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KZ-MONADS AND KAN INJECTIVITY

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Dedicated to the memory of Marta Bunge (1938-2022)

Abstract: We introduce the notion of Kan injectivity in 2-categories and study its properties. For an adequate 2-category $\mathcal{K}$, we show that every set of morphisms $\mathcal{H}$ induces a KZ-monad on $\mathcal{K}$ whose 2-category of pseudoalgebras is the locally full sub-2-category of all objects (left) Kan injective with respect to $\mathcal{H}$ and morphisms preserving Kan extensions. The main ingredient is the construction of a (pseudo)chain whose appropriate "convergence" is ensured by a small object argument.

Keywords: 2-category, Kan injectivity, KZ-monad, small object argument.
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Introduction

A very classical problem in category theory goes under the name of the *orthogonal subcategory problem.* For $\mathcal{H}$ a class of maps in a category $\mathbb{C}$, we ask whether the full subcategory of orthogonal objects $\mathcal{H}^{\perp}$ is reflective in $\mathbb{C}$, that is, $\mathcal{H}^{\perp}$ is the category of algebras of an idempotent monad.

There are several reasons to study orthogonal subcategories and their reflectivity, because many situations in mathematics can be reduced to an orthogonality class of objects. For example, the set $\mathcal{H}$ of maps that specifies the orthogonality class can be understood as a set of axioms that the objects in the orthogonality class must satisfy (see [AHS06] for a theoretical approach to this motto, or [AR94, 1.33] for some practical examples of how it functions). Thus, orthogonality offers a categorical tool to axiomatise convenient subcategories.

The orthogonal subcategory problem has a long standing tradition and was approached by several authors. Peter Freyd and Max Kelly [FK72] provided what later became a standard reference on the topic. In [Kel80],

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Kelly unified the work of earlier authors, by providing a beautiful solution for this problem in a broad setting by means of the colimit of a transfinite sequence. This construction is quite in the same spirit of the celebrated *small object argument*. In a more recent account, the work of Jiří Adámek and Jiří Rosický [AR94, Chap. 1.C] gives a detailed description of the transfinite sequence in locally presentable categories. Their technique is very influential for our treatment.

The aim of this paper is to establish a similar result for a 2-dimensional variation of the orthogonal subcategory problem which captures many relevant constructions of 2-dimensional category theory. We will direct our study to the interplay between Kan-injectivity and lax-idempotent monads (i.e KZ-monads). They are natural substitutes for orthogonality and idempotent monads when working in 2-categories.

This work generalises the seminal work of [ASV15] and introduces Kan Injectivity in 2-categories. An object X is (left) Kan injective with respect to a map h if every $f : \mathrm{dom}(h) \to X$ can be extended to the codomain of h through a 2-cell

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\xi_f}{\Longrightarrow}"] & A' \arrow[dl, "f/h"] \\ X & \end{tikzcd}$$

and such an extension is universal among the possible extensions; more precisely, $(f/h, \xi_f)$ is the (left) Kan extension of f along h. Given a class $\mathcal{H}$ of 1-cells, we can form the locally full sub-2-category $\mathbf{LInj}(\mathcal{H})$ of all objects left Kan injective with respect to $\mathcal{H}$ and 1-cells preserving the corresponding Kan extensions. There are two natural notions of Kan Injectivity, the strongest one demanding that ξ_f is invertible. We will show how they relate to each other in subsection 1.2, concluding that both notions give rise to the same Kan injective sub-2-categories. It is known that some relevant 2-categories can be described via Kan injectivity (Example 1.5). We aim to push this observation and show that a vast class of interesting 2-categories can be described via *Kan injectivity axioms*. To do so, we will link Kan injective sub-2-categories to KZ-monads.

The concept of KZ-monad in a 2-category (also known as lax-idempotent monad or KZ-doctrine), presented by Anders Kock in [Koc95] generalises the one of idempotent monad in ordinary categories. In [BF06] Marta Bunge and

Jonathan Funk characterised the 2-adjunctions giving rise to KZ-monads. In [MW12], Francisco Marmolejo and Richard Wood showed that a KZ-monad in a 2-category and its algebras may be presented in terms of left Kan extensions. In particular, their results can essentially be summarised as in Theorem 2.1. This theorem was previously shown for the particular case of order-enriched categories in [CS11], and is an important tool in the proof of Theorem 4.3.

Our **main result** (Theorem 4.3) shows that, in a 2-category $\mathcal{K}$ whose objects satisfy a convenient notion of smallness (see Definition 4.1), for every set $\mathcal{H}$ of morphisms of $\mathcal{K}$ the inclusion

$$\mathbf{LInj}(\mathcal{H}) \hookrightarrow \mathcal{K}$$

is the right adjoint 2-functor of a KZ-adjunction. To this end, we construct, for each object X, a transfinite (pseudo)chain (see Construction 3) leading to the components of the unit of the KZ-adjunction. $\mathbf{LInj}(\mathcal{H})$ is then essentially the corresponding category of (pseudo)algebras. This chain generalizes the Kan injective reflection chain presented in [ASV15] for order-enriched categories. Here, the main factor allowing us to take off from the locally thin context of [ASV15] is the use of a special colimit, which we call *coequinserter*.

The **structure of the paper** goes as follows. In Section 1 we start by introducing weak Kan injectivity and Kan injectivity (Definition 1.1). We put the notions into context, making the due comparison to the literature and proving the closedness under (bi)limits of $\mathbf{LInj}(\mathcal{H})$ (Proposition 1.3), a soundness result towards the main theorem. In Proposition 1.10, we show that, in any 2-category with bicocomma objects, for every class of maps $\mathcal{H}$ there exists a class of maps $\bar{\mathcal{H}}$ such that $\mathbf{WLInj}(\mathcal{H}) = \mathbf{LInj}(\bar{\mathcal{H}})$. We also show that every class of morphisms saturated under Kan-injectivity contains all lari 1-cells and is closed under composition, bicocomma, bipushouts and wide bipushouts (Proposition 1.13).

The subsequent three sections build the technology needed to prove our main theorem. In Section 2, after recalling some results due to Marmolejo and Wood on the structure of KZ-monads, we formulate the result (Theorem 2.1) which will serve as a basis for the proof of our main theorem. We finish this section with Corollary 2.5 stating that, for every class of 1-cells $\mathcal{H}$, if the inclusion $\mathbf{LInj}(\mathcal{H}) \hookrightarrow \mathcal{K}$ is the right part of a KZ-adjunction, then the 2-category of pseudoalgebras of the corresponding KZ-monad is essentially $\mathbf{LInj}(\mathcal{H})$.

Section 3 gives an explicit construction of a pseudochain (Construction 3.4) which provides the candidate left biadjoint to the forgetful functor $\mathbf{LInj}(\mathcal{H}) \hookrightarrow \mathcal{K}$. It is shown that in this pseudochain $(x_{ij})_{i\leq j}$ every x_{0i} belongs to the Kan injective saturation of $\mathcal{H}$.

Section 4 contains our main theorem, which is the following.

Theorem (4.3)**.** Let $\mathcal{K}$ be a 2-category with bicolimits and small objects. Then, for any set $\mathcal{H}$ of 1-cells in $\mathcal{K}$, the inclusion 2-functor $\mathbf{LInj}(\mathcal{H}) \hookrightarrow \mathcal{K}$ is the right part of a KZ-adjunction. Moreover, $\mathbf{LInj}(\mathcal{H})$ is the corresponding Eilenberg-Moore category, up to equivalence of 2-categories.

Section 5, our last section, applies the machinery developed in the paper to study a broad class of 2-categories defined over Lex, the 2-category of categories with finite limits. The main result (Theorem 5.9) of the section relates Kan injectivity with the theory of lex-colimits by Garner and Lack [GL12] and offers an alternative characterization of Φ-exactness.

1. Kan Injectivity

1.1. Left Kan injectivity – weak and strong. Let $\mathcal{K}$ be a 2-category, and $f\colon A \to X$ and $h\colon A \to A'$ two 1-cells in $\mathcal{K}$. Recall that the **left Kan extension of** f **along** h is defined as a 1-cell $f/h\colon A' \to X$ together with a 2-cell,

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\xi_f}{\Longrightarrow}" near start] & A' \arrow[dl, "f/h"] \\ X & \end{tikzcd} \tag{1}$$

such that for any other 1-cell $g\colon A' \to X$ with a 2-cell $\alpha\colon f \Rightarrow g\circ h$ there exists a unique 2-cell $\overline{\alpha}\colon f/h \Rightarrow g$ such that we have the equality $\alpha = (\bar{\alpha} \circ h) \cdot \xi_f$:

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dd, "f"'] \arrow[ddrr, phantom, "\overset{\xi_f}{\Longrightarrow}" near start] & & A' \arrow[ddll, bend right, "f/h"'] \arrow[ddll, bend left, "g"] \arrow[ddll, phantom, "\Downarrow\bar{\alpha}"] \\ & & \\ X & & \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[rr, "h"] \arrow[dd, "f"'] \arrow[ddrr, phantom, "\overset{\alpha}{\Longrightarrow}"] & & A' \arrow[ddll, bend left, "g"] \\ & & \\ X & & \end{tikzcd}$$

Of course, such a 1-cell f/h is defined up to isomorphism.

A 1-cell $p\colon X \to X'$ **preserves the left Kan extension** $(f/h, \xi_f)$ if the pair $(p(f/h), p \circ \xi_f)$ forms a left Kan extension of pf along h, i.e. there is an invertible 2-cell $(pf)/h \cong p \circ f/h$ satisfying the following equation.

$$
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "f"'] & A' \arrow[ddl, bend right, "(pf)/h"'] \arrow[ddl, bend left, "p\circ f/h"] \\
X \arrow[d, "p"'] \arrow[r, phantom, "\overset{\xi_{pf}}{\Longrightarrow}\ \overset{\cong}{\Searrow}"] & \\
X' &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "f"'] & A' \arrow[dl, bend left, "f/h"] \\
X \arrow[d, "p"'] \arrow[ur, phantom, "\overset{\xi_f}{\Longrightarrow}"] & \\
X' &
\end{tikzcd}
$$

Throughout the paper we will make use of the notion of left Kan injectivity given below. We also present the notion of weakly left Kan injectivity, which will be discussed in this section.

Definition 1.1.

(1) An object $X \in \mathcal{K}$ is **weakly left Kan injective** with respect to a family of 1-cells $\mathcal{H}$ if, for all $h\colon A \to A'$ in $\mathcal{H}$ and any $f\colon A \to X$ in $\mathcal{K}$ the left Kan extension $(f/h, \xi_f)$ of f along h exists, see (1).

By the general theory of Kan extensions, this amounts to say that the representable functor $\mathcal{K}(-, X)\colon \mathcal{K} \to \mathbf{Cat}$ maps every 1-cell of $\mathcal{H}$ to a right adjoint 1-cell.

(2) We say that $X \in \mathcal{K}$ is **left Kan injective** with respect to $\mathcal{H}$ if it is weakly left Kan injective and, moreover, the 2-cells ξ_f are invertible. This amounts to say that the representable functor $\mathcal{K}(-, X)$ maps every 1-cell of $\mathcal{H}$ to a *rali* in **Cat** (i.e. a right adjoint with invertible unit).*

(3) A 1-cell $p\colon X \to X'$ of $\mathcal{K}$ is **(weakly) left Kan injective** with respect to $\mathcal{H}$ if its domain and codomain are so and p preserves left Kan extensions along 1-cells in $\mathcal{H}$.

(4) We can form a locally full sub-2-category $\mathbf{WLInj}(\mathcal{H})$ of $\mathcal{K}$ with objects all weakly left Kan injectives with respect to $\mathcal{H}$ and 1-cells between them which preserve left Kan extensions along maps in $\mathcal{H}$. Similarly, we define

$$\mathbf{LInj}(\mathcal{H})$$

** rali* stands for *right adjoint left inverse*; analogously, *lari* stands for *left adjoint right inverse*, i.e. a left adjoint with invertible unit. Similarly, we write lali and rari for left/right adjoints with invertible counit.

restricting objects to left Kan injectives with respect to $\mathcal{H}$.

Bunge and Funk [BF99] studied certain KZ-doctrines, called admissible, and characterised their algebras in terms of weakly left Kan injectivity, considering pointwise left Kan extensions (see also [Str81]). As we will see in the next section, we may characterise the algebras of any KZ-doctrine in terms of left Kan injectivity, and this fact is an important tool in our paper.

Remark 1.2. Consider the diagram below, where X and X' are left Kan injective with respect to $h\colon A \to A'$, and $\overline{h_X} := (-)/h$ is the left adjoint of $\mathcal{K}(h, X)$. A 1-cell $p\colon X \to X'$ preserves left Kan extensions along h if and only if it satisfies an appropriate Beck-Chevalley condition, namely, the following square commutes up to isomorphism:

$$\begin{tikzcd}
\mathcal{K}(A', X) \arrow[d, "\mathcal{K}(A',p)"'] \arrow[dr, phantom, "\cong"] & \mathcal{K}(A, X) \arrow[l, "\overline{h_X}"'] \arrow[d, "\mathcal{K}(A,p)"] & X \arrow[d, "p"] \\
\mathcal{K}(A', X') & \mathcal{K}(A, X') \arrow[l, "\overline{h_{X'}}"] & X'
\end{tikzcd}$$

This characterization concerning Kan injectivity leads to a nice behaviour of Kan injective sub-2-categories with respect to bilimits and pseudolimits, which we describe in the following proposition.

Proposition 1.3. The inclusion 2-functor $\mathbf{LInj}(\mathcal{H}) \hookrightarrow \mathcal{K}$ creates bilimits and pseudolimits.

Proof: Let us consider a pseudofunctor $D\colon I \to \mathbf{LInj}(\mathcal{H})$ (with I a small 2-category) and a weight $W\colon I \to \mathbf{Cat}$ (strict 2-functor).

(1) For any object $i \in I$, $Di \in \mathbf{LInj}(\mathcal{H})$, i.e. $\mathcal{K}(h, Di) =: {h_{Di}}^*$ is a rali (let us denote with $\overline{h_{Di}} \dashv {h_{Di}}^*$ the adjunction).

(2) For any 1-cell $u\colon i \to j \in I$, Du is Kan injective, i.e.

$$\begin{tikzcd}
\mathcal{K}(A, D_i) \arrow[r, "\overline{h_{Di}}"] \arrow[d, "Du\circ-"'] \arrow[dr, phantom, "\cong"] & \mathcal{K}(B, D_i) \arrow[d, "Du\circ-"] \\
\mathcal{K}(A, D_j) \arrow[r, "\overline{h_{Dj}}"'] & \mathcal{K}(B, D_j)
\end{tikzcd}$$

Let us note that these isomorphisms make $\overline{h_{D-}}$ into a pseudonatural transformation, the composition and the other axioms follow by the universal property of Kan extensions.

It is easy to check that also ${h_{D-}}^*$ is a pseudonatural transformation. In particular we have $\overline{h_{D-}} \dashv {h_{D-}}^*$ in the 2-category $[I, \mathbf{Cat}]$ of pseudofunctors, pseudonatural transformations and modifications (which also makes ${h_{D-}}^*$ a rali). It is well-known that hom-functors into $\mathbf{Cat}$ preserve adjunctions (see [Gra74, Proposition I,6.3]). Then, setting $H := \overline{h_{D-}} \circ -$ and $H^* := {h_{D-}}^* \circ -$, we get an adjunction

$$\begin{tikzcd} {[I,\mathbf{Cat}](W,\mathcal{K}(A,D-))} \arrow[r, bend left, "H"] \arrow[r, phantom, "\perp"] & {[I,\mathbf{Cat}](W,\mathcal{K}(B,D-))} \arrow[l, bend left, "H^*"] \end{tikzcd} \quad \text{in } \mathbf{Cat}.$$

Now, let us assume that the W-weighted bi/pseudolimit of D exists in $\mathcal{K}$, i.e.

(1) **Pseudolimit:** There exists an object $L_p \in \mathcal{K}$ such that

$$\mathcal{K}(A, L_p) \cong [I, \mathbf{Cat}](W, \mathcal{K}(A, D-))$$

is an **isomorphism** of categories for any A.

(2) **Bilimit:** There exists an object $L_b \in \mathcal{K}$ such that

$$\mathcal{K}(A, L_b) \simeq [I, \mathbf{Cat}](W, \mathcal{K}(A, D-))$$

is an **equivalence** of categories for any A.

Since both isomorphisms and equivalences of categories preserve adjunctions, in both cases we get a lali (for $L = L_p$ or $L = L_b$)

$$\begin{tikzcd} \mathcal{K}(A,L) \arrow[r, bend left, "\overline{h}"] \arrow[r, phantom, "\perp"] & \mathcal{K}(B,L) \arrow[l, bend left, "-\circ h"] \end{tikzcd}$$

Let us consider projections $l^i_w \colon L \to Di$, i.e. the image of the object $w \in Wi$ under the i-component of the universal pseudonatural transformation $Wi \to \mathcal{K}(L, Di)$ (the one corresponding to 1_L). Let us consider the diagram below

$$\begin{tikzcd}
\mathcal{K}(A,L) \arrow[rrr, "\overline{h_L}"] \arrow[ddd, "l^i_w\circ-"'] \arrow[dr, dash, "\sim"] \arrow[ddr, phantom, "\cong"] & & & \mathcal{K}(B,L) \arrow[ddd, "l^i_w\circ-"] \arrow[ddl, phantom, "\cong"] \\
& & (1) & \\
& {[W,\mathcal{K}(A,D-)]} \arrow[r, "H"] \arrow[ddl, "e_{i,w}"'] & {[W,\mathcal{K}(B,D-)]} \arrow[uur, dash, "\sim"'] \arrow[ddr, "e_{i,w}"] & \\
& & (2) & \\
\mathcal{K}(A,Di) \arrow[rrr, "\overline{h_{Di}}"'] & & & \mathcal{K}(B,Di)
\end{tikzcd}$$

where $e_{i,w}$ is the functor taking a pseudonatural transformation $\alpha\colon W \to \mathcal{K}(A,D-)$ and evaluates its i-component at the object $w \in Wi$ (see below)

$$e_{i,w}\colon \alpha \longmapsto \alpha_i(w) \in \mathcal{K}(A,Di)$$

and we wrote $[W,\mathcal{K}(A,D-)]$ for the category $[I,\mathbf{Cat}](W,\mathcal{K}(A,D-))$. Let us recall that H sends a pseudonatural transformation $\alpha\colon W \to \mathcal{K}(A,D-)$ to

$$W \xrightarrow{\ \alpha\ } \mathcal{K}(A,D-) \xrightarrow{\ \overline{h_{D-}}\ } \mathcal{K}(B,D-).$$

Hence, it is straightforward to check that the diagram (2) commutes and so the whole diagram above (since diagram (1) commutes by definition of $\overline{h_L}$). This shows that each l^i_w is left Kan injective. ∎

All of this reasoning works also when we have $-\circ h$ only a left adjoint and not lali, hence also $\mathbf{WLInj}(\mathcal{H})$ is closed under weighted bi/pseudolimit. This also follows from the fact shown below that every weakly left Kan injective sub-2-category is left Kan injective (Proposition 1.10).

Next we list some examples concerning Kan injective sub-2-categories.

Example 1.4 (Categories with finite colimits are weakly Kan Injective)**.** Let **Rex** be the 2-category of small categories with finite colimits and functors preserving them. Define, in **Cat**,

$$\mathcal{H} = \{\top : D \to 1 \mid D \text{ is a finite category}\}.$$

It then follows from [ML13, X.7.1] that $\mathbf{Rex} \simeq \mathbf{WLInj}(\mathcal{H})$. Of course, because finite colimits are generated by finite coproducts and coequalizers, $\mathcal{H}$ can be reduced to three arrows $D \to 1$, with

$$D = 0,\ \boxed{a\bullet\ \ \bullet b},\ \boxed{\bullet \underset{\to}{\overset{\to}{\ }} \bullet},$$

determining the existence of an initial object, binary coproducts and coequalizers, respectively. For any class $\mathcal{D}$ of finite categories, using a similar argument, we can get the category of categories with any colimit of shape in $\mathcal{D}$.

Example 1.5 (Categories with finite colimits are Kan Injective)**.** Similarly to the previous discussion, we will describe **Rex** as a left Kan injective sub-2-category of **Cat**. In order to do so, for all finite categories D, call $\hat{D}$ the category obtained from D by freely adding a terminal object and call $\iota_D : D \to \hat{D}$ the canonical inclusion. Define,

$$\mathcal{H} = \{\iota_D : D \to \hat{D} \mid D \text{ is a finite category}\}.$$

It then follows from [Rie17, 3.1.8, 6.3.10] that $\mathbf{Rex} \simeq \mathbf{LInj}(\mathcal{H})$. Naturally, as in the above example, $\mathcal{H}$ can be reduced to a class containing only three arrows.

Example 1.6 (Orthogonality)**.** In the context of ordinary categories, that is, locally discrete 2-categories, the notion of Kan-injectivity is just the classical definition of orthogonality. In this case, $\mathbf{LInj}(\mathcal{H})$ is the full subcategory of all objects orthogonal to $\mathcal{H}$ usually denoted by $\mathcal{H}^{\perp}$.

Example 1.7 (Fullness)**.** If $\mathcal{H}$ is made of lax epimorphisms (i.e. for every $h : A \to A'$ in $\mathcal{H}$ and every X, the functor $\mathcal{K}(h, X) : \mathcal{K}(A', X) \to \mathcal{K}(A, X)$ is fully faithful), then $\mathbf{LInj}(\mathcal{H})$ is a full sub-2-category. Indeed, for every map p between Kan injective objects, from the fact that $((pf)/h)h \cong pf \cong p(f/h)h$, it will follow that $(pf)/h \cong p(f/h)$. A detailed study on lax epimorphisms may be seen in [NS22].

Example 1.8 (Order enriched categories)**.** Known examples abound in the 2-category of posets and other order enriched categories. For instance, in the category $\mathbf{Top}_0$ of T_0 topological spaces and continuous maps, the category of continuous lattices and maps preserving directed suprema and infima is $\mathbf{RInj}(\mathcal{H})$ for $\mathcal{H}$ the class of (topological) embeddings, where **RInj** referes to right Kan injectivity in the expected sense. In the category **Loc** of locales and localic maps, the category of stably locally compact locales with convenient maps is $\mathbf{LInj}(\mathcal{H})$ for $\mathcal{H}$ the class of flat embeddings (see [Joh02]). These and other examples may be encountered in [ASV15] and [CS17].

Remark 1.9 (A comparison with enriched weakness)**.** In [LR12], Rosicky and Lack introduce a very interesting notion of injectivity, which is parametric with respect to a class of maps. Let us recall it and briefly to compare it with our notion. Let $\mathcal{V}$ be a reasonably nice category to enrich on and let $\mathcal{E}$

be a class of maps in $\mathcal{V}$. Let $\mathcal{K}$ be a category enriched over $\mathcal{V}$ and $\mathcal{H}$ be a class of 1-cells; then they define

$$\mathbf{Inj}_{\mathcal{E}}(\mathcal{H})$$

to be the full subcategory of $\mathcal{K}$ of those objects X such that $\mathcal{K}(-,X)$ maps $\mathcal{H}$ to $\mathcal{E}$. This definition resonates with ours. Indeed, let us consider the particular choice $\mathcal{V} = \mathbf{Cat}$ and $\mathcal{E} = \mathsf{ra}, \mathsf{rali}$, where ra and rali stand for the classes of right adjoints and of right adjoint left inverses, respectively.

It is clear that on the level of objects, $\mathbf{Inj}_{\mathsf{ra}}(\mathcal{H})$ and $\mathbf{Inj}_{\mathsf{rali}}(\mathcal{H})$ coincide with our notions of Kan injectives. Yet, there is a huge difference on the choice of the 1-cells, which, in our case, leads to a, in general, non-full sub-2-category.

1.2. A comparison between weak Kan-injectivity and Kan-injectivity. The following proposition allows us to restrict to left Kan injectivity without losing generality.

Proposition 1.10. Let $\mathcal{H}$ be a class of maps in a 2-category $\mathcal{K}$ with (bi-)cocomma objects, then there exists a class of maps $\overline{\mathcal{H}}$ such that $\mathbf{WLInj}(\mathcal{H}) = \mathbf{LInj}(\overline{\mathcal{H}})$.

1.11 (The mapping cone trick)**.** Concerning examples 1.4 and 1.5, we can guess a construction of $\overline{\mathcal{H}}$ from $\mathcal{H}$. Indeed, in that case, from each arrow $\top : D \to 1$ one can obtain the *mapping cone* $\iota_D \colon D \to \hat{D}$ via the (bi-)cocomma object below.

$$\begin{tikzcd} D \arrow[r, "\top"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\rho}{\Longrightarrow}"] & 1 \arrow[d, dashed, "j"] \\ D \arrow[r, dashed, "\iota_D"'] & \hat{D} \end{tikzcd}$$

We show in the proof of Proposition 1.10 that this is an instance of a general property. A very similar idea and result appears in [Str14, Sec. 2].

Proof of Proposition 1.10: For every map $h \in \mathcal{H}$, we construct the mapping cone $C(h)$ over h as the bi-cocomma object below.

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\rho}{\Longrightarrow}"] & A' \arrow[d, dashed, "j"] \\ A \arrow[r, dashed, "i_h"'] & C(h) \end{tikzcd}$$

Then, we define $\overline{\mathcal{H}}$ to be the class of all i_h with $h \in \mathcal{H}$. Let us now show that an object X is weakly left Kan injective with respect to $\mathcal{H}$ if and only if it is left Kan injective with respect to $\overline{\mathcal{H}}$. In particular, we will show that an object X is weakly left Kan injective to a $h \in \mathcal{H}$ if and only if it is left Kan injective to i_h.

(1) We start by showing that if X is left Kan injective with respect to i_h, then it is weakly left Kan injective with respect to h.

Let $f\colon A \to X$ be a 1-cell in $\mathcal{K}$. Since X is left injective with respect to i_h, there exists the left Kan extension f/i_h with the associated 2-cell $\xi_f^{i_h}$ an isomorphism. Then, we can set $f/h := f/i_h \circ j$ and ξ_f^h as the pasting diagram below:

$$\begin{tikzcd}
A \arrow[rr, "h"] \arrow[drr, "i_h"'] \arrow[dd, "f"'] & & A' \arrow[d, "j"] \\
 & \overset{\xi_f^{i_h}}{\Longrightarrow} \quad \overset{\rho}{\Longrightarrow} & C(h) \arrow[dll, dashed, "f/i_h"] \\
X. & &
\end{tikzcd}$$

Now, we will show that f/h and ξ_f^h defined in this way satisfy the universal property of the left Kan extension. Let $g\colon A' \to X$ be a 1-cell in $\mathcal{K}$ together with a 2-cell

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[dr, "f"'] & A' \arrow[d, "g"] \arrow[dl, phantom, "\overset{\beta}{\Longrightarrow}"] \\
 & X
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\beta}{\Longrightarrow}"] & A' \arrow[d, "g"] \\
A \arrow[r, "f"'] & X
\end{tikzcd}$$

By the universal property of the bi-cocomma object, the 2-cell β is equivalent to

a 1-cell $\overline{g}\colon C(h) \to X$ and invertible 2-cells

$$\begin{tikzcd}
A \arrow[r, "i_h"] \arrow[drr, bend right, "f"'] & C(h) \arrow[dr, "\overline{g}"] \arrow[d, phantom, "\cong"] & \\
 & \, & X
\end{tikzcd}
\qquad
\begin{tikzcd}
A' \arrow[d, "j"'] \arrow[ddr, bend left, "g"] & \\
C(h) \arrow[dr, "\overline{g}"'] \arrow[r, phantom, "\cong"] & \, \\
 & X
\end{tikzcd}$$

whose pasting with ρ gives β. Then, using the universal property of the left Kan extension f/i_h, we get that these data is equivalent to have

$$\text{a 1-cell } \overline{g}\colon C(h) \to X \text{ such that } \overline{g} \circ j \cong g \text{ and}$$
$$\text{a 2-cell } \overline{\beta}\colon f/i_h \to \overline{g} \text{ such that}$$

$$\begin{tikzcd}
A \arrow[r, "i_h"] & C(h) \arrow[d, bend right=40, "f/i_h"'] \arrow[d, bend left=40, "\overline{g}"] \arrow[d, phantom, "\overset{\overline{\beta}}{\Rightarrow}"] \\
& X
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "i_h"] \arrow[d, "i_h"'] \arrow[dr, "f"] \arrow[dr, phantom, shift right=4ex, "{\xi_f^{i_h}}^{-1} \Uparrow"] & C(h) \arrow[d, "\overline{g}"] \arrow[dl, phantom, shift left=3ex, "\cong"] \\
C(h) \arrow[r, "f/i_h"'] & X.
\end{tikzcd}$$

Let us notice that this means that $\overline{\beta} i_h$ is completely determined by the universal 2-cell $\xi_f^{i_h}$ and the isomorphism $f \cong \overline{g} \circ i_h$. Then, using the 2-dimensional property of the bi-cocomma object we get that $\overline{\beta}$ corresponds to the two 2-cells $\overline{\beta} i_h$ and $\overline{\beta} j$. Therefore, the data above corresponds to

$$\text{a 1-cell } \overline{g}\colon C(h) \to X \text{ with } \overline{g} \circ j \cong g \text{ and}$$
$$\text{a 2-cell } \overline{\beta}_j\colon f/i_h \circ j \to \overline{g} \circ j \text{ such that}$$

$$\begin{tikzcd}
A \arrow[rr, "h"] \arrow[d, equal] \arrow[drr, phantom, "\overset{\rho}{\Longrightarrow}"] & & A' \arrow[d, "j"] \\
A \arrow[rr, "i_h"] \arrow[dr, "i_h"'] \arrow[drr, "f"] & & C(h) \arrow[d, "\overline{g}"] \arrow[dl, phantom, "\cong"] \\
& C(h) \arrow[r, "f/i_h"'] \arrow[ur, phantom, "{\xi_f^{i_h}}^{-1} \Uparrow"] & X
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\rho}{\Longrightarrow}"] & A' \arrow[d, "j"] \arrow[dr, "j"] & \\
A \arrow[r, "i_h"'] & C(h) \arrow[r, phantom, "\overset{\overline{\beta}_j}{\Longrightarrow}"] \arrow[dr, "f/i_h"'] & C(h) \arrow[d, "\overline{g}"] \\
& & X
\end{tikzcd}$$

Putting together all of these steps we get that, given a 1-cell $g\colon A' \to X$ and a 2-cell $\beta\colon f \to g \circ h$, there exists a unique 2-cell $\tilde{\beta}\colon f/h \to g$ (with $\tilde{\beta}$ the composition of $\bar{\beta}_j$ with the isomorphism $\bar{g} j \cong g$ above) such that

$$
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, equal] & A' \arrow[dd, bend left, "g"] \\
A \arrow[r, phantom, "\overset{\beta}{\Longrightarrow}"] \arrow[dr, bend right, "f"'] & \\
& X
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\rho}{\Longrightarrow}"] & A' \arrow[d, dashed, "j"] \arrow[dd, bend left=60, "g"] \\
A \arrow[r, "i_h"'] \arrow[dr, bend right, "f"'] \arrow[dr, phantom, "\overset{\xi_f^{i_h}}{\Longrightarrow}"] & C(h) \arrow[d, dashed, "f/i_h"] \arrow[d, phantom, shift left=4, "\tilde{\beta}"] \\
& X.
\end{tikzcd}
$$

(2) Now we show that if X is weakly left Kan injective with respect to h, then it is left Kan injective with respect to i_h.

Let $f\colon A \to X$ be a 1-cell in $\mathcal{K}$. Since X is weakly left injective with respect to h, there exists the left Kan extension $(f/h, \xi_f^h)$. Then, by the universal property of the bi-cocomma object, there exists a unique (up-to-isomorphism) f/i_h such that

$$
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\xi_f^h}{\Longrightarrow}"] & A' \arrow[d, bend left, "f/h"] \\
A \arrow[r, bend right, "f"'] & X
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\rho}{\Longrightarrow}"] & A' \arrow[d, "j"] \arrow[dr, bend left, "f/h"] \arrow[dr, phantom, "\cong"] & \\
A \arrow[r, "i_h"] \arrow[drr, bend right, "f"'] \arrow[drr, phantom, "\overset{\xi_f^{i_h}}{\Longrightarrow}"] & C(h) \arrow[dr, dashed, "\exists! f/i_h"] & \\
& & X,
\end{tikzcd}
$$

where also $\xi_f^{i_h}$ is an isomorphism. Let us prove now that f/i_h and $\xi_f^{i_h}$ have the universal property of a left Kan extension.

Let $t\colon C(h) \to X$ be a 1-cell. We want to show that to give a 2-cell $\gamma\colon (f/i_h)i_h \Rightarrow ti_h$ is equivalent to give a 2-cell $\overline{\gamma}\colon f/i_h \Rightarrow t$ with $\overline{\gamma} \circ i_h = \gamma$. By the universal property of the bi-cocomma object, to have a 2-cell $\overline{\gamma}\colon f/i_h \Rightarrow t$ is equivalent to give 2-cells $\gamma_{i_h}(= \gamma)$ and γ_j

such that

$$
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\rho}{\Longrightarrow}"] & A' \arrow[d, "j"] & \\
A \arrow[r, "i_h"] \arrow[dr, "i_h"'] & C(h) \arrow[dr, "t"] \arrow[d, phantom, "\overset{\gamma_{i_h}}{\Longrightarrow}"] & \\
& C(h) \arrow[r, "f/i_h"'] & X
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\rho}{\Longrightarrow}"] & A' \arrow[d, "j"'] \arrow[dr, "j"] & \\
A \arrow[r, "i_h"'] & C(h) \arrow[r, phantom, "\overset{\gamma_j}{\Longrightarrow}"] \arrow[dr, "f/i_h"'] & C(h) \arrow[d, "t"] \\
& & X.
\end{tikzcd}
$$

We show that this is equivalent to give a 2-cell $\gamma = \gamma_{i_h}$ with $\overline{\gamma} i_h = \gamma$, by showing that γ_j is determined by γ_{i_h}. This will complete the proof that X is left Kan injective with respect to i_h. Indeed, pasting with $\xi_f^{i_h}$, expanding the identity on $f/i_h \circ j$ through the isomorphism $f/i_h \circ j \cong f/h$, and using the definition of f/i_h, we obtain the following equality

$$
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\rho}{\Longrightarrow}"] & A' \arrow[d, "j"] & \\
A \arrow[r, "i_h"] \arrow[dr, "i_h"] \arrow[ddrr, bend right, "f"'] & C(h) \arrow[ddr, bend left, "t"] \arrow[dr, phantom, "\overset{\gamma_{i_h}}{\Rightarrow}"] & \\
& C(h) \arrow[dr, dashed, "f/i_h"] \arrow[d, phantom, "\Uparrow \xi_f^{i_h}"] & \\
& & X
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, equal] & A' \arrow[dd, "f/h"] \arrow[r, "j"] \arrow[rr, bend left, "j"] & C(h) \arrow[ddl, "f/i_h"] \arrow[r, phantom, "\overset{\gamma_j}{\Longrightarrow}"] & C(h) \arrow[ddll, bend left, "t"] \\
A \arrow[dr, "f"'] \arrow[r, phantom, "\overset{\xi_f^h}{\Longrightarrow}"] & \arrow[r, phantom, "\cong"] & {} & \\
& X & &
\end{tikzcd}
$$

showing that γ_j is determined by $\gamma = \gamma_{i_h}$ via the universality of the left Kan extension of f along h.

Finally, using the description of the Kan extensions given above, we can see the equality for 1-cells as well.

Let $p\colon X \to Y$ be left Kan injective with respect to $\overline{\mathcal{H}}$. Then, for any $h\colon A \to A' \in \mathcal{H}$ and $f\colon A \to X \in \mathcal{K}$,

$$
\begin{aligned}
p \circ f/h &\cong p \circ f/i_h \circ j && \text{(by construction above)}\\
&\cong (pf)/i_h \circ j && \text{(because } p \in \mathbf{LInj}(\overline{\mathcal{H}}))\\
&\cong (pf)/h && \text{(by construction above).}
\end{aligned}
$$

On the other hand, let us consider $p\colon X \to Y \in \mathbf{WLInj}(\mathcal{H})$. For any $i_h \in \overline{\mathcal{H}}$ and any $f\colon A \to X \in \mathcal{K}$, through the universal property of the co-comma object $C(h)$,

$$p \circ f/i_h \text{ corresponds to } p \circ f/h$$
$$\text{and } (pf)/i_h \text{ to } (pf)/h.$$

Since $p \in \mathbf{WLInj}(\mathcal{H})$, we get $p \circ f/h \cong (pf)/h$ and therefore $p \circ f/i_h \cong (pf)/i_h$. ■

1.3. Saturated classes. Kan injectivity determines a Galois connection between locally full sub-2-categories and classes of 1-cells. More precisely, given a locally full sub-2-category $\mathcal{A}$, denote by $\mathcal{A}^{\mathbf{LInj}}$ the class of all 1-cells with respect to which all objects and 1-cells of $\mathcal{A}$ are left Kan injective. Then, we have that $\mathcal{A} \subseteq \mathcal{B}$ implies $\mathcal{B}^{\mathbf{LInj}} \subseteq \mathcal{A}^{\mathbf{LInj}}$; we also have that $\mathcal{H} \subseteq \mathcal{I}$ implies $\mathbf{LInj}(\mathcal{I}) \subseteq \mathbf{LInj}(\mathcal{H})$, and

$$\mathcal{A}^{\mathbf{LInj}} \subseteq \mathcal{H} \text{ if and only if } \mathcal{A} \subseteq \mathbf{LInj}(\mathcal{H}).$$

These considerations justify the definition below.

Definition 1.12 ($\mathcal{H}^{\mathrm{sat}}$)**.** The *saturation of* $\mathcal{H}$ with respect to Kan-injectivity is defined by,

$$\mathcal{H}^{\mathrm{sat}} := (\mathbf{LInj}(\mathcal{H}))^{\mathbf{LInj}}.$$

It follows from the previous discussion that we have $\mathbf{LInj}(\mathcal{H}^{\mathrm{sat}}) = \mathbf{LInj}(\mathcal{H})$. The following proposition shows that $\mathcal{H}^{\mathrm{sat}}$ is closed under certain constructions. This result will be used along the paper, in particular, in Lemma 3.5 and Proposition 5.5.

Proposition 1.13. $\mathcal{H}^{\mathrm{sat}}$ is closed under the following constructions:

(1) (Laris) Any lari 1-cell $l\colon A \to B$ belongs to $\mathcal{H}^{\mathrm{sat}}$.
(2) (Isomorphisms) If $h \in \mathcal{H}^{\mathrm{sat}}$ and there exists an isomorphism $h \cong h'$, then $h' \in \mathcal{H}^{\mathrm{sat}}$.
(3) (Compositions) Given a pair of composable 1-cells $f\colon A \to B$ and $g\colon B \to C$, if $f, g \in \mathcal{H}^{\mathrm{sat}}$, then $gf \in \mathcal{H}^{\mathrm{sat}}$.
(4) (Reflections) If $h \in \mathcal{H}^{\mathrm{sat}}$ and there are pseudocommutative squares

$$\begin{tikzcd} A \arrow[d, "s"'] \arrow[dr, phantom, "\cong"] & B \arrow[l, "l_1"'] \arrow[d, "h"] \\ A' & B' \arrow[l, "l_2"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, "r_1"] \arrow[d, "s"'] \arrow[dr, phantom, "\cong"] & B \arrow[d, "h"] \\ A' \arrow[r, "r_2"'] & B' \end{tikzcd} \tag{2}$$

where l_1 and l_2 are laris with right adjoints r_1 and r_2, respectively, then $s \in \mathcal{H}^{\text{sat}}$.

(5) (Bicocomma objects and bipushouts). If in

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "r"'] \arrow[dr, phantom, "\Longrightarrow"] & A' \arrow[d, "s"] \\ B \arrow[r, "\overline{h}"'] & C \end{tikzcd} \tag{3}$$

$h \in \mathcal{H}^{\text{sat}}$, then $\overline{h} \in \mathcal{H}^{\text{sat}}$, provided that (3) is a bicocomma object or an invertible 2-cell forming a bipushout.

(6) (Wide bipushouts). If the diagram

$$\begin{tikzcd} A \arrow[rr, "h_i"] \arrow[dr, "h"'] & \cong & A_i \arrow[dl, "d_i"] \\ & B & \end{tikzcd}$$

represents a wide bipushout of a family of 1-cells h_i with all of them in $\mathcal{H}^{\text{sat}}$, then $h \in \mathcal{H}^{\text{sat}}$.

Proof:

(1) **Laris:** For any $X \in \mathcal{K}$ and any lari 1-cell $l\colon A \to B$, we want to show that X is Kan injective with respect to l, i.e. $\mathcal{K}(l, X)$ is rali. This is true because the 2-functor $\mathcal{K}(-, X)$ send lari 1-cells to rali 1-cells (see [Gra74, Remark I,6.5]).

(2) **Isomorphisms:** Clearly, for any $X \in \mathcal{K}$, if $h \cong h'$, then also $\mathcal{K}(h, X) \cong \mathcal{K}(h', X)$. Hence, if X is Kan injective with respect to h, then X is also Kan injective with respect to h'.

(3) **Composition:** This follows since the composition of ralis is a rali.

(4) **Reflections:** Let us consider the pseudocommutative squares (2), and let X be left Kan injective with respect to h, i.e. $h^* := \mathcal{K}(h, X)$ is a rali. We want to show that $\mathcal{K}(s, X)$ is a rali as well. Applying $\mathcal{K}(-, X)$ to the pseudocommutative square with l_1 and l_2 we get the

pseudocommutative square below.

$$\begin{tikzcd}[column sep=large, row sep=large]
\mathcal{K}(A,X) \arrow[r, "{l_1}^*"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[d, dashed, bend right, "?"'] & \mathcal{K}(B,X) \arrow[l, bend right, "{r_1}^*"'] \arrow[d, bend left, "(-)/h=:\overline{h}"] \arrow[d, phantom, shift left=2, "\vdash"] \\
\mathcal{K}(A',X) \arrow[u, "s^*"] \arrow[r, "{l_2}^*"] \arrow[r, phantom, shift right=3, "\top"] \arrow[ur, phantom, "\cong"] & \mathcal{K}(B',X) \arrow[u, "h^*"'] \arrow[l, bend left, "{r_2}^*"]
\end{tikzcd}$$

We want to find a left adjoint to s^* with invertible unit. We claim that ${r_2}^* \circ \overline{h} \circ {l_1}^*$ is the required left adjoint. Let us consider two maps $g\colon A \to X$ and $g'\colon A' \to X$, then,

$$\begin{array}{rcl}
{r_2}^* \circ \overline{h} \circ {l_1}^* g & \longrightarrow & g' \\ \hline
\overline{h}{l_1}^* g & \longrightarrow & {l_2}^* g' \\ \hline
{l_1}^* g & \longrightarrow & h^*{l_2}^* g' \\ \hline
{l_1}^* g & \longrightarrow & {l_1}^* s^* g' \\ \hline
g & \longrightarrow & s^* g'
\end{array}$$

We note that in this chain of bijections we used two adjunctions, the isomorphisms ${l_1}^* s^* \cong h^* {l_1}^*$ and that since r_1 is rari, then ${l_1}^*$ is fully faithfull. Clearly this bijection is natural, so we have left to check only that the unit of this adjunction is invertible. Setting $g' := {r_2}^*\overline{h}{l_1}^* g$, following the bijections above we obtain

$$\begin{aligned}
g \to s^*{r_2}^*\overline{h}{l_1}^* g &\cong {r_1}^* h^* \overline{h} {l_1}^* g && (\text{by } r_2 s \cong h r_1) \\
&\cong {r_1}^* {l_1}^* g && (\text{by } h^* \text{ rali}) \\
&\cong g && (\text{by } r_1 \text{ rari}).
\end{aligned}$$

(5) **Bipushouts:** Consider the diagram below, we want to show that if X is Kan injective with respect to h, and the square (3) is a bipushout, then X is also Kan injective with respect to k. The diagram shows how to construct the candidate Kan extension of s using the universal property of the bipushout.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\cong"] & \bullet \arrow[d, "k"] \arrow[ddr, bend left, "s"] & \\
A' \arrow[r, "g"'] & \bullet & \\
& & X
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] & \bullet \arrow[ddr, bend left, "s"] \arrow[d, phantom, "\cong"] & \\
A' \arrow[drr, dashed, bend right, "(sf)/h"'] & {} & \\
& & X
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\cong"] & \bullet \arrow[d, "k"] \arrow[ddr, bend left, "s"] \arrow[ddr, phantom, bend left=10, "\cong"] & \\
A' \arrow[r] \arrow[drr, bend right, "(sf)/h"'] & \bullet \arrow[dr, dashed, "s/k"] \arrow[d, phantom, "\cong"] & \\
& {} & X
\end{tikzcd}
$$

If we follow this approach to show the univeral property of the Kan extension the proof would be very technical. Instead, we follow a more formal approach. In the diagram below, the situation above is formulated in therms of h^* having a left adjoint. Recall that the diagram in the middle must be a bipullback, and we can thus construct the dashed functor on the right.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\
A' \arrow[r, "g"'] \arrow[ur, phantom, "\ulcorner" very near end] & B'
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{K}(B', X) \arrow[r, "k^*"] \arrow[d, "g^*"'] \arrow[dr, phantom, "\lrcorner" very near start] & \mathcal{K}(B, X) \arrow[d, "f^*"] \\
\mathcal{K}(A', X) \arrow[r, shift left, "h^*"] \arrow[r, phantom, "\top"] & \mathcal{K}(A, X) \arrow[l, dashed, shift left, "(-)/h"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{K}(B, X) \arrow[dr, dashed] \arrow[drr, bend left, "\mathrm{id}"] \arrow[ddr, bend right, "(-)/h \circ f^*"'] & & \\
& \mathcal{K}(B', X) \arrow[r, "k^*"] \arrow[d, "g^*"'] & \mathcal{K}(B, X) \arrow[d, "f^*"] \\
& \mathcal{K}(A', X) \arrow[r, "h^*"'] & \mathcal{K}(A, X)
\end{tikzcd}
$$

We now want to show that the dashed arrow provides a left adjoint for k^*. We shall call $(-)//k$ the dashed functor. By the universal property of the bipullback, we already have the invertible map $1 \to k^* \circ (-)//k$, which will be our unit. To construct the counit, we consider the diagram below, and use the 2-dimensional part of the universal property of the bipullback to obtain the desired 2-cell $(-)//k \circ k^* \to 1$.

$$
\begin{tikzcd}
\mathcal{K}(B', X) \arrow[d, "k^*"'] \arrow[ddrr, bend left, "k^*"] & & \\
\mathcal{K}(B, X) \arrow[dr, dashed, "(-)//k"'] \arrow[r, phantom, "\overset{\Longrightarrow}{\eta_k^{-1} \circ k^*}"] & {} & \\
& \mathcal{K}(B', X) \arrow[r, "k^*"] \arrow[d, "g^*"'] & \mathcal{K}(B, X) \arrow[d, "f^*"] \\
& \mathcal{K}(A', X) \arrow[r, "h^*"'] & \mathcal{K}(A, X)
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{K}(B', X) \arrow[d, "k^*"'] \arrow[dr, "g^*"] \arrow[drr, bend left, "id"] & & & \\
\mathcal{K}(B, X) \arrow[r, phantom, "\cong"] \arrow[dr, "f^*"] \arrow[dd, "(-)//k"'] & \mathcal{K}(A', X) \arrow[r, phantom, "\overset{\epsilon_h \circ g^*}{\Longrightarrow}"] \arrow[d, "h^*"] & \mathcal{K}(B', X) \arrow[r, "k^*"] \arrow[dd, "g^*"] & \mathcal{K}(B, X) \arrow[dd, "f^*"] \\
& \mathcal{K}(A, X) \arrow[dr, "(-)/h"] \arrow[d, phantom, "\cong"] & & \\
\mathcal{K}(B', X) \arrow[rr, "g^*"'] & {} & \mathcal{K}(A', X) \arrow[r, "h^*"'] & \mathcal{K}(A, X)
\end{tikzcd}
$$

Moreover, given a 1-cell $p : X \to X'$ which is left Kan injective with respect to h, using the construction above of $(-)/k := (-)//k$ and Remark 1.2, we conclude that p is also left Kan injective with respect to k.

Bi-cocomma objects: We follow the same argument of the second part of Proposition 1.10. Indeed, in the notation of that proposition, if X was Kan injective with respect to h (as opposed to weak Kan injective) the result is true a fortiori. Also, in the proof we never use the fact that the 1-cell $A \to A$ is the identity, it could be any 1-cell. This delivers the proof.

(6) **Wide bipushouts:** The proof is completely similar to the one for bipushouts. Using the left Kan injectivity of X with respect to all h_i by means of the hom-functor $\mathcal{K}(-,X)$, we obtain a wide bipullback and, as a consequence, a left adjoint of $\mathcal{K}(h,X)$ making it a rali:

$$\begin{tikzcd}
\mathcal{K}(A_i,X) \arrow[rr, "\mathcal{K}(h_i,X)"] & & \mathcal{K}(A,X) \\
& \mathcal{K}(B,X) \arrow[ul, "\mathcal{K}(h,X)"'] \arrow[ur, "\mathcal{K}(d_i,X)"] & \\
& \mathcal{K}(A,X) \arrow[uul, bend left, "(-)/h_i"] \arrow[u, dashed, "(-)/h"'] \arrow[uur, bend right, "\mathrm{id}"'] &
\end{tikzcd}$$

That is, for each $s : B \to X$, the 1-cell s/h is obtained by the universality of the wide bipushout:

$$\begin{tikzcd}
A \arrow[rr, "h_i"] \arrow[dr, "h"'] \arrow[ddr, bend right, "s"'] & \cong & A_i \arrow[dl, "d_i"] \arrow[ddl, bend left, "s/h_i"] \\
\cong & B \arrow[d, dashed, "s/h"] & \cong \\
& X &
\end{tikzcd} \tag{4}$$

■

2. KZ-monads presented via Kan-injectivity

Idempotent monads over a category $\mathbb{C}$ are precisely those whose categories of algebras are full reflective subcategories of $\mathbb{C}$. Thus, an idempotent monad may be presented by orthogonality with respect to the family $(\delta_X \colon X \to \bar{X})_{X\in\mathbb{C}}$ of reflections into the corresponding reflective subcategory. In this section, we see that, analogously, a KZ-monad may be presented by left Kan injectivity with respect to a family of 1-cells $(\delta_X \colon X \to \bar{X})_{X\in\mathcal{K}}$, where every $\bar{X}$ is essentially a pseudoalgebra. These facts will have an important role in Section 4.

We recall from [Koc95] that a KZ-monad, also known as a lax-idempotent pseudomonad or KZ-doctrine, can be described as a pseudomonad with unit

δ and multiplication μ such that μ is a right adjoint to $T\delta$ (and a left adjoint to δ_T) with convenient coherence relations.

As in [BF99], by a *KZ-adjunction* we mean a pseudoadjunction whose induced pseudomonad is a KZ-monad.

The following theorem, which, for the particular case of order-enriched categories, was given in [CS11, Theorem 3.4], is, for the general context, essentially contained in [MW12], as we explain in the proof.

Theorem 2.1.

(1) Let $\mathcal{A}$ be a locally full (and locally replete) sub-2-category of the 2-category $\mathcal{K}$, and let

$$d_X \colon X \to DX,\ X \in \mathcal{K},$$

be a family of 1-cells with $\mathcal{A} \subseteq \mathbf{LInj}(\{d_X \colon X \to DX \mid X \in \mathcal{K}\})$ and such that:

(a) For all $X \in \mathcal{K}$, $DX \in \mathcal{A}$, and, for every $f \colon X \to A$ with $A \in \mathcal{A}$, $f/d_X \in \mathcal{A}$.

(b) Every d_X is dense, i.e. the left Kan extension of d_X along itself is given by the 1_{DX} and an invertible 2-cell.

Then, the inclusion $\mathcal{A} \hookrightarrow \mathcal{K}$ is the right part of a KZ-adjunction in $\mathcal{K}$.

(2) Conversely, every KZ-monad $\mathbb{D}$ may be induced by the data in (1) where $d : \mathrm{Id}_{\mathcal{K}} \to D$ is the unit.

Remark 2.2. Under assumption (a), condition (b) is equivalent to the following condition used in [CS11]:

(b′) $(fd_X)/d_X \cong f$ for all $f : DX \to A$ in $\mathcal{A}$.

Indeed, assuming (b), with $f \in \mathcal{A}$, we have that $(fd_X)/d_X \cong f(d_X/d_X) \cong f$.

Proof of Theorem 2.1: (1) Recall, from [MW12, Definition 3.1], that a *left Kan pseudomonad* $\mathbb{D}$ consists of the following data:

(1) for every $X \in \mathcal{K}$, a 1-cell

$$d_X \colon X \to DX;$$

(2) for every 1-cell $f \colon X \to DY$, a left Kan extension of f along d_X

$$\begin{tikzcd} A \arrow[r, "d_A"] \arrow[dr, "f"'] & DA \arrow[d, "f^D"] \arrow[dl, phantom, "\xRightarrow{D_f}"] \\ & DB \end{tikzcd}$$

with $\mathbb{D}_f$ invertible;

(3) for every $f\colon X \to Y$ and $g\colon Z \to DX$, $(f^{\mathbb{D}} \circ g)^{\mathbb{D}} \cong f^{\mathbb{D}} \circ g^{\mathbb{D}}$;
(4) every d_X is dense.

Marmolejo and Wood proved in [MW12, Theorem 4.1] that this data induces a KZ-monad $\mathbb{D} = (D, d, m)$.[†] Following the proof of their theorem, we see that the given D is extended to the endo-pseudofunctor $D\colon \mathcal{K} \to \mathcal{K}$, and d is extended to a strong transformation which is the unit of the pseudomonad. It is clear that, under the hypotheses of our Theorem 2.1, the family $d_X, X \in \mathcal{K}$, fulfils the conditions defining a left Kan pseudomonad, where $f^{\mathbb{D}}$ is an existing left Kan extension f/d_X. The pseudofunctor $D\colon \mathcal{K} \to \mathcal{K}$ is defined on 1-cells by $Df = (d_Y \circ f)^{\mathbb{D}} \cong (d_Y \circ f)/d_X$, which lies in $\mathcal{A}$. Thus, D admits a corestriction $D_{\mathcal{A}}$ to $\mathcal{A}$. Moreover, from Remark 2.2, for every $f\colon X \to A$ with $A \in \mathcal{A}$, the morphism $f/d_X\colon DX \to A$ is the unique 1-cell of $\mathcal{A}$, up to isomorphism, such that $(f/d_X) \circ d_X \cong f$. Consequently, the inclusion functor of $\mathcal{A}$ into $\mathcal{K}$ is the right 2-functor of a KZ-adjunction

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook, bend right] \arrow[r, phantom, "\perp"] & \mathcal{K} \arrow[l, bend right, "D_{\mathcal{A}}"'] \end{tikzcd}$$

whose induced pseudomonad is $\mathbb{D}$.

(2) This is Theorem 4.2 of [MW12] (and its proof). ■

The next theorem describes the category of pseudoalgebras of a KZ-monad by means of left Kan injectivity.

Theorem 2.3 ([CS11], [MW12], see also [KR77] and [BF99])**.** The 2-category of pseudoalgebras and homomorphisms of a KZ-monad is, up to 2-equivalence, the sub-2-category $\mathbf{LInj}(\mathcal{U})$ where $\mathcal{U}$ is made of all components of the unit of the pseudomonad.

Proof: For order-enriched categories, this was proven in [CS11]. For the general context it immediatly follows from [MW12], by combining the description made by Marmolejo and Wood, in Section 3 of that paper, of the category of algebras $\mathbb{D}$-$\overline{\mathrm{Alg}}$ for $\mathbb{D}$ a left Kan pseudomonad, and the fact, given by them in Section 5, Theorem 5.1, that it is, up 2-equivalence, the category of algebras of the *lax*-idempotent pseudomonad determined by $\mathbb{D}$. ■

We have just seen that the category of pseudoalgebras of a KZ-monad is essentially a Kan injective sub-2-category of $\mathcal{K}$. A natural question is:

[†]Marmolejo and Wood studied the dual situation: *right* Kan and *colax-idempotent* pseudomonads.

When does a Kan injective sub-2-category is 2-equivalent to the category of pseudoalgebras for a KZ-monad? For ordinary categories this reduces to the famous Orthogonal Subcategory Problem (introduced in [FK72]) asking when is an orthogonal subcategory the category of algebras of an idempotent monad. For order-enriched categories, an answer of the *Kan Injective Subcategory Problem* was given in [ASV15]. The next two sections are dedicated to give an answer in the general 2-dimensional context.

We end this section by showing that a Kan injective sub-2-category of $\mathcal{K}$ whose inclusion into $\mathcal{K}$ is the right part of a KZ-adjunction is always KZ-monadic, that is, the category of pseudoalgebras of the corresponding KZ-monad, up to 2-equivalence. We will make use of the following lemma (proved in [CS11, Proposition 2.13] for the particular case of order-enriched categories), which shows how left injectivity interacts with lali 1-cells.

Lemma 2.4. Every sub-2-category $\mathbf{LInj}(\mathcal{H})$ is closed under lalis, that is: for any pseudocommutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "l_1"'] \arrow[dr, phantom, "\cong"] & B \arrow[d, "l_2"] \\
X \arrow[r, "g"'] & Y,
\end{tikzcd}$$

with f a 1-cell of $\mathbf{LInj}(\mathcal{H})$ and l_1, l_2 lalis, then also g belongs to $\mathbf{LInj}(\mathcal{H})$.

Proof: We first show that X belongs to $\mathbf{LInj}(\mathcal{H})$. Given any $h\colon C \to C'$ in $\mathcal{H}$ and any $p\colon C \to X$, we need to prove that there exists a Kan extension p/h with an invertible universal 2-cell. Since A is left Kan injective with respect to h we can consider the following 2-cell, where $l := l_1 \dashv r$ and ϵ is the counit of the adjunction (which is an isomorphism since l is lali).

$$\begin{tikzcd}
C \arrow[rrr, "h"] \arrow[dr, "p"] \arrow[ddd, "p"'] & & & C' \arrow[ddl, bend left, "(rp)/h"] \\
& X \arrow[dr, "r"] \arrow[ddl, equal] \arrow[rr, phantom, "\overset{\xi^h_{rp}}{\Longrightarrow}"] & & \\
& \arrow[r, phantom, "\overset{\epsilon^{-1}}{\Longrightarrow}"] & A \arrow[dll, bend left, "l"] & \\
X & & &
\end{tikzcd}$$

The pasting diagram makes $l \circ (rp)/h$ a left Kan extension of p along h (with universal 2-cell invertible, since ξ^h_{rp} is):

$$\begin{aligned} l \circ (rp)/h &\cong (lrp)/h && \text{(since left adjoints preserves left Kan extensions)} \\ &\cong p/h && \text{(since } lr \cong 1\text{).} \end{aligned}$$

Let us now consider the pseudocommutative square (with $l_i \dashv r_i$ for $i = 1, 2$)

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "l_1"'] \arrow[dr, phantom, "\cong"] & B \arrow[d, "l_2"] \\ X \arrow[r, "g"'] & Y. \end{tikzcd}$$

By the first part we already know that X and Y are left Kan injective with respect to $\mathcal{H}$. We have left to prove that g preserves Kan extensions, i.e. for any $h\colon C \to C'$ in $\mathcal{H}$ and any $t\colon C \to X$, then $g \circ t/h \cong (gt)/h$. Indeed,

$$\begin{aligned} g \circ t/h &\cong g \circ l_1 \circ (r_1 t)/h && \text{(by (1) applied to } X\text{)} \\ &\cong l_2 \circ f \circ (r_1 t)/h && \text{(by the pseudocommutativity of the square)} \\ &\cong l_2 \circ (f r_1 t)/h && \text{(because } f \in \mathbf{LInj}(\mathcal{H})\text{)} \\ &\cong (l_2 f r_1 t)/h && \text{(bcecause left adjoints preserve Kan extension)} \\ &\cong (g l_1 r_1 t)/h && \text{(by } g l_1 \cong l_2 f\text{)} \\ &\cong (gt)/h && \text{(by } l_1 r_1 \cong 1\text{).} \end{aligned}$$

■

Corollary 2.5. For every class of 1-cells $\mathcal{H}$, if the inclusion $\mathbf{LInj}(\mathcal{H}) \hookrightarrow \mathcal{K}$ is the right part of a KZ-adjunction, then the 2-category of pseudoalgebras of the corresponding KZ-monad is 2-equivalent to $\mathbf{LInj}(\mathcal{H})$.

Proof: By the two above theorems, and using their notation, we have just to prove that $\mathbf{LInj}(\{d_X \mid X \in \mathcal{K}\})$ is contained in $\mathbf{LInj}(\mathcal{H})$. We start proving that, given $X \in \mathbf{LInj}(\{d_X \mid X \in \mathcal{K}\})$, then $1_X/d_X\colon \bar{X} \to X$ is lali, in particular $1_X/d_X \dashv d_X$. We set ϵ as the inverse of the universal 2-cell

$$\begin{tikzcd} X \arrow[r, "d_X"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\sim}{\Longrightarrow}"] & \bar{X} \arrow[dl, "1_X/d_X"] \\ X. & \end{tikzcd}$$

Moreover, we can define $\eta\colon 1_{\bar{X}} \Rightarrow d_X \circ 1_X/d_X$ using that $1_{\bar{X}}$ is a Kan extension (since d_X is dense). More precisely, we define η as the 2-cell corresponding to

$$
\begin{tikzcd}
X \arrow[r, "d_X"] \arrow[d, "d_X"'] & \bar{X} \arrow[dl, "d_X \circ 1/d_X"] \arrow[dl, phantom, shift right=3, "\Longrightarrow"'] \\
\bar{X} &
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
X \arrow[r, "d_X"] \arrow[dr, equal] \arrow[dr, phantom, shift left=3, "\overset{\epsilon^{-1}}{\Longrightarrow}"] & \overline{X} \arrow[d, "1/d_X"] & \\
& X \arrow[r, "d_X"] & \overline{X}
\end{tikzcd}
$$

One triangle identity follows directly from the definitions of ϵ and η and the second one from the (2-dimensional) universal property of $1_X/d_X$.

Then, since $\bar{X} \in \mathbf{LInj}(\mathcal{H})$, by Lemma 2.4 we get that also $X \in \mathbf{LInj}(\mathcal{H})$. Moreover, given $u\colon X \to Y$ in $\mathbf{LInj}(\{d_X\})$, we can consider the diagrams

$$
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "d_X"'] \arrow[dr, phantom, "\cong"] & Y \arrow[d, equal] \\
\bar{X} \arrow[r, "u/d_X"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\bar{X} \arrow[r, "u/d_X"] \arrow[d, "1_X/d_X"'] \arrow[dr, phantom, "\cong"] & Y \arrow[d, equal] \\
X \arrow[r, "u"'] & Y
\end{tikzcd}
$$

which are mates. Then, since $u/d_X \in \mathbf{LInj}(\mathcal{H})$, using again Lemma 2.4, we get that also $u \in \mathbf{LInj}(\mathcal{H})$. ■

Remark 2.6. For a KZ-monad, let $\mathcal{U}$ be the class of the units. Between the sub-2-category of all (pseudo)algebras and its full sub-2-category consisting of all free algebras we may encounter several relevant sub-2-categories. This is the topic of the paper [HS17], dealing with the order-enriched context.

3. The (pseudo)chain construction

The transfinite chain described here is a 2-dimensional enhancement of the *orthogonal reflection construction* [AR94, 1.37]. The Pos-enriched version analog of this chain was presented in [ASV15, Construction 5.2].

The archetype of a transfinite construction of this kind is the one of Quillen's Small Object Argument. A deep general study on transfinite constructions of free algebras on ordinary categories was made in [Kel80]. In the transfinite construction of [ASV15], besides the conical colimits used in the ordinary case, coinserters were applied. Here, we use a new ingredient, named coequinserter, whose definition (in its strict version) is given next. It is a special 2-colimit which may be obtained as the composition of a coinserter with a coequifier.

Definition 3.1. Given a 2-cell

$$\begin{tikzcd} & B \arrow[dr, "f"] \arrow[dd, phantom, "\Downarrow\gamma"] & \\ A \arrow[ur, "h"] \arrow[dr, "h"'] & & C, \\ & B \arrow[ur, "g"'] & \end{tikzcd}$$

a **coequinserter** of γ consists of a 1-cell $i\colon C \to Q$ and a 2-cell

$$\begin{tikzcd} & C \arrow[dr, "i"] \arrow[dd, phantom, "\Downarrow\phi"] & \\ B \arrow[ur, "f"] \arrow[dr, "g"'] & & Q \\ & C \arrow[ur, "i"'] & \end{tikzcd}$$

such that

$$\begin{tikzcd} & B \arrow[dr, "f"] \arrow[dd, phantom, "\Downarrow\gamma"] & & \\ A \arrow[ur, "h"] \arrow[dr, "h"'] & & C \arrow[r, "i"] & Q \\ & B \arrow[ur, "g"'] & & \end{tikzcd} = \begin{tikzcd} & & C \arrow[dr, "i"] \arrow[dd, phantom, "\Downarrow\phi"] & \\ A \arrow[r, "h"] & B \arrow[ur, "f"] \arrow[dr, "g"'] & & Q \\ & & C \arrow[ur, "i"'] & \end{tikzcd}$$

with the following universal properties:

(a) For any other 1-cell $u\colon C \to R$ and 2-cell $\epsilon\colon uf \Rightarrow ug$ such that $u\gamma = \epsilon h$, there exists a unique $t\colon Q \to R$ such that $ti = u$ and $t\phi = \epsilon$.

(b) For any pair of 1-cells $u, v\colon Q \to R$ and 2-cell $\theta\colon ui \Rightarrow vi$ such that

$$\begin{tikzcd} & C \arrow[dr, "i"] \arrow[dd, phantom, "\Downarrow\phi"] & & \\ B \arrow[ur, "f"] \arrow[dr, "g"'] & & Q \arrow[r, "u"] \arrow[d, phantom, "\Downarrow\theta"] & R \\ & C \arrow[r, "i"'] \arrow[ur, "i"] & Q \arrow[ur, "v"'] & \end{tikzcd} = \begin{tikzcd} & C \arrow[r, "i"] \arrow[dr, "i"] \arrow[dd, phantom, "\Downarrow\phi"] & Q \arrow[dr, "u"] \arrow[d, phantom, "\Downarrow\theta"] & \\ B \arrow[ur, "f"] \arrow[dr, "g"'] & & Q \arrow[r, "v"'] & R, \\ & C \arrow[ur, "i"'] & & \end{tikzcd}$$

then there exists a unique 2-cell $\overline{\theta}\colon u \Rightarrow v$ with $\overline{\theta} i = \theta$.

Remark 3.2 (Coequinserters from coinserters and coequifiers)**.** In a 2-category with coinserters and coequifiers, we can construct a coequinserter as follows.

First, we consider the coinserter of $f, g \colon B \to C$,

$$\begin{tikzcd}
& C \arrow[dd, phantom, "\Downarrow \chi"] \arrow[dr, "e"] & \\
B \arrow[ur, "f"] \arrow[dr, "g"'] & & D. \\
& C \arrow[ur, "e"'] &
\end{tikzcd}$$

Then, let $q \colon C \to Q$ be the coequifier of $\chi \circ h$ and $e \circ \gamma$:

$$\begin{tikzcd}
& B \arrow[r, "f"] \arrow[dd, phantom, "\Downarrow \chi\circ h"] & C \arrow[dd, phantom, "\Downarrow e\circ\gamma"] \arrow[dr, "e"] & \\
A \arrow[ur, "h"] \arrow[dr, "h"'] & & & D. \\
& B \arrow[r, "g"'] & C \arrow[ur, "e"'] &
\end{tikzcd}$$

One can check that the coequinserter is given by the 1-cell $qe \colon C \to Q$ and the 2-cell $q \circ \chi \colon (qe)f \Rightarrow (qe)g$.

Notation 3.3 (Pseudochains)**.** For any limit ordinal i, let $\mathbf{i}$ be the ordered set of all ordinals $j \leq i$ looked as a locally discrete 2-category. By an *i-pseudochain* in a 2-category $\mathcal{K}$ we mean a pseudofunctor

$$X \colon \mathbf{i} \to \mathcal{K}\,.$$

We denote $X(j \leq k)$ by $X_j \xrightarrow{x_{jk}} X_k$ and take $x_{jj} = 1_{X_j}$. Analogously, we may consider a pseudochain indexed by all ordinals, considering a pseudofunctor from the category **Ord**.

In a 2-category $\mathcal{K}$ with (weighted) bicolimits, given a set of 1-cells $\mathcal{H}$, we are going to construct, for every object $X \in \mathcal{K}$, a pseudochain which will allow us (in Section 4) to obtain the free (pseudo)algebras of a KZ-monad induced by the inclusion $\mathbf{LInj}(\mathcal{H}) \hookrightarrow \mathcal{K}$.

Construction 3.4 (The Kan injective pseudochain)**.** Let $\mathcal{K}$ be a 2-category with weighted bicolimits and let $\mathcal{H}$ be a set of 1-cells in $\mathcal{K}$. Given an object X we construct a pseudochain (see 3.3) of objects X_i ($i \in \mathbf{Ord}$). We denote the connecting maps by $x_{ji} \colon X_j \to X_i$ for all $j \leq i$ (we will omit the subscript when clear from context).

The first step is the given object $X_0 := X$. The limit steps X_i, for i a limit ordinal, are defined by bicolimits of i-pseudochains:

$$X_i := \mathrm{bicolim}_{j<i} X_j.$$

Isoleted steps: given X_i with i even, we define both X_{i+1} and X_{i+2}. The idea is that the $i+1$ step approximates the 1-dimensional property of a Kan injective object and the $i+2$ step the 2-dimensional one.

(1) To define X_{i+1} and the connecting map $x_{i,i+1} \colon X_i \to X_{i+1}$, consider all the spans

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & A' \\ X_i & \end{tikzcd} \tag{5}$$

where $h \in \mathcal{H}$ and f is arbitrary. We take the conical bicolimit of the diagram of all spans of the form (5), being X_i a fixed object of the diagram and with $A = \mathrm{dom}(h)$ and $A' = \mathrm{cod}(h)$ running all $h \in \mathcal{H}$. This bicolimit may be obtained as a wide bipushout of all bipushouts of f along h as in (5).

We set $x_{i,i+1}$ and $f//h$ the coprojections of the bicolimit, and the 1-cell $x_{i,i+1}$ is the wanted new connecting map in the pseudochain:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\cong"] & A' \arrow[d, "f//h"] \\ X_i \arrow[r, "x_{i,i+1}"'] & X_{i+1} \end{tikzcd} \tag{6}$$

(2) Here we define X_{i+2} and the connected map $x_{i+1,i+2} : X_{i+1} \to X_{i+2}$. For every 2-cell

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\gamma}{\Longrightarrow}"] & A' \arrow[d, "g"] \\ X_j \arrow[r, "x_{j,i+1}"'] & X_{i+1} \end{tikzcd}$$

with $h \in \mathcal{H}$ and j even, we consider the 2-cell

$$
\begin{tikzcd}
A \arrow[rr, "h"] \arrow[d, "h"'] \arrow[dr, "f"] & & A' \arrow[dr, "g"] & \\
A' \arrow[r, phantom, "\cong"] \arrow[dr, "f//h"'] & X_j \arrow[rr, "x_{j,i+1}"] \arrow[d, dashed, "x_{j,j+1}"] \arrow[ur, phantom, "\overset{\gamma}{\Longrightarrow}"] & & X_{i+1} \\
& X_{j+1} \arrow[urr, "x_{j+1,i+1}"'] \arrow[ur, phantom, "\cong"] & &
\end{tikzcd}
$$

and its bi-coequinserter $c_\gamma\colon X_{i+1} \to C_\gamma$ with universal 2-cell

$$
\begin{tikzcd}
& X_{j+1} \arrow[rr, "x_{j+1,i+1}"] & & X_{i+1} \arrow[dr, "c_\gamma"] & \\
A' \arrow[ur, "f//h"] \arrow[drr, "g"'] & & \arrow[d, phantom, "\Downarrow \chi_\gamma"] & & C_\gamma \\
& & X_{i+1} \arrow[urr, "c_\gamma"'] & &
\end{tikzcd}
.
$$

We define the morphism $x_{i+1,i+2}\colon X_{i+1} \to X_{i+2}$ through the wide bi-pushout of all these c_γ:

$$
\begin{tikzcd}
X_{i+1} \arrow[r, "c_\gamma"] \arrow[dr, "x_{i+1,i+2}"'] \arrow[dr, phantom, shift left=3, "\cong"] & C_\gamma \arrow[d, "d_\gamma"] \\
& X_{i+2}.
\end{tikzcd}
\tag{7}
$$

In the following lemma, which is going to be useful in the proof of Theorem 4.3, we show that, for every ordinal i, x_{0i} belongs to the Kan injectivity saturation $\mathcal{H}^{\mathrm{sat}}$, see Subsection 1.3.

Lemma 3.5. In the Kan injective pseudochain, for every ordinal i, the sub-2-category $\mathbf{LInj}(\mathcal{H})$ is left Kan injective with respect to $x_{0i}\colon X_0 \to X_i$, i.e.

$$\mathbf{LInj}(\mathcal{H}) \subseteq \mathbf{LInj}(\{x_{0i} \mid X \in \mathcal{K}\}).$$

This determines, for each $p_0 : X_0 \to P$ with $P \in \mathbf{LInj}(\mathcal{H})$, a pseudococone $p_i : X_i \to P$ such that

$$p_i \cong p_0/x_{0i}.$$

Proof: The proof is by transfinite induction on ordinals.

Limit step. Assume the property hold for all $i < \kappa$, where κ is a limit ordinal. Then, by construcion of X_κ, there is a unique (up-to-iso) 1-cell $p_\kappa \colon X_\kappa \to P$ such that $p_i \cong p_\kappa x_{i\kappa}$, for all $i < \kappa$.

$$\begin{tikzcd}
X_0 \arrow[r, "x_{0i}"] \arrow[rr, bend left, "x_{0\kappa}"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\cong"] & X_i \arrow[r, "x_{i\kappa}"] \arrow[dl, "p_i"] \arrow[u, phantom, "\cong"] \arrow[d, phantom, "\cong"] & X_\kappa \arrow[dll, bend left, "p_\kappa"] \\
P & &
\end{tikzcd}$$

We want to show that $p_\kappa \cong p_0/x_{0\kappa}$. Given a 2-cell (r, α) as below, for every $i < \kappa$, since $p_i \cong p_0/x_{0i}$ by inductive hypothesis, we have a unique 2-cell $\alpha_i \colon p_i \Rightarrow r x_{i\kappa}$ such that

$$\begin{tikzcd}
& X_i \arrow[dr, "x_{i\kappa}"] \arrow[d, phantom, "\cong"] & \\
X_0 \arrow[ur, "x_{0i}"] \arrow[rr, "x_{0\kappa}"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & & X_\kappa \arrow[dll, bend left, "r"] \\
P & &
\end{tikzcd}
=
\begin{tikzcd}
X_0 \arrow[r, "x_{0i}"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\cong"] & X_i \arrow[r, "x_{i\kappa}"] \arrow[dl, "p_i"] \arrow[d, phantom, "\overset{\alpha_i}{\Rightarrow}"] & X_\kappa \arrow[dll, bend left, "r"] \\
P & &
\end{tikzcd}$$

But then, the 2-dimensional aspect of the universality of the bicolimit of the pseudochain $(X_i)_{i<\kappa}$ ensures the existence of a unique 2-cell $\overline{\alpha} \colon p_\kappa \Rightarrow r$, with

$$\alpha_i = \begin{tikzcd}
X_i \arrow[r, "x_{i\kappa}"] \arrow[d, "p_i"'] \arrow[dr, phantom, "\cong"] & X_\kappa \arrow[dl, "p_\kappa"'] \arrow[dl, bend left, "r"] \arrow[dl, phantom, "\Downarrow\overline{\alpha}"] \\
P &
\end{tikzcd}
\text{ for all } i < \kappa \text{ and, thus, } \alpha =
\begin{tikzcd}
X_0 \arrow[r, "x_{0\kappa}"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\cong"] & X_\kappa \arrow[dl, "p_\kappa"'] \arrow[dl, bend left, "r"] \arrow[dl, phantom, "\Downarrow\overline{\alpha}"] \\
P &
\end{tikzcd}.$$

The unicity of $\bar{\alpha}$ is clear. Consequently, $p_k \cong p_0/x_{0\kappa}$. Moreover, let us consider $u \colon P \to Q$ in $\mathbf{LInj}(\mathcal{H})$ and $p \colon X_0 \to P$ in $\mathcal{K}$. By induction hypothesis we assume that u is left Kan injective with respect to $x_{i\kappa}$ for all $i < k$, i.e. $(up)/x_{0i} \cong u(p/x_{0i})$. We want to show that u is also in $\mathbf{LInj}(\{x_{0\kappa}\})$.

$$\begin{aligned}
((up)/x_{0k})x_{ik} &\cong (up)_\kappa x_{ik} \cong (up)_i \cong (up)/x_{0i} && \text{(by part above for } up) \\
&\cong u(p/x_{0i}) && \text{(by induction hypothesis)} \\
&\cong u(p/x_{0k})x_{ik} && \text{(by part above for } p).
\end{aligned}$$

Hence, by the universal property of the bicolimit, $(up)/x_{0k} \cong u(p/x_{0k})$.

Isolated step. Let i be an even ordinal such that every x_{0j} with $j \leq i$ belongs to the Kan injective saturation of $\mathcal{H}$. We treat the two cases of the construction separately.

$i+1$. As seen in the construction of the pseudochain, $x_{i,i+1}\colon X_i \to X_{i+1}$ is a wide bipushout of bipushouts of morphisms along 1-cells of $\mathcal{H}$. Then, by Proposition 1.13, P is Kan injective with respect to $x_{i,i+1}$. Here we see in detail how to obtain $p_{i+1} \cong p_i/x_{i,i+1}$. Combining this with the inductive hypothesis on x_{0i}, we get that $p_{i+1} \cong p_i/x_{i,i+1}$.

Recall the bicolimit diagram (6) used in the construction of the pseudochain. By its universality, we obtain $p_{i+1} : X_{i+1} \to P$ as below:

$$\xi^h_{p_i f} = \begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\cong"] & A' \arrow[d, "f//h"] \arrow[ddr, bend left, "(p_i f)/h"] \arrow[ddr, phantom, "\cong"] & \\ X_i \arrow[r, "x_{i,i+1}"'] \arrow[drr, bend right, "p_i"'] \arrow[drr, phantom, "\cong"] & X_{i+1} \arrow[dr, dashed, "p_{i+1}"] & \\ & & P \end{tikzcd} \tag{8}$$

We want to show that the bottom triangle forms a left Kan extension of p_i along $x_{i,i+1}$, that is, $p_{i+1} \cong p_i/x_{i,i+1}$.

To do so, consider a 2-cell (r, α) as in the diagram below, and let $\tilde{\alpha}$ be the pasting of the invertible 2-cell $p_{i+1}x_{i,i+1} \Rightarrow p_i$ with α:

$$\tilde{\alpha} = \begin{tikzcd} & X_i \arrow[r, "x_{i,i+1}"] \arrow[dl, "x_{i,i+1}"'] \arrow[dd, "p_i"] \arrow[ddl, phantom, "\cong"] & X_{i+1} \arrow[ddl, bend left, "r"] \arrow[ddl, phantom, "\overset{\alpha}{\Longrightarrow}"] \\ X_{i+1} \arrow[dr, "p_{i+1}"'] & & \\ & P & \end{tikzcd}$$

Moreover, for every span (h, f), let $\bar{\alpha}_{hf}$ be the unique 2-cell for which we have the following equality, determined by the universality of $\xi^{h}_{p_i f}$,

$$
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\cong"] & A' \arrow[rr, bend left, "f//h"] \arrow[ddl, bend left, "(p_i f)/h"'] \arrow[rr, phantom, "\overset{\bar{\alpha}_{hf}}{\Longrightarrow}"] & & X_{i+1} \arrow[dlll, "r"] \\
X_i \arrow[d, "p_i"'] & & & \\
P & & &
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\cong"] & A' \arrow[d, "f//h"] \\
X_i \arrow[r, "x_{i,i+1}"] \arrow[d, "p_i"'] \arrow[dr, phantom, "\overset{\alpha}{\Longrightarrow}"] & X_{i+1} \arrow[dl, "r"] \\
P &
\end{tikzcd}
$$

and put

$$
\tilde{\alpha}_{hf} =
\begin{tikzcd}
 & X_{i+1} \arrow[dr, "p_{i+1}"] & \\
A' \arrow[ur, "f//h"] \arrow[rr, "(p_i f)/h"] \arrow[dr, "f//h"'] \arrow[rr, phantom, shift left=3, "\cong"] \arrow[rr, phantom, shift right=3, "\Downarrow\bar{\alpha}_{hf}"] & & P \\
 & X_{i+1}. \arrow[ur, "r"'] &
\end{tikzcd}
$$

The 2-cells $\tilde{\alpha}$ and $\tilde{\alpha}_{hf}$ obey to the conditions under which we can apply the two dimensional aspect of the universality of the bicolimit given by (6). Consequently, there is a unique $\bar{\alpha} : p_{i+1} \Rightarrow r$ with $\bar{\alpha} x_{i,i+1} = \tilde{\alpha}$ and $\bar{\alpha}(f//h) = \tilde{\alpha}_{hf}$.

Hence, pasting with isomorphisms and using the property of α_{hf}, we see that $\overline{\alpha}$ satisfy also the following equation.

$$
\alpha =
\begin{tikzcd}
X_i \arrow[r, "x_{i+1}"] \arrow[d, "p_i"'] \arrow[dr, phantom, "\cong"] & X_{i+1} \arrow[dl, bend right, "p_{i+1}"'] \arrow[dl, bend left, "r"] \arrow[dl, phantom, "\overline{\alpha}"] \\
P &
\end{tikzcd}
$$

and that it is unique.

Concerning 1-cells, let $u\colon P \to Q$ be in $\mathbf{LInj}(\mathcal{H})$ and set $q := up$. Adding u to diagram (8), we have $(up_i)/h \cong u(p_i/h)$ and, $up_i \cong q_i$. Thus up_{i+1} and q_{i+1} take isomorphic values when composed with $x_{i,i+1}$ and $f//h$. Consequently, $q_{i+1} \cong up_{i+1}$, that is, $(up)/x_{0,i+1} \cong u(p/x_{0,i+1})$.

$i+2$. Let γ be a 2-cell with j even and $j \leq i$ as below.

$$
\begin{tikzcd}
A \arrow[rr, "h"] \arrow[d, "h"'] \arrow[drr, phantom, "\overset{\gamma}{\Longrightarrow}"] & & A' \arrow[d, "s"] \\
A' \arrow[r, "f//h"'] & X_{j+1} \arrow[r, "x"] & X_{i+1}
\end{tikzcd}
$$

Since $p_{j+1}(f//h) \cong (p_j f)/h$ is a left Kan extension, see diagram (8), there exists a unique 2-cell $\bar{\gamma}\colon p_{j+1}(f//h) \Rightarrow p_{i+1}s$ such that

$$
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, "h"] & A' \arrow[ddr, bend left, "s"] & \\
X_j \arrow[r, phantom, "\cong"] \arrow[dr, "x_{j,j+1}"] \arrow[ddr, bend right, "p_j"'] & A' \arrow[d, "f//h"] \arrow[r, phantom, "\overset{\gamma}{\Longrightarrow}"] & {} \\
{} \arrow[r, phantom, "\cong"] & X_{j+1} \arrow[r, "x"] \arrow[d, dashed, "p_{j+1}"] \arrow[r, phantom, shift left=6, ""] & X_{i+1} \arrow[dl, "p_{i+1}"] \arrow[dl, phantom, "\cong", shift right=3] \\
& P &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\cong"] & A' \arrow[d, dashed, "f//h"] \arrow[dr, "s"] & \\
X_j \arrow[r, "x_{j,j+1}"] \arrow[dr, "p_j"'] \arrow[dr, phantom, shift left=3, "\cong"] & X_{j+1} \arrow[r, Rightarrow, "\bar{\gamma}"] \arrow[d, dashed, "p_{j+1}"] & X_{i+1} \arrow[dl, "p_{i+1}"] \\
& P &
\end{tikzcd}
\tag{9}
$$

Therefore, by the 1-dimensional aspect of the universality of the bi-coequinserter, there is a unique (up-to-iso) 1-cell $p_\gamma\colon C_\gamma \to P$ such that $p_\gamma c_\gamma \cong p_{i+1}$ and

$$
\begin{tikzcd}
 & X_{j+1} \arrow[r, "x"] \arrow[dd, phantom, "\Downarrow \chi_\gamma"] & X_{i+1} \arrow[d, dashed, "c_\gamma"] \arrow[dr, bend left, "p_{i+1}"] \arrow[dr, phantom, "\cong"] & \\
A' \arrow[ur, "f//h"] \arrow[dr, "s"'] & & C_\gamma \arrow[r, dashed, "p_\gamma"] & P \\
 & X_{i+1} \arrow[ur, dashed, "c_\gamma"] \arrow[urr, bend right, "p_{i+1}"'] \arrow[urr, phantom, "\cong"] & &
\end{tikzcd}
=
$$

$$
\begin{tikzcd}
 & X_{j+1} \arrow[r, "x"] \arrow[dr, "p_{j+1}"] \arrow[dd, phantom, "\Downarrow \overline{\gamma}"] \arrow[dr, phantom, "\cong"] & X_{i+1} \arrow[d, "p_{i+1}"] \\
A' \arrow[ur, "f//h"] \arrow[dr, "s"'] & & P \\
 & X_{i+1} \arrow[ur, "p_{i+1}"'] &
\end{tikzcd}
$$

Having these for all the possible 2-cells γ gives rise to a unique (upt-to-iso) morphism $p_{i+2}\colon X_{i+2} \to P$ making the following diagram pseudocommutative.

$$
\begin{tikzcd}
X_{i+1} \arrow[r, "c_\gamma"] \arrow[dr] \arrow[ddr, bend right, "p"'] \arrow[dr, phantom, "\cong"] & C_\gamma \arrow[d, dashed, "d_\gamma"] \arrow[ddr, bend left, "n_\gamma"] & \\
 & X_{i+2} \arrow[dr, dotted, "p_{i+2}"] \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\cong"] & {} \\
 & {} & P
\end{tikzcd}
$$

We now would like to conclude that $p_{i+2} \cong p_0/x_{0,i+2}$, and to do so, consider any 2-cell (r, μ) as below,

$$
\begin{tikzcd}
X_0 \arrow[r, "x_{0i}"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\cong"] & X_{i+2} \arrow[dl, dotted, "p_{i+2}"] \\
P &
\end{tikzcd}
\qquad
\begin{tikzcd}
X_0 \arrow[r, "x_{0i}"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\overset{\mu}{\Rightarrow}"] & X_{i+2} \arrow[dl, "r"] \\
P &
\end{tikzcd}
$$

Since $p_{i+1} \cong p_0/x_{0,i+1}$, there is a unique 2-cell $\overline{\mu}\colon p_{i+1} \Rightarrow rx_{i+1,i+2}$ such that

$$
\begin{tikzcd}
X_0 \arrow[r, "x_{0,i+1}"] \arrow[dr, "x"', "\cong"] \arrow[dd, "p_0"'] & X_{i+1} \arrow[d, "x_{i+1,i+2}"] \\
\arrow[r, phantom, "\Rightarrow\mu"] & X_{i+2} \arrow[dl, "r"] \\
P &
\end{tikzcd}
=
\begin{tikzcd}
X_0 \arrow[r, "x_{0,i+1}"] \arrow[dd, "p_0"'] & X_{i+1} \arrow[dr, "x_{i+1,i+2}"] \arrow[ddl, "p_{i+1}"'] \arrow[d, phantom, "\cong"] & \\
& \arrow[r, phantom, "\Rightarrow\overline{\mu}"] & X_{i+2} \arrow[dll, "r"] \\
P & &
\end{tikzcd}
$$

So, for every γ as in the beginning of this step, we obtain a 2-cell $\tilde{\mu}$ defined as the pasting diagram below.

$$
\tilde{\mu} :=
\begin{tikzcd}
& C_\gamma \arrow[dr, "p_\gamma"] \arrow[d, phantom, "\cong"] & \\
X_{i+1} \arrow[ur, "c_\gamma"] \arrow[rr, "p_{i+1}"] \arrow[drr, "x_{i+1,i+2}"] \arrow[d, "c_\gamma"'] & \arrow[dr, phantom, "\Downarrow\bar{\mu}"] & P \\
C_\gamma \arrow[rr, "d_\gamma"'] \arrow[r, phantom, "\cong"] & & X_{i+2} \arrow[u, "r"']
\end{tikzcd}
$$

We want to show that $\tilde{\mu}$ satisfies the required condition for the 2-dimensional universal property of the bi-coequinserter, i.e. to prove that the two pasting diagrams below are equal.

$$
\begin{tikzcd}
A' \arrow[r, "x(f//h)"] \arrow[dr, "s"'] & X_{i+1} \arrow[dr, "c_\gamma"] \arrow[d, phantom, "\Downarrow\chi_\gamma"] & & \\
& X_{i+1} \arrow[r, "c_\gamma"] \arrow[dr, "c_\gamma"'] & C_\gamma \arrow[dr, "p_\gamma"] \arrow[d, phantom, "\Downarrow\tilde{\mu}"] & \\
& & C_\gamma \arrow[r, "rd_\gamma"'] & P
\end{tikzcd}
$$

$$
\begin{tikzcd}
A' \arrow[r, "x(f//h)"] \arrow[dr, "s"'] & X_{i+1} \arrow[r, "c_\gamma"] \arrow[dr, "c_\gamma"'] \arrow[d, phantom, "\Downarrow\chi_\gamma"] & C_\gamma \arrow[dr, "p_\gamma"] \arrow[d, phantom, "\Downarrow\tilde{\mu}"] & \\
& X_{i+1} \arrow[r, "c_\gamma"'] & C_\gamma \arrow[r, "rd_\gamma"'] & P
\end{tikzcd}
$$

Since these two 2-cells have as domain

$$p_\gamma c_\gamma x_{j+1,i+1}(f//h) \cong p_{i+1} x_{j+1,i+1}(f//h) \cong p_{j+1}(f//h) \cong (p_j f)/h$$

which is a left Kan extension, then to show that they are equal it sufficies to show that precomposing with h we obtain the same 2-cell. Indeed, the next three pasting diagrams give all the same 2-cell, by the definition of χ_γ applied twice:

$$\begin{tikzcd}
A \arrow[r, "h"] & B \arrow[rr, "x\circ f//h"] \arrow[dr, "s"'] & & X_{i+1} \arrow[dr, "c_\gamma"] & & \\
& & X_{i+1} \arrow[rr, "c_\gamma"] \arrow[dr, "c_\gamma"'] \arrow[ur, phantom, "\Downarrow\chi_\gamma"] & & C_\gamma \arrow[dr, "p_\gamma"] & \\
& & & C_\gamma \arrow[rr, "rd_\gamma"'] \arrow[ur, phantom, "\Downarrow\tilde{\mu}"] & & P
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "h"'] \arrow[dr, phantom, shift left=3, "\Downarrow\gamma"] & X_j \arrow[dr, "x"] & & & \\
& B \arrow[r, "s"'] & X_{i+1} \arrow[r, "c_\gamma"] \arrow[dr, "c_\gamma"'] \arrow[dr, phantom, shift left=3, "\Downarrow\tilde{\mu}"] & C_\gamma \arrow[dr, "p_\gamma"] & \\
& & & C_\gamma \arrow[r, "rd_\gamma"'] & P
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "h"] & B \arrow[r, "x\circ f//h"] \arrow[dr, "s"'] \arrow[dr, phantom, shift left=3, "\Downarrow\chi_\gamma"] & X_{i+1} \arrow[r, "c_\gamma"] \arrow[dr, "c_\gamma"'] \arrow[dr, phantom, shift left=3, "\Downarrow\tilde{\mu}"] & C_\gamma \arrow[dr, "p_\gamma"] & \\
& & X_{i+1} \arrow[r, "c_\gamma"'] & C_\gamma \arrow[r, "rd_\gamma"'] & P
\end{tikzcd}$$

Consequently, we may apply the two-dimensional aspect of the universality of each bi-coequinserter c_γ, obtaining a unique 2-cell $\hat{\mu}_\gamma$ such that $\hat{\mu}_\gamma \circ c_\gamma = \tilde{\mu}$.

$$\begin{tikzcd}
& X_{i+2} \arrow[dr, bend left, "p_{i+2}"] \arrow[d, phantom, "\cong"] & \\
C_\gamma \arrow[ur, bend left, "d_\gamma"] \arrow[rr, bend left=10, "p_\gamma"] \arrow[rr, bend right=20, "rd_\gamma"'] \arrow[rr, phantom, "\Downarrow\hat{\mu}_\gamma"] & {} & P
\end{tikzcd}$$

Because the equality $\hat{\mu_\gamma} \circ c_\gamma = \tilde{\mu}$ holds for all γ, by the two dimensional universal property of the wide bipushout there exists a unique

$$\overset{\bullet}{\mu} \colon p_{i+2} \Rightarrow r$$

such that $\overset{\bullet}{\mu} \circ d_\gamma = \hat{\mu}_\gamma$. Consequently, $\overset{\bullet}{\mu}\; x_{0,i+2} = \overset{\bullet}{\mu}\; d_\gamma c_\gamma x_{0,i+1} = \hat{\mu}_\gamma c_\gamma x_{0,i+1} = \mu$. The unicity of $\overset{\bullet}{\mu}$ is a routine check.

Concerning the Kan injectivity of 1-cells, let $u : P \to Q$ be in $\mathbf{LInj}(\mathcal{H})$. We want to show that $up_{i+2} \cong q_{i+2}$. In diagram (9), put $\bar{\gamma}_P := \bar{\gamma}$ and, analogously, use the notation $\bar{\gamma}_Q$ for the $\bar{\gamma}$ corresponding to Q, instead of P. Since $u : P \to Q$ preserves left Kan extensions, $u\bar{\gamma}_P = \bar{\gamma}_Q$. Then, we have that

$$up_\gamma \xi_\gamma = u\bar{\gamma}_P = \bar{\gamma}_Q$$

and

$$up_\gamma c_\gamma \cong up_{i+1} \cong q_{i+1}$$

showing that $up_\gamma \cong q_\gamma$. Since this holds for all γ, by the universality of the wide bipushout (7), we conclude that $up_{i+2} \cong q_{i+2}$. ■

4. KZ-monadicity via the pseudochain

Along this section $\mathcal{K}$ is a 2-category with (weighted) bicolimits.

A key requisite in the classical Small Object Argument and Orthogonal Subcategory Problem is a convenient concept of smallness for objects. Here we make use of the following notion:

Definition 4.1. An object A is λ-*small*, for λ an infinite regular cardinal, if the 2-functor

$$\mathcal{K}(A, -) \colon \mathcal{K} \to \mathbf{Cat}$$

preserves bicolimits of λ-pseudochains.

Explicitly: For every λ-pseudochain $(X_i)_{i<\lambda}$, with bicolimit coprojections $l_i : X_i \to L$, we have:

(1) every morphism $a \colon A \to L$ factorises through some X_i (up-to-iso);

$$\begin{tikzcd} & L \\ A \arrow[ur, "a"] \arrow[r, "a'"'] \arrow[ur, phantom, shift right=3, "\cong"] & X_i \arrow[u, "l_i"'] \end{tikzcd}$$

(2) for every 2-cell of the form

$$\begin{tikzcd} & X_i \arrow[dr, "l_i"] \arrow[dd, phantom, "\Downarrow\alpha"] & \\ A \arrow[ur, "f"] \arrow[dr, "g"'] & & L \\ & X_{i'} \arrow[ur, "l_{i'}"'] & \end{tikzcd}$$

there is some $j \geq i, i'$ and a 2-cell $\overline{\alpha}$ such that

$$\begin{tikzcd} & X_i \arrow[dr] \arrow[drr, bend left, "l_i"] \arrow[dd, phantom, "\Downarrow\overline{\alpha}"] & & \\ A \arrow[ur, "f"] \arrow[dr, "g"'] & & X_j \arrow[r, "l_j"] & L \\ & X_{i'} \arrow[ur] \arrow[urr, bend right, "l_{i'}"'] & & \end{tikzcd} = \alpha.$$

An object X is said to be *small* if it is λ-small for some infinite regular cardinal λ.

Remark 4.2. An example of λ-small object is the notion of λ-bipresentable object studied in great detail in [DLO22]. Recall that an object A of $\mathcal{K}$ is said to be **λ-bipresentable** if the 2-functor $\mathcal{K}(A, -) \colon \mathcal{K} \to \mathbf{Cat}$ preserves filtered bicolimits in the sense of [DLO22, 2.1.3]. Notice that in 1-dimensional category theory the two notions collapse due to [AR94, 1.6]. The 2-dimensional aspects of such a result are unknown at the current state of art.

Theorem 4.3. Let $\mathcal{K}$ be a 2-category with bicolimits and small objects. Then, for any set $\mathcal{H}$ of 1-cells in $\mathcal{K}$, the inclusion 2-functor

$$\mathbf{LInj}(\mathcal{H}) \hookrightarrow \mathcal{K}$$

is the right part of a KZ-adjunction. Moreover, $\mathbf{LInj}(\mathcal{H})$ is the corresponding Eilenberg-Moore category, up to equivalence of 2-categories.

Proof: Since $\mathcal{H}$ is a set and every object of $\mathcal{K}$ is small, there is some infinite regular cardinal κ such that all domains and codomains of morphisms of $\mathcal{H}$ are κ-small.

We will use Theorem 2.1, setting $\mathcal{A} := \mathbf{LInj}(\mathcal{H})$, $DX := X_\kappa$ and d_X as the 1-cells $x_{0,\kappa}\colon X = X_0 \to X_\kappa$, to prove that the inclusion 2-functor

$$\mathbf{LInj}(\mathcal{H}) \hookrightarrow \mathcal{K}$$

is the right part of a KZ-adjunction. In Lemma 3.5, we have already proved that $\mathbf{LInj}(\mathcal{H})$ is a sub-2-category of $\mathbf{LInj}(\{x_{0\kappa} \mid X \in \mathcal{K}\})$. Therefore, we just need to prove the following two properties:

(1) For all $X \in \mathcal{K}$, $X_\kappa \in \mathbf{LInj}(\mathcal{H})$ and, for any $p\colon X \to P$ with $P \in \mathbf{LInj}(\mathcal{H})$, the morphism $p/x_{0,\kappa}$ belongs to $\mathbf{LInj}(\mathcal{H})$.

(2) Every $x_{0,\kappa}$ is dense, i.e. the triangle

$$\begin{tikzcd} X_0 \arrow[r, "x_{0,\kappa}"] \arrow[d, "x_{0,\kappa}"'] \arrow[dr, phantom, "\cong"] & X_\kappa \arrow[dl, "1_{X_\kappa}"] \\ X_\kappa & \end{tikzcd}$$

presents 1_{X_κ} as a left extension of $x_{0,\kappa}$ along itself.

Let us prove these properties.

(1) First, we will prove that $X_\kappa \in \mathbf{LInj}(\mathcal{H})$. Given $h\colon A \to A' \in \mathcal{H}$ and $f\colon A \to X_\kappa$, since A is κ-small and $X_\kappa = \operatorname{bicolim}_{j<\kappa} X_j$, there is some even ordinal i such that $f \cong x_{i,\kappa} \circ f'$ with $f'\colon A \to X_i$. We claim that $f/h := x_{i+1,\kappa} \circ f'//h$ and the invertible 2-cell

$$\xi_f^h := \begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f'"] \arrow[dd, bend right=60, "f"'] \arrow[dr, phantom, "\cong"] & A' \arrow[d, "f'//h"] \\ X_i \arrow[r] \arrow[d, "x_{i,\kappa}"] \arrow[dr, phantom, "\cong"] & X_{i+1} \arrow[dl, "x_{i+1,\kappa}"] \\ X_\kappa & \end{tikzcd}$$

provides a left Kan extension of f along h. To prove this, let us consider a 2-cell

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[dr, "f"'] \arrow[r, phantom, shift right=4, "\overset{\beta}{\Longrightarrow}"] & A' \arrow[d, "g"] \\ & X_\kappa. \end{tikzcd}$$

Since A' is κ-small, g factorises through some X_j, i.e. $g \cong x_{j,\kappa}g'$. Without loss of generality, we may assume that this j is even and

$i \leq j$. This way, we can consider the 2-cell β' given by the pasting below.

$$\begin{tikzcd}
 & X_i \arrow[r, "x_{i,j}"] \arrow[rrd] \arrow[d, phantom, "\cong"] & X_j \arrow[rd, "x_{j,\kappa}"] \arrow[d, phantom, "\cong" near start] & \\
A \arrow[ru, "f'"] \arrow[rd, "h"'] \arrow[rrr, "f" description] & \Downarrow\beta & & X_\kappa \\
 & A' \arrow[r, "g'"'] \arrow[rru, "g" description] & X_j \arrow[ru, "x_{j,\kappa}"'] \arrow[u, phantom, "\cong" near start] &
\end{tikzcd}$$

Therefore, since A is κ-small, there is some $m \geq j$ (which we may assume even) and a 2-cell $\overline{\beta}$ such that

$$\beta' = \begin{tikzcd}
 & X_i \arrow[r] & X_j \arrow[rd] \arrow[rrd] & & \\
A \arrow[ru, "f'"] \arrow[rd, "h"'] & \Downarrow\overline{\beta} & & X_{m+1} \arrow[r] \arrow[u, phantom, "\cong"] \arrow[d, phantom, "\cong"] & X_\kappa \\
 & A' \arrow[r, "g'"'] & X_j \arrow[ru] \arrow[rru] & &
\end{tikzcd}$$

The 2-cell $\overline{\beta}$, modulo pasting with the invertible 2-cell $x_{i+1,m+1} \circ f'//h \circ h \cong x_{i,m+1} \circ f'$, is of the form of the 2-cells γ considered in the construction of X_{m+2} (see (2) of Construction 3.4), so let us consider the bi-coequinserter associated to it, which we denote with

$$\begin{tikzcd}
 & X_{i+1} \arrow[r] & X_{m+1} \arrow[rd, "c_{\overline{\beta}}"] & \\
A' \arrow[ru, "f'//h"] \arrow[rd, "g'"'] & \Downarrow\chi_{\overline{\beta}} & & C_{\overline{\beta}} \\
 & X_j \arrow[r] & X_{m+1}. \arrow[ru, "c_{\overline{\beta}}"'] &
\end{tikzcd}$$

We put

$$\tilde{\beta} := \begin{tikzcd}
 & X_{i+1} \arrow[r] \arrow[rrrrr, bend left, "x_{i+1,\kappa}"] \arrow[dd, phantom, "\Downarrow \chi_{\overline{\beta}}"] & X_{m+1} \arrow[dr, dashed, "c_{\overline{\beta}}"] \arrow[drr, bend left] \arrow[dr, phantom, shift left=4, "\cong"] & & & \\
A' \arrow[ur, "f'//h"] \arrow[dr, "g'"'] & & & C_{\overline{\beta}} \arrow[r] & X_{m+2} \arrow[r] & X_\kappa \\
 & X_j \arrow[r] \arrow[rrrrru, bend right, "x_{j,\kappa}"'] & X_{m+1} \arrow[ur, dashed, "c_{\overline{\beta}}"] \arrow[urr, bend right] \arrow[ur, phantom, shift right=4, "\cong"] & & &
\end{tikzcd}$$

which is a 2-cell $x_{i+1,\kappa} f'//h \Rightarrow x_{j,\kappa} g' \cong g$.

Warning. In the next equations we will write $\simeq$ between two 2-cells whether the equality holds when pasting with the connecting isomorphisms of the pseudochain X_i and the isomorphisms $x_{i,\kappa} f' \cong f$ and $x_{j,\kappa} g' \cong g$. We write this instead of the proper equality of pasting diagrams to make the proof more readable.

Pasting $\widetilde{\beta}$ with ξ^h_f, we get, by successively using the definitions of $\chi_{\overline{\beta}}$ and $\overline{\beta}$:

$$\begin{tikzcd}
 & X_i \arrow[r] \arrow[d, phantom, "\cong"] & X_{i+1} \arrow[r] \arrow[dd, phantom, "\Downarrow \chi_{\overline{\beta}}"] & X_{m+1} \arrow[dr, dashed, "c_{\overline{\beta}}"] \arrow[drr, bend left] \arrow[dr, phantom, shift left=4, "\cong"] & & & \\
A \arrow[ur, "f'"] \arrow[r, "h"'] & A' \arrow[ur, dashed, "f//h"] \arrow[dr, "g'"'] & & & C_{\overline{\beta}} \arrow[r] & X_{m+2} \arrow[r] & X_\kappa \\
 & & X_j \arrow[r] & X_{m+1} \arrow[ur, dashed, "c_{\overline{\beta}}"] \arrow[urr, bend right] \arrow[ur, phantom, shift right=4, "\cong"] & & &
\end{tikzcd} \simeq$$

$$\begin{tikzcd}
 & X_i \arrow[r] \arrow[dd, phantom, "\Downarrow \overline{\beta}"] & X_{i+1} \arrow[dr] & & & \\
A \arrow[ur, "f'"] \arrow[dr, "h"'] & & & X_{m+1} \arrow[r] \arrow[rr, bend left, "\cong"] \arrow[rr, bend right, "\cong"'] & C_{\overline{\beta}} \arrow[r] & X_{m+2} \arrow[r] & X_\kappa \\
 & A' \arrow[r, "g'"'] & X_j \arrow[ur] & & &
\end{tikzcd} \simeq$$

$$\begin{tikzcd}
 & X_i \arrow[r] \arrow[dd, phantom, "\Downarrow \overline{\beta}"] & X_{i+1} \arrow[dr] & & \\
A \arrow[ur, dashed, "f'"] \arrow[dr, "h"'] & & & X_{m+1} \arrow[r] & X_\kappa \\
 & A' \arrow[r, "g'"'] & X_j \arrow[ur] & &
\end{tikzcd} \simeq \beta.$$

It is clear that $\tilde{\beta}$ is the unique 2-cell such that pasting with ξ_f^h gives β, which provides the required property of the left Kan extension.

Second, let us consider a 1-cell $p\colon X_0 \to P$ with $P \in \mathbf{LInj}(\mathcal{H})$. We know that p gives rise to a pseudococone $p_i\colon X_i \to P$ satisfying the conditions in Lemma 3.5. Now, we want to show that the morphism $p_\kappa\colon X_\kappa \to P$ belongs to $\mathbf{LInj}(\mathcal{H})$, i.e. for every $f\colon A \to X_\kappa$ and $h\colon A \to A' \in \mathcal{H}$

$$p_\kappa f/h \cong (p_\kappa f)/h.$$

Since A is κ-small, there is some even ordinal i and $f'\colon A \to X_i$ such that $f \cong x_{i,\kappa} f'$. From the first part, we know that $f/h \cong x_{i+1,\kappa} f'//h$. Therefore,

$$\begin{aligned}
p_\kappa f/h &\cong p_\kappa x_{i+1,\kappa} f'//h && \text{(by first part)}\\
&\cong p_{i+1} f'//h && \text{(by pseudococone condition)}\\
&\cong (p_i f')/h && \text{(by construction of } p_{i+1}\text{, see diagram (8)))}\\
&\cong (p_\kappa x_{i,\kappa} f')/h && \text{(by pseudococone condition)}\\
&\cong (p_\kappa f)/h && \text{(by } f \cong x_{i,\kappa} f'\text{).}
\end{aligned}$$

(2) Let us now consider X_κ, which is in $\mathbf{LInj}(\mathcal{H})$ as we proved in the previous point. Setting $p_0 := x_{0,\kappa}$ and $P := X_\kappa$, by Lemma 3.5, we get a pseudococone $p_i\colon X_i \to X_\kappa$ with $p_i \cong p_0/x_{0,i}$. We will show that, for any $i < \kappa$,

$$p_i \cong x_{i,\kappa}$$

which implies that $p_\kappa \cong 1_{X_\kappa}$ and $1_{X_\kappa} \cong x_{0,\kappa}/x_{0,\kappa}$. As usual, we will proceed inductively.

Limit step. It follows directly by bicolimit properties.

Step $i+1$. By construction $p_{i+1}\colon X_{i+1} \to P = X_\kappa$ is the unique (up-to-iso) morphism such that, for any h and f,

$$\xi^h_{p_i f} = \quad \begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\cong"] & A' \arrow[d, dashed, "f//h"] \arrow[ddr, bend left, "(p_i f)/h"] & \\ X_i \arrow[r] \arrow[drr, bend right, "p_i"'] & X_{i+1} \arrow[dr, dashed, "p_{i+1}"] \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\cong"] & {} \\ & {} & P = X_\kappa \end{tikzcd}$$

Now, we will prove that $x_{i+1,\kappa}$ has the same universal property of p_{i+1}. Let us recall that in point (1) we proved that $(x_{i,\kappa}f)/h \cong x_{i+1,\kappa}f//h$. Since by inductive hypothesis we have $p_i \cong x_{i,\kappa}$, then

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\cong"] & A' \arrow[d, dashed, "f//h"] \arrow[ddr, bend left, "(x_{i\kappa}f)/h"'] \arrow[ddr, bend left=40, "(p_i f)/h"] & \\ X_i \arrow[r] \arrow[drr, bend right=10, "x_{i\kappa}"] \arrow[drr, bend right=25, "p_i"'] & X_{i+1} \arrow[dr, dashed] \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\cong"] & {} \\ & {} & X_\kappa \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\cong"] & A' \arrow[d, dashed, "f//h"] \arrow[ddr, bend left, "(p_i f)/h"] & \\ X_i \arrow[r] \arrow[drr, bend right, "p_i"'] & X_{i+1} \arrow[dr, dashed, "p_{i+1}"] \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\cong"] & {} \\ & {} & X_\kappa. \end{tikzcd}$$

Therefore, $x_{i+1,\kappa} \cong p_{i+1}$ since it satisfies the same universal property.

Step $i+2$. Since

$$\begin{tikzcd} X_{i+1} \arrow[r, "c_\gamma"] \arrow[dr, "x_{i+1,i+2}"'] \arrow[dr, phantom, bend left=20, "\cong"] & C_\gamma \arrow[d, "d_\gamma"] \\ & X_{i+2} \end{tikzcd}$$

is a wide bipushout, it suffices to show that for every γ,

$$p_{i+2}d_\gamma \cong x_{i+2,\kappa}d_\gamma.$$

But $(C_\gamma, c_\gamma, \chi_\gamma)$ is a bi-coequinserter, so we can check this isomorphism precomposing with c_γ and χ_γ.

Let us start with c_γ. We have that

$$p_{i+2}d_\gamma c_\gamma \cong x_{i+2,\kappa}d_\gamma c_\gamma \quad\iff\quad p_{i+2}x_{i+1,i+2} \cong x_{i+2,\kappa}x_{i+1,i+2}$$

and both members of the second equality are isomorphic to p_{i+1}: $p_{i+2}x_{i+1,i+2} \cong p_{i+1}$ because p_i is a pseudococone, and $x_{i+2,\kappa}x_{i+1,i+2} \cong x_{i+1,\kappa} \cong p_{i+1}$ by inductive hypothesis.

We have left to prove the equality precomposing with the 2-cell χ_γ, i.e. the 2-cell

$$\begin{tikzcd}
& X_{j+1} \arrow[rr, "x_{j+1,i+1}"] \arrow[dr, phantom, "\Downarrow\chi_\gamma"] & & X_{i+1} \arrow[dr, dashed, "c_\gamma"] \arrow[drrr, bend left, "x_{i+1,\kappa}"] \arrow[drr, phantom, "\cong"] & & & \\
A' \arrow[ur, "f//h"] \arrow[drr, "g"'] & & & & C_\gamma \arrow[r, "d_\gamma"'] & X_{i+2} \arrow[r, "p_{i+2}"'] & X_\kappa \\
& & X_{i+1} \arrow[urr, "c_\gamma"'] & & & &
\end{tikzcd}$$

equals the 2-cell

$$\begin{tikzcd}
& X_{j+1} \arrow[rr, "x_{j+1,i+1}"] \arrow[dr, phantom, "\Downarrow\chi_\gamma"] & & X_{i+1} \arrow[dr, "c_\gamma"] \arrow[drrr, bend left, "x_{i+1,\kappa}"] \arrow[drr, phantom, "\cong"] & & & \\
A' \arrow[ur, "f//h"] \arrow[drr, "g"'] & & & & C_\gamma \arrow[r, "d_\gamma"'] & X_{i+2} \arrow[r, "x_{i+2,\kappa}"'] & X_\kappa. \\
& & X_{i+1} \arrow[urr, "c_\gamma"'] & & & &
\end{tikzcd}$$

These 2-cells have domain $x_{i+1,\kappa}x_{j+1,i+1}f//h \cong x_{j+1,\kappa}f//h \cong (x_{j,\kappa}f)/h$ which is a left Kan extension. Therefore, this equation is true if and only if it remains true pasting with the universal 2-cell $\xi^h_{x_{j,\kappa}f}$. Indeed,

$$\begin{tikzcd}
& X_j \arrow[dr, "x_{j,j+1}"] \arrow[rrrrrrd, bend left, "x_{j,\kappa}"] \arrow[d, phantom, "\cong"] & & & & & & \\
A \arrow[ur, "f"] \arrow[r, "h"'] & A' \arrow[r, "f//h"'] \arrow[drr, bend right, "g"'] & X_{j+1} \arrow[rr, "x_{j+1,i+1}"] \arrow[dr, phantom, "\Downarrow\chi_\gamma"] & & X_{i+1} \arrow[dr, dashed, "c_\gamma"] \arrow[drrr, "x_{i+1,\kappa}"] \arrow[drr, phantom, "\cong"] & & & \\
& & & X_{i+1} \arrow[rr, "c_\gamma"'] & & C_\gamma \arrow[r, "d_\gamma"'] & X_{i+2} \arrow[r, "p_{i+2}"'] & X_\kappa
\end{tikzcd}$$

$$= \text{(by definition of } \chi_\gamma\text{)}$$

$$\begin{tikzcd}
& X_j \arrow[dr, "x_{j,j+1}"] \arrow[rrrrd, bend left, "x_{j,\kappa}"] \arrow[d, phantom, "\Downarrow\gamma"] & & & & \\
A \arrow[ur, "f"] \arrow[dr, "h"'] & {} & X_{i+1} \arrow[rrr, "x_{i+1,\kappa}"] \arrow[dr, dashed, "c_\gamma"] \arrow[drr, phantom, "\cong"] & & & X_\kappa \\
& A' \arrow[ur, "g"] & & C_\gamma \arrow[r, "d_\gamma"'] & X_{i+2} \arrow[ur, "p_{i+2}"'] &
\end{tikzcd}$$

$$= \text{(similarly as above)}$$

$$
\begin{tikzcd}
 & X_j \arrow[dr, "x_{j,j+1}"] \arrow[d, phantom, "\cong"] \arrow[drrrrrr, bend left, "x_{j,\kappa}"] & & & & & & \\
A \arrow[ur, "f"] \arrow[r, "h"'] & A' \arrow[r, "f//h"'] \arrow[drr, bend right, "g"'] & X_{j+1} \arrow[rr, "x_{j+1,i+1}"] \arrow[rr, phantom, bend left, "\cong"] \arrow[dr, phantom, "\Downarrow \chi_\gamma"] & & X_{i+1} \arrow[dr, "c_\gamma"] \arrow[drrr, "x_{i+1,\kappa}"] \arrow[drr, phantom, "\cong"] & & & \\
 & & & X_{i+1} \arrow[rr, "c_\gamma"'] & & C_\gamma \arrow[r, "d_\gamma"'] & X_{i+2} \arrow[r, "x_{i+2,\kappa}"'] & X_\kappa .
\end{tikzcd}
$$

This concludes the proof of the first part of the theorem, i.e. the inclusion 2-functor $\mathbf{LInj}(\mathcal{H}) \hookrightarrow \mathcal{K}$ is the right part of a KZ-adjunction.

Finally, it follows from Corollary 2.5 that $\mathbf{LInj}(\mathcal{H})$ is the corresponding 2-category of pseudoalgebras of the induced KZ-monad. ■

5. Lex colimits, distributive laws and Kan injectivity

5.1. Distributivity. In this section we are given a pseudomonad (S, s, m) and a KZ pseudomonad (T, t, n) on a 2-category $\mathcal{K}$ with weighted bicolimits and a pseudodistributive law

$$d\colon ST \Rightarrow TS.$$

For the theory of pseudodistributive laws over KZ doctrines, we refer to [Wal19, Mar99]. Recall that, as shown in [Wal19, Thm. 35 (a)(b), or Cor. 50], this amounts to a lift $(\hat{T}, \hat{t}, \hat{n})$ of T to the category of (pseudo)algebras for S as in the diagram below, moreover $\hat{T}$ is KZ too and its unit $\hat{t}$ coincides with the unit of T.

$$
\begin{tikzcd}
S\text{-Alg} \arrow[r, "\hat{T}"] \arrow[d] & S\text{-Alg} \arrow[d] \\
\mathcal{K} \arrow[r, "T"'] & \mathcal{K}
\end{tikzcd}
$$

Moreover, because we are assuming that T is KZ, there is at most one such a d [Wal19, Def 33, Thm. 44 and Cor. 49] and it has to coincide with the left Kan extension below.

$$
\begin{tikzcd}
 & T \arrow[dl, "sT"'] \arrow[dr, "Ts"] & \\
ST \arrow[rr, "d \equiv Ts/sT"'] & & TS
\end{tikzcd}
$$

This situation is pretty common in practice. Our guiding example is that in which S is the free completion under finite limits in Cat, while T is a completion under a family of colimits. The reader might observe that in this specific example S is coKZ, and we have not listed this one as a working assumption. We will come back to this later.

Definition 5.1 (Three interesting classes of maps)**.** Consider the diagrams below collecting the data of the distributivity law d between the pseudomonads S and T.

$$\begin{tikzcd} & 1 \arrow[dl, "s"'] \arrow[dr, "t"] & \\ S & & T \end{tikzcd} \qquad \begin{tikzcd} & T \arrow[dl, "sT"'] \arrow[dr, "Ts"] & \\ ST \arrow[rr, "d"'] & & TS \end{tikzcd} \qquad \begin{tikzcd} & S \arrow[dl, "St"'] \arrow[dr, "tS"] & \\ ST \arrow[rr, "d"'] & & TS \end{tikzcd}$$

We can then define three classes of maps in S-Alg.

(1) $\mathcal{H}_{St}$ contains the maps $St : S \to ST$.
(2) $\mathcal{H}_{tS}$ contains the maps $tS : S \to TS$. Notice that those coincide with the unit $\hat{t}$ on free S-algebras and thus are in the 2-category S-Alg.
(3) $\mathcal{H}_{\hat{t}}$ contains the maps $\hat{t} : 1 \to \hat{T}$.

Remark 5.2. On a technical level, the rest of the section will be devoted to discuss the diagram below, that is to discuss the relation between the 2-categories of Kan injectives with respect to these classes of 1-cells.

$$\begin{tikzcd} \mathbf{LInj}(\mathcal{H}_{St}) \arrow[dr] & \mathbf{LInj}(\mathcal{H}_{tS}) \arrow[l, "d^*"'] \arrow[d] & \mathbf{LInj}(\mathcal{H}_{\hat{t}}) \arrow[l] \arrow[dl] & \hat{T}\text{-Alg} \arrow[l, "\simeq"'] \arrow[dll] \\ & S\text{-Alg} & & \end{tikzcd}$$

As hinted by the diagram we will show that d yields a forgetful functor from $\mathbf{LInj}(\mathcal{H}_{St})$ to $\mathbf{LInj}(\mathcal{H}_{tS})$ and that the two classes $\mathcal{H}_{St}$ and $\mathcal{H}_{\hat{t}}$ specify the relatable 2-categories of Kan injectives. Already at this stage we can infer that $\mathbf{LInj}(\mathcal{H}_{\hat{t}})$ is equivalent to $\hat{T}$-Alg. Indeed, the lift of a KZ monad will be KZ and thus its category of pseudoalgebras coincides with the Kan injectives with respect to the unit as observed in Theorem 2.3.

Proposition 5.3. The precomposition with d defines a forgetful 2-functor

$$d^* \colon \mathbf{LInj}(\mathcal{H}_{tS}) \to \mathbf{LInj}(\mathcal{H}_{St}).$$

Proof: The key aspect of this proof is to show that in the diagram below, when X is Kan injective with respect to tS, the precomposition with d gives us the left Kan extension f/St.

$$\begin{tikzcd} S \arrow[r, "St"] \arrow[dr, "tS"] \arrow[dd, "f"'] & ST \arrow[d, "d"] \\ & TS \arrow[dl, dashed] \\ X & \end{tikzcd}$$

Indeed, if we show this, we have that every $\mathbf{LInj}(\mathcal{H}_{tS})$ lies in $\mathbf{LInj}(\mathcal{H}_{St})$ and we will see that a routine idea delivers the functoriality on the spot. Now, recall that by [Wal19, Thm 44, Cor. 49] d must coincide with tS/St. We are thus left with the composition

$$(f/tS) \circ (tS/St)$$

and we want to show that it coincides with f/St. But recall that tS is the unit of $\hat{T}$ on a free algebra, and that $\hat{T}$ is KZ. We can thus apply the Kancellation rule [DLL18, Rem 2.20] and deduce that

$$(f/tS) \circ (tS/St) \cong f/St,$$

which is the thesis. Notice that to apply [DLL18, Rem 2.20], we need St to be admissible in the sense of Bunge and Funk [BF06, Def. 4.3.2], this is true by [Wal19, Lemma 41]. ■

Remark 5.4. Because $\mathcal{H}_{tS} \subset \mathcal{H}_{\hat{t}}$, it follows from subsection 1.3 that we have $\mathbf{LInj}(\mathcal{H}_{\hat{t}}) \subseteq \mathbf{LInj}(\mathcal{H}_{tS})$.

Proposition 5.5. If S is KZ, then $\mathbf{LInj}(\mathcal{H}_{tS}) = \mathbf{LInj}(\mathcal{H}_{\hat{t}})$.

Proof: Together with the previous remark, it is enough to show that $\mathcal{H}_{\hat{t}} \subset \mathcal{H}_{tS}^{\mathrm{sat}}$. By inspecting the diagram below, that is witnessing the fact that every S algebra is reflective in its free completion,

$$\begin{tikzcd}
Y \arrow[rr, "s_Y"'] \arrow[dd, "\hat{t}_Y = t_Y"'] & & SY \arrow[ll, bend right, "a"'] \arrow[dd, "t_{SY}"] \\
& & \\
TY \arrow[rr, "Ts_Y"] & & TSY \arrow[ll, bend left, "Ta"]
\end{tikzcd}$$

We see that $\hat{t}$ can be obtained from tS via one of the steps that saturates $\mathcal{H}_{tS}$ in Proposition 1.13 and thus the two classes have the same saturation. ■

5.2. Lex colimits. The technology of lex colimits was introduced by Garner and Lack [GL12] to describe a large class of structures where colimits interact with limits *lex*-ly. The paradigmatic example of this behavior is Grothendieck topoi, where this phenomenon is called descent.

Let us recall very briefly their definitions to set the notation of the subsection and introduce the reader to the topic. We work in Cat, the 2-category of small (but possibly large) categories, functors between them and modifications. This choice will rule out the very interesting case of Grothendieck topoi, and could be avoided by paying the price of a very detailed foundational analysis. We prefer to stick to small categories because the treatment will be largely cleaner.

Construction 5.6. For Φ a set of weights $W\colon I^{\mathrm{op}} \to \mathrm{Set}$ and C a category, we can consider the category $\Phi_l(C)$ as the full subcategory of the presheaf category $\mathrm{Psh}(C)$ consisting of all Φ-weighted colimits of representables (see [GL12, Sec. 3]). This construction defines a KZ doctrine on Cat, which lifts to a KZ monad over Lex, the 2-category of small categories with finite limits and functors preserving them.

$$\begin{tikzcd}
\mathrm{Lex} \arrow[r, "\hat{\Phi}_l"] \arrow[d] & \mathrm{Lex} \arrow[d] \\
\mathrm{Cat} \arrow[r, "\Phi_l"'] & \mathrm{Cat}
\end{tikzcd}$$

Of course, this perfectly fits with the narrative the previous subsection, indeed Lex is coKZ doctrinal over Cat.

Definition 5.7 (Φ-lex-cocompleteness, [GL12])**.** C is Φ-*lex-cocomplete* if it is lex and has colimits of shape Φ, that is if it is lex and is a Φ_l algebra. This gives us the 2-category Φ-LexAlg, of lex categories supporting a Φ_l structure and functors preserving both finite limits and Φ-colimits.

Definition 5.8 (Φ-exactness, [GL12])**.** Now a Φ-lex-cocomplete category is said to be Φ-*exact* if its algebra structure $\Phi_l(C) \to C$ is lex, which amounts to saying that C bears a structure of pseudo-algebra for the pseudomonad Φ_l on Lex. This gives us the 2-category Φ-Ex, which is another name for $\hat{\Phi}_l$-Alg.

Of course, it is easy to see that we have a fully faithful forgetful 2-functor which is only acknowledging that the requirement of being Φ-exact is stronger than being lex and Φ-cocomplete.

$$\begin{tikzcd}
\Phi\text{-LexAlg} \arrow[dr] & & \Phi\text{-Ex} \arrow[ll, "U"'] \arrow[dl] \\
& \mathrm{Lex} &
\end{tikzcd}$$

We are now read to apply our machinery to this situation, and get the following theorem.

Theorem 5.9. The following are equivalent.

(1) C is Φ-exact.
(2) C is Kan injective in Lex to all the maps $D \to \Phi_l(D)$.

Remark 5.10. This theorem follows directly from the previous subsection, but we shall compare it with the content of [GL12, 3.4]. The (1) $\Leftrightarrow$ (2) part of the theorem above may seem to say the same thing of [GL12, 3.4]. But their Kan extensions are taken in Cat, while in our paper we compute them in Lex. The forgetful functor from Lex to Cat does not seem to preserve left Kan extensions in general, their result is thus surprising from this point of view.

Theorem 5.11. Φ-LexAlg is equivalent to $\mathbf{LInj}(\mathcal{H}_{St})$ and the forgetful functor U is precisely d^*.

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