjects can be constructed with respect to an arbitrary coherent system of subobjects, provided it is sufficiently compatible with the pretorsion theory.

4. The compatibility conditions

The various constructions of a stable category as in [11, 3, 4] underline clearly which properties are essential to get the expected result. In this section, we focus on these properties and show that they hold in our main cases of interest. We use freely the notation of Section 3, without recalling it.

Definition 4.1. *Let $\mathcal{C}$ be a category with a strict initial object, provided with both a pretorsion theory and a coherent system of subobjects. These data are called* compatible *when the following conditions are satisfied.*

- **(CC1):** *The class of trivial objects is closed under distinguished subobjects.*
- **(CC2):** *Given a morphism $f\colon S \cup T \longrightarrow B$, with S, T two distinguished subobjects of A, f is trivial as soon as its restrictions on S and T are trivial.*
- **(CC3):** *The functor ϕ preserves distinguished subobjects.*
- **(CC4):** *Given a pullback square*



  with s a distinguished monomorphism, v is a distinguished epimorphism.
- **(CC5):** *Let s be a distinguished monomorphism such that $\phi(s)$ is an isomorphism; then s is an isomorphism.*

Let us infer some consequences of such a situation.

Proposition 4.2. *In the conditions of Definition 4.1, the initial object is trivial.*

Proof: Trivial objects exist by axiom (PT2) for a pretorsion theory. We conclude by axiom (CS1) for a coherent system of subobjects, and the compatibility condition (CC1). ∎

Proposition 4.3. *In the conditions of Definition 4.1, if sf is trivial with $s\colon S \rightarrowtail A$ a distinguished subobject, then f is trivial.*

Proof: The composite sf factors as gh through a trivial object D. Therefore f factors through the distinguished subobject $g^{-1}(S)$ of D, which is trivial by condition (CC1). ∎

Proposition 4.4. *In the conditions of Definition 4.1, the class of torsion free objects is closed under distinguished subobjects.*

Proof: Consider the following diagram, where A is torsion free and S is distinguished in A.

$$\begin{array}{ccccc}
\tau(S) & \overset{\varepsilon_S}{\rightarrowtail} & S & \overset{\eta_S}{\twoheadrightarrow} & \phi(S) \\
{\scriptstyle \tau_S}\downarrow & & {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \phi(s)} \\
\tau(A) & \underset{\varepsilon_A}{\rightarrowtail} & A & = & \phi(A)
\end{array}$$

The composite $s\varepsilon_S = \phi(s)\eta_S\varepsilon_S$ is trivial, thus ε_S is trivial because s is a distinguished subobject. Therefore $\mathsf{id}_S\varepsilon_S$ is trivial, proving that id_S factors through the $\mathcal{Z}$-cokernel η_S. So the epimorphism η_S is a section, thus an isomorphism. ∎

Proposition 4.5. *In the conditions of Definition 4.1, the canonical morphisms $\eta_A\colon A \twoheadrightarrow \phi(A)$ are distinguished epimorphisms.*

Proof: Simply choose $s = \mathsf{id}_{\phi(A)}$ in condition (CC4). ∎

Proposition 4.6. *In the conditions of Definition 4.1, given the pullback diagram of condition (CC4), one has $S \cong \phi(P)$ and $v \cong \eta_P$.*

Proof: Consider the following diagram, where S is isomorphic to $\phi(S)$ by Proposition 4.4

Theorem 5.6. *In $\mathsf{Stab}(\mathcal{E})$, choosing as torsion and torsion free objects the same objects as in $\mathcal{E}$, one gets a torsion theory on $\mathsf{Stab}(\mathcal{E})$ which turns the functor $\sigma\colon \mathcal{E} \longrightarrow \mathsf{Stab}(\mathcal{E})$ in a torsion functor.*

Proof: Consider a morphism $A \longrightarrow B$ in $\mathsf{Stab}(\mathcal{E})$, represented by (S_0, S_1, f) in $\mathsf{DisPar}(\mathcal{E})$. If A is torsion and B is torsion free, then S_1 is torsion by Corollary 4.9 and thus f is trivial by axiom (PT1) in $\mathcal{E}$ (see Definition 3.1). By Lemma 5.3, axiom (PT1) holds in $\mathsf{Stab}(\mathcal{E})$.

The core of the proof is about axiom (PT2). We fix an object $A \in \mathcal{E}$ and must prove that the image by the functor σ of its canonical $\mathcal{Z}$-exact sequence in $\mathcal{E}$

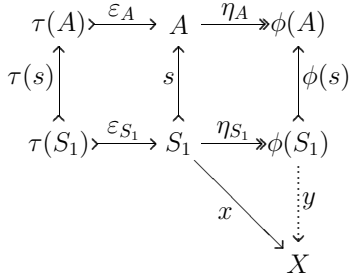
$$\tau(A) \overset{\varepsilon_A}{\rightarrowtail} A \overset{\eta_A}{\twoheadrightarrow} \phi(A)$$

is an exact sequence in $\mathsf{Stab}(\mathcal{E})$.

First, the kernel part. Consider a morphism from X to A in $\mathsf{Stab}(\mathcal{E})$ represented by a morphism (S_0, S_1, x) in $\mathsf{DisPar}(\mathcal{E})$ and whose composite with η_A is zero. That composite is represented by $(S_0, S_1, \eta_A x)$; by Lemma 5.3, saying that it is zero means that $\eta_A x$ is trivial. Therefore x factors uniquely in $\mathcal{E}$ as $x = \varepsilon_A y$. It follows at once that (S_0, S_1, y) yields a factorization of (S_0, S_1, x) through ε_A in $\mathsf{Stab}(\mathcal{E})$.

To prove the uniqueness, consider another factorization (T_0, T_1, z). Write U_0, U_1 for the two distinguished subobjects in a congruence diagram exhibiting the equivalence between $(S_0, S_1, \varepsilon_A y)$ and $(T_0, T_1, \varepsilon_A z)$. Since $\varepsilon_A y$ and $\varepsilon_A z$ coincide on U_1, so do y and z because ε_A is a monomorphism. The rest follows easily from Lemma 4.3.

To prove the cokernel part, let us choose in $\mathsf{Stab}(\mathcal{E})$ a morphism from A to X, represented by (S_0, S_1, x) in $\mathsf{DisPar}(\mathcal{E})$, and whose composite with ε_A is zero. We consider then the situation



- f and g coincide on U_1;
- the restriction of f to $U_0 \cap S_1$ is trivial;
- the restriction of g to $U_0 \cap T_1$ is trivial.

It is lengthy, but routine computation, to observe that this yields a congruence (see [10]) on $\mathsf{DisPar}(\mathcal{E})$. The stable category $\mathsf{Stab}(\mathcal{E})$ is the quotient of $\mathsf{DisPar}(\mathcal{E})$ by that congruence. We write $\pi\colon \mathsf{DisPar}(\mathcal{E}) \longrightarrow \mathsf{Stab}(\mathcal{E})$ for the quotient functor and $\sigma\colon \mathcal{E} \longrightarrow \mathsf{Stab}(\mathcal{E})$ for the composite $\pi\iota$. For objects and morphisms in $\mathcal{E}$, we shall generally write A and f in $\mathsf{Stab}(\mathcal{E})$ instead of $\sigma(A)$ and $\sigma(f)$

Let us now prove the various properties of a stable category, as announced at the end of Section 3.

Proposition 5.1. *The category* $\mathsf{Stab}(\mathcal{E})$ *admits* $\mathbf{0}$ *as a zero object.*

Proof: Writing $0_A\colon \mathbf{0} \longrightarrow A$ for the unique morphism in $\mathcal{E}$, the two partial distinguished morphisms

$$(A, \mathbf{0}, \mathsf{id}_{\mathbf{0}})\colon A \longrightarrow\!\!\!\!\!\rightarrow \mathbf{0},\ (\mathbf{0}, \mathbf{0}, 0_A)\colon \mathbf{0} \longrightarrow\!\!\!\!\!\rightarrow A$$

are the only possible ones in $\mathsf{DisPar}(\mathcal{E})$, since $\mathbf{0}$ is strict initial. It is immediate to observe that they are inverse isomorphisms in $\mathsf{Stab}(\mathcal{E})$. ∎

Corollary 5.2. *The zero morphism from* A *to* B *in* $\mathsf{Stab}(\mathcal{E})$ *is represented by* $(A, \mathbf{0}, 0_B)$. ∎

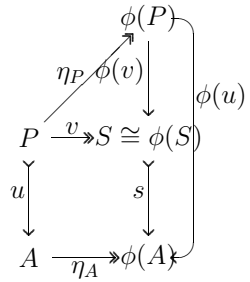
Lemma 5.3. *A morphism* $(S_0, S_1, f)\colon A \longrightarrow B$ *in* $\mathsf{DisPar}(\mathcal{E})$ *is mapped on a zero morphism by the quotient functor* π *if and only if* f *is trivial.*

Proof: Consider a congruence diagram as above, expressing the possible equivalence of (S_0, S_1, f) and $(A, \mathbf{0}, 0_B)$. Again since $\mathbf{0}$ is strict initial in $\mathcal{E}$, we must have $U_1 = \mathbf{0}$ and thus $U_0 = A$, which forces at once the result. ∎

Corollary 5.4. *A morphism* $f\colon A \longrightarrow B$ *in* $\mathcal{E}$ *is mapped on a zero morphism in* $\mathsf{Stab}(\mathcal{E})$ *if and only if it is trivial.* ∎

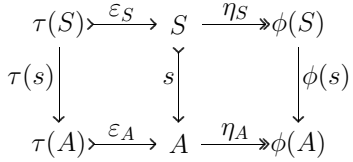
Proposition 5.5. *An object* $A \in \mathcal{E}$ *is mapped on a zero object in* $\mathsf{Stab}(\mathcal{E})$ *if and only if it is trivial.*

Proof: Expressing that the two partial morphisms in Proposition 5.1 yield inverse isomorphisms in $\mathsf{Stab}(\mathcal{E})$ reduces, by Lemma 5.3, to the identity on A being trivial. But if the identity on A factors through a trivial object X, A is a retract of X and therefore is trivial (see Corollary 2.8 in[12]). ∎



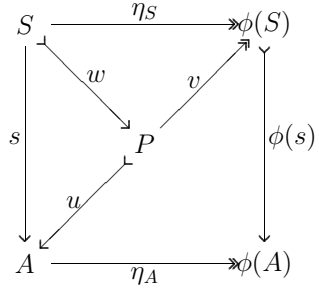
By condition (CC3), $\phi(s)$ and $\phi(u)$ are distinguished subobjects. By Proposition 2.2, $\phi(v)$ is a distinguished subobject as well. By condition (CC4), v is a distinguished epimorphism factoring through the distinguished subobject $\phi(v)$, thus $\phi(v)$ is an isomorphism. Therefore $\phi(P) \cong \phi(S)$ and $v \cong \eta_P$. ∎

Proposition 4.7. *In the conditions of Definition 4.1, consider the following diagram*



where the two rows are canonical $\mathcal{Z}$*-exact sequences and* s *is a distinguished monomorphism. Then* $\tau(s)$ *and* $\phi(s)$ *are distinguished monomorphisms and both squares are pullbacks.*

Proof: Let us begin with the right hand square. We know already, by condition (CC3), that $\phi(s)$ is a distinguished monomorphism. We consider the pullback (P, u, v) of $\phi(s)$ along η_A and the corresponding factorization w.



Thus u is a distinguished subobject by (CS3) and w as well, by Proposition 2.2. By Proposition 4.6, $\phi(P) \cong \phi(S)$ and $v \cong \eta_P$. But this implies that $\phi(w)$ is isomorphic to the identity on $\phi(S)$. By condition (CC5), w is an isomorphism and the outer square is a pullback.

For the left hand square, if $su = \varepsilon_A v$, composing with η_A yields a trivial morphism $\eta_A\varepsilon_A v = \phi(s)\eta_S u$. Since $\phi(s)$ is a distinguished subobject, it follows from Proposition 4.3 that $\eta_S u$ is trivial. Thus u factors uniquely through ε_S. That factorization is also a factorization of v through $\tau(s)$, because ε_A is a monomorphism. Thus the left hand square is a pullback. And since s is a distinguished subobject, so is $\tau(s)$ by axiom (CS3). ∎

Corollary 4.8. *In the conditions of Definition 4.1, the functor τ preserves distinguished subobjects.* ∎

Corollary 4.9. *In the conditions of definition 4.1, the class of torsion objects is closed under distinguished subobjects.*

Proof: In the diagram of Proposition 4.7, if A is a torsion object, ε_A is an isomorphism and thus by pullback, ε_S as well. ∎

Corollary 4.10. *In the conditions of Definition 4.1, the functor τ preserves the union and the intersection of distinguished subobjects.*

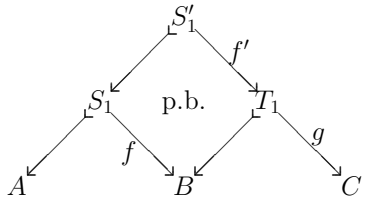
Proof: Considering again the diagram of Proposition 4.7, τ acts by pullbacks along ε_A at the level of distinguished subobjects of A. One concludes by axiom (CS4). ∎

Corollary 4.11. *In the conditions of Definition 4.1, the functor ϕ preserves the union and the intersection of distinguished subobjects.*

Proof: Let R and S be two distinguished subobjects of A. By condition (CC3), $\phi(S)$ and $\phi(R)$ are distinguished subobjects of ϕ_A, thus also their union $\phi(S)\cup\phi(R)$, by axiom (CS2). By Proposition 4.7, the inverse images of $\phi(S)$ and $\phi(R)$ along η_A are S and R. Thus by axiom (CS4), the inverse image of $\phi(A)\cup\phi(B)$ along η_A is $S\cup R$. By Proposition 4.6, $\phi(S\cup R) \cong \phi(S)\cup\phi(R)$. An analogous argument holds for the intersection. ∎

Proposition 4.12. *In the conditions of Definition 4.1, consider the following diagram*

- the composition law is made via a pullback:



$$(T_0, T_1, g) \circ (S_0, S_1, f) = (S_0 \cup S_0', S_1', gf')$$

with $S_0' = f^{-1}(T_0)$ and $S_1' = f^{-1}(T_1)$.

Checking the category axioms is just routine computation.

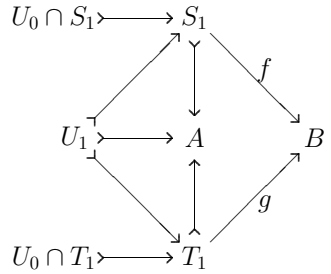
There is an obvious functor $\iota\colon \mathcal{E} \longrightarrow \mathsf{DisPar}(\mathcal{E})$, which is the identity on objects and maps an arrow $f\colon A \longrightarrow B$ on $(\mathbf{0}, A, f)$.

The partial morphisms should be thought as morphisms defined only on S_1, but that will be extended by zero on S_0 in the stable category, in order to become defined on the whole of A.

Next we construct the stable category, denoted by $\mathsf{Stab}(\mathcal{E})$. For this, we declare equivalent two parallel morphisms in $\mathsf{DisPar}(\mathcal{E})$

$$(S_0, S_1, f), (T_0, T_1, g)\colon A \rightrightarrows B$$

when there exists a so-called "congruence" diagram



(Congruence diagram)

with the properties:

- U_0 and U_1 are distinguished subobjects of A;
- $U_0 \cup U_1 = A$;
- U_1 is contained in both S_1 and T_1;

and $g(y) = x$. We get further $x = t\big(h(y)\big)$ and t is surjective, thus bijective, on the objects. But t is also full by saturation, thus is an isomorphism.

(CC5) Let $S\colon \mathcal{S} \rightarrowtail \mathcal{A}$ be a saturated subcategory such that $\phi(S)$ is an isomorphism. Write S as a canonical inclusion. For every object $a \in \mathcal{A}$, there is thus an object $a' \in \mathcal{S}$ such that $\phi(a)$ and $\phi(a')$ are isomorphic in $\phi(\mathcal{A})$. By saturation of $\mathcal{S}$ in $\mathcal{A}$, this forces $a \in \mathcal{S}$. Thus $\mathcal{S}$ and $\mathcal{A}$ have the same objects and $\mathcal{S}$ is full in $\mathcal{A}$, as a saturated subcategory. Therefore $\mathcal{S} = \mathcal{A}$.

Let us now switch to the case of a Grothendieck topos. In [7], it is observed that in a topos of presheaves, the (groupoid/skeletal) pretorsion theory has canonical $\mathcal{Z}$-exact sequences computed pointwise as in Set. But being a monomorphism, a saturated subcategory, a pullback, a groupoid, a skeletal category are notions expressed in terms of finite limits and unions. Since limits and colimits (and thus unions) are computed pointwise in a topos of presheaves, we get the expected result in every topos of presheaves, just because it holds pointwise in Set.

Again in [7] it is observed that the canonical $\mathcal{Z}$-exact sequences for the (groupoid/skeletal) pretorsion theory in a topos of sheaves are obtained by constructing these sequences in the corresponding topos of presheaves, and applying the associated sheaf functor. Since this last functor preserves finite limits and colimits (and thus unions), we conclude the proof in the case of every Grothendieck topos. ∎

5. The stable category

To avoid repeating it each time, let us put:

Blanket assumption for this whole section
Let $\mathcal{E}$ be a category with a strict initial object, provided with a pretorsion theory and a coherent system of distinguished subobjects which are compatible.

First, let us construct the category of distinguished partial morphisms of $\mathcal{E}$, which we shall denote as $\mathsf{DisPar}(\mathcal{E})$.

- the objects are those of $\mathcal{E}$;
- a morphism from A to B is a triple (S_0, S_1, f) where
  - S_0 and S_1 are distinguished subobjects of A;
  - $S_0 \cup S_1 = A$;
  - $f\colon S_1 \longrightarrow B$ is a morphism in $\mathcal{E}$;
  - f restricted to $S_0 \cap S_1$ is a trivial morphism.

$$\begin{array}{ccccc} X & \xrightarrow{k} & S & \xrightarrow{p} & Y \\ {\scriptstyle x}\downarrow & & {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle y} \\ \tau(A) & \xrightarrow{\varepsilon_A} & A & \xrightarrow{\eta_A} & \phi(A) \end{array}$$

where the bottom row is a canonical $\mathcal{Z}$-exact sequence, the vertical morphisms are distinguished monomorphisms and the two squares are pullbacks. Then the upper row is a canonical $\mathcal{Z}$-exact sequence as well.

Proof: By Proposition 4.6, we have at once $p \cong \eta_S$. By Proposition 4.7, the pullback of s along ε_A is $\tau(S)$. Thus up to isomorphism, $X \cong \tau(S)$ and $x \cong \varepsilon_S$. ∎

Example 4.13. *Let $\mathcal{C}$ be a category with a strict initial object. Any pretorsion theory on $\mathcal{C}$ where the initial object is trivial is compatible with the indiscrete coherent system of subobjects (see 2.6).*

Proof: Trivial because $\mathbf{0}$ is strict initial. ∎

Example 4.14. *If $\mathcal{C}$ is a lextensive category, a pretorsion theory on $\mathcal{C}$ and the coherent system of complemented subobjects (see 2.6) are compatible provided that the class of trivial objects is closed under complemented subobjects and binary coproducts.*

Proof: The assumptions take care of conditions (CC1) and (CC2). Lemma 4 in [5] implies our Propositions 4.6, 4.7 and 4.12, from which conditions (CC3), (CC4) and (CC5). ∎

Example 4.15. *In the category of preordered objects in an exact coherent category, the (equivalence/partial order) pretorsion theory and the coherent system of saturated subobjects are compatible.*

Proof: When useful, we use the more convenient notation $\mathbb{A} = (A, R)$ for a preordered object, and $\mathbb{S} = (S, R_S)$ for a subobject of (A, R) provided with the induced preorder. We use freely the internal logic of the ambient coherent category $\mathcal{C}$.

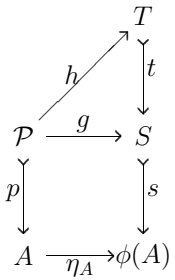
(CC1) holds because $\mathbf{0}$ is strict initial.

(CC2) A morphism $f\colon A \longrightarrow B$ is trivial when it factors through (B, Δ_B). From which the result, since by saturation, the preorder relation of $S \cup T$ is the union of the preorder relations of S and T.

(CC3) Consider $s\colon (S,R_S)\rightarrowtail(A,R)$ saturated. First, $\phi(s)$ is a monomorphism. Two elements $\overline{x}$, $\overline{y}$ of $\phi(S,R_S)$ identified by $\phi(s)$ correspond to two elements x, y in $S\subseteq A$ such that the pair (x,y) lies in $R\cap R^\circ$ (see Example 3.3). But since (S,R_S) is saturated in (A,R), (S,R_S°) is saturated in (A,R°) and the pair (x,y) lies in $R_S\cap R_S^\circ$. This implies $\overline{x}=\overline{y}$. Thus $\phi(s)$ is a monomorphism, which we shall write as an inclusion.

Next, choose $(\overline{x},\overline{y})\in\eta_A(R)$ in $\phi(A,R)$ and assume that $\overline{x}\in\phi(S,R_S)$ or $\overline{y}\in\phi(S,R_S)$. This means the existence of elements x, y in A, mapped respectively to $\overline{x}$ and $\overline{y}$, with thus $(x,y)\in R$ and one of the two elements x, y in S. By saturation, both elements x, y are in S and thus both elements $\overline{x}$, $\overline{y}$ are in $\phi(S)$.

(CC4) Consider the following pullback diagram, with S saturated in $\phi(A,R)$. We know already that P is saturated in A. There is no restriction in writing these distinguished monomorphisms as canonical inclusions.



Suppose that g factors as th through a saturated subobject T. Given an element $x\in S$, there is an element $y\in A$ such that $\eta_A(y)=x$, thus y lies in the pullback P and $g(y)=x$. This means that $x=t\big(h(y)\big)$, so that the morphism of preordered objects t is an isomorphism in the ambient category $\mathcal{C}$. Since moreover T is saturated in S, t is an isomorphism of preordered objects, and so, g is a distinguished epimorphism.

(CC5) Let again $s\colon (S,R_S)\rightarrowtail(A,R)$ be a saturated subobject, which we write as a canonical inclusion. If $\phi(s)$ is an isomorphism, for every $a\in A$, there exists $a'\in S$ such that $\eta_A(a)=\eta_S(a')$. This means that the pair (a,a') is in $R\cap R^\circ$ with $a'\in S$. Since S is saturated in A, $a\in S$ and s is an isomorphism. ∎

Example 4.16. *In the category of internal categories in a Grothendieck topos $\mathcal{E}$, the (groupoid/ skeletal) pretorsion theory and the coherent system of saturated internal subcategories are compatible.*

Proof: Let us first write down the proof in Cat.

(CC1) A trivial category is one with only automorphisms. A saturated subcategory has only automorphisms as well, because it is full.

]CC2] A functor $F\colon \mathcal{S}\cup\mathcal{T}\longrightarrow\mathcal{B}$ is trivial when it factors through the subcategory of automorphisms of $\mathcal{B}$. Unions of saturated subcategories are computed as in Set at both the level of objects and the level of arrows (see 2.8, from which the result).

In [6], it is proved that $\phi(\mathcal{A})$ is constructed from $\mathcal{A}$ as the quotient which identifies isomorphic objects. An arrow $a\longrightarrow b$ in $\phi(\mathcal{A})$, as for every quotient in Cat, is represented by a finite chain of arrows in $\mathcal{A}$, which become composable in the quotient. But in this specific case, it is proved in [6] that a unique "reduced" representation exists: a chain which does not contain any identity arrow nor any consecutive pair of arrows which are composable in $\mathcal{A}$, and where thus – by definition of the quotient – the codomain of each arrow of the chain is isomorphic to the domain of the next one, while the domain of the first arrow is isomorphic to a and the codomain of the last arrow is isomorphic to b. The identities in $\phi(\mathcal{A})$ are represented by the empty sequences.

(CC3) Consider a saturated subcategory $S\colon \mathcal{S}\rightarrowtail\mathcal{A}$. First, two objects of $\mathcal{S}$ identified in $\phi(\mathcal{A})$ are isomorphic in $\mathcal{A}$, thus also in $\mathcal{S}$ by saturation. This proves that $\phi(S)$ is injective on the objects. Next, if two arrows in reduced form in $\phi(\mathcal{S})$ are identified in $\phi(\mathcal{A})$, they are equal by uniqueness of the reduced form. Thus $\phi(S)$ is a monomorphism. Moreover given an arrow of $\phi(\mathcal{A})$ expressed in reduced form, if one of the objects involved in the chain is in $\mathcal{S}$, by saturation, so are all the objects and all the arrows of the chain. Thus $\phi(\mathcal{S})$ is saturated in $\phi(\mathcal{A})$.

(CC4) We refer to the same diagram as for proving condition (CC4) in Example 4.15, using for clarity calligraphic letters and upper case letters to indicate respectively the corresponding categories and functors. The subcategory $\mathcal{S}$ is thus saturated in $\phi(\mathcal{A})$ and by (CS3), its pullback $\mathcal{P}$ is saturated in $\mathcal{A}$. Let us view S and P as inclusions of full subcategories. Since $\eta_{\mathcal{A}}$ is surjective on objects, an object $x\in\mathcal{S}$ is equal in $\phi(\mathcal{A})$ to an object of the form $\eta_{\mathcal{A}}(y)$, with thus y an object of $\mathcal{A}$. But then y lies in the pullback $\mathcal{P}$