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NON SYMMETRIC CAUCHY KERNEL, CRYSTALS AND LAST PASSAGE PERCOLATION

OLGA AZENHAS, THOMAS GOBET AND CÉDRIC LECOUVEY

Abstract: We use non-symmetric Cauchy kernel identities to get the law of last passage percolation models in terms of Demazure characters. The construction is based on some restrictions of the RSK correspondence that we rephrase in a unified way which is compatible with crystal basis theory.

Keywords: Cauchy identity, Demazure characters, crystals, Coxeter monoids, percolation models.
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1. Introduction

The Cauchy kernel identity is a classical corner stone in the theory of symmetric functions and characters of the linear groups over the complex field. Given two sets of indeterminates $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ it asserts that

$$\prod_{i=1}^{m}\prod_{j=1}^{n}\frac{1}{1-x_iy_j} = \sum_{\lambda\in\mathcal{P}_{\max(m,n)}} s_\lambda(X)s_\lambda(Y)$$

where $\mathcal{P}_{\max(m,n)}$ is the set of partitions with at most $\max(m,n)$ parts and, for each such partition λ, $s_\lambda(X)$ and $s_\lambda(Y)$ are the Schur polynomials in the indeterminates X and Y, respectively. In fact the Schur functions $s_\lambda(X)$ and $s_\lambda(Y)$ can be interpreted as the characters of the irreducible finite-dimensional representations of highest weight λ for the linear Lie algebras $\mathfrak{gl}_m(\mathbb{C})$ and $\mathfrak{gl}_n(\mathbb{C})$. The aforementioned Cauchy identity can then be regarded as the character of the $\mathfrak{gl}_m \times \mathfrak{gl}_n$ bi-module $S(\mathbb{C}^m \times \mathbb{C}^n)$ where $S(\mathbb{C}^m \times \mathbb{C}^n)$ is the symmetric tensor space associated to $\mathbb{C}^m \times \mathbb{C}^n$. This can be proved in a very elegant way (see [13, 33]) by using the Robinson-Schensted-Knuth correspondence. Recall this is a one-to-one map ψ between the set $\mathcal{M}_{m,n}$ of matrices M with m rows, n columns and entries in $\mathbb{Z}_{\geq 0}$, and the pairs (P, Q) of semistandard tableaux both with the same shape λ where P and Q have entries in $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$, respectively. The RSK correspondence has many interesting properties. In particular, for each matrix M in $\mathcal{M}_{m,n}$, the greatest integer which can be obtained by summing up the entries in all the possible paths starting at position $(1, n)$ and ending at position $(m, 1)$ with steps $\longleftarrow$ or $\downarrow$ coincides[*] with the longest row in the tableaux P, Q such that $\psi(M) = (P, Q)$. It is then natural to study percolation models based on the RSK correspondence where random matrices whose entries follow independent geometric laws are considered (see [5] for a recent exposition). This type of model has been deeply studied by Johansson in [17], who proved that the fluctuations of the previous last passage percolation, once correctly normalized, are controlled by the Tracy-Widom distribution (defined from the study of the largest eigenvalues of random Hermitian matrices). The RSK correspondence admits various generalizations which can also be used to get interesting last passage percolation models. These models involve symmetric polynomials or generalizations of

[*]We here consider the paths which are compatible with the version of RSK that will be used in the paper.

symmetric polynomials, in particular characters of representations of Lie algebras other than $\mathfrak{gl}_n$ (which are also symmetric with respect to the associated Weyl group). We refer the reader to [7] for a recent survey and numerous new interesting results in this direction. In a connected area, the dual Cauchy identity also yields rich random structures as those studied in the recent papers [3, 31].

In this paper, we shall follow a different approach and consider percolation models based on the non-symmetric Cauchy kernel as initially studied by Lascoux in [23]. It was also later considered in [15] just as computations on polynomials. This means that the ordinary Cauchy identity will be replaced by its non-symmetric analogue

$$\prod_{1\leq j\leq i\leq n} \frac{1}{1-x_i y_j} = \sum_{\mu\in\mathbb{Z}^n_{\geq 0}} \overline{\kappa}^{\mu}(X)\kappa_{\mu}(Y) \tag{1}$$

where $\overline{\kappa}^{\mu}(X)$ and $\kappa_{\mu}(Y)$ are this time Demazure atoms and Demazure characters (see § 2.3.4 below for complete definitions) in the indeterminates X and Y (with $m = n$). It is important to emphasize here that these polynomials are not symmetric in X and Y. They only correspond to characters of representations for subalgebras of the enveloping algebra $U(\mathfrak{gl}_n)$. It was proved in [23] that the identity (1) can be obtained by restricting the RSK correspondence ψ to the set of lower triangular matrices†. Since then, different other proofs have been proposed, in particular in [1] (using the combinatorics of skyline diagrams) and [8] (using the combinatorics of crystal bases). The seminal paper [23] also established generalizations of the formula (1) where positions with nonzero entries are authorized in the matrices outside their lower part. These so-called extended staircase formulas (see § 3.2 and § 3.3) were then obtained just by computations on polynomials and thus not related to the RSK correspondence. This connection was partially done in [2] where other truncated staircases formulas are also proved to be compatible with the RSK correspondence using the combinatorics of skyline diagrams [28, 29] and Fomin's growth diagrams [12, 33]. This corresponds to the case where nonzero entries are authorized only in positions (i, j) with $n - p \leq i \leq j \leq q$, for p and q two nonnegative integers such that $n \geq q \geq p \geq 1$.

The goal of our paper is two-fold. First, we establish all the existing variants of the non-symmetric Cauchy Kernel identities in the setting of crystal

†In fact, the convention of our paper differs from that in [23] which considers matrices with nonzero entries in positions (i, j) with $1 \leq i + j \leq n + 1$ rather than lower-triangular matrices.

basis theory and make it compatible with the RSK construction based on bi-crystals. Recall here that crystals are oriented graphs which can be interpreted as the combinatorial skeletons of irreducible finite-dimensional representations of $\mathfrak{gl}_n$. We refer the reader to [6] and the references therein for a recent exposition. Crystal bases were introduced by Lusztig (for any finite root system) [27] and Kashiwara (for classical root systems) [18] in 1990. The graph structure arises from the action of the so-called Kashiwara operators, which are certain renormalizations of the Chevalley operators. It was later proved that crystals coincide with Littelmann's graphs defined by using his path model [25]. Crystal theory allows one to get an illuminating interpretation of the RSK-correspondence and thus, in particular, of the Cauchy identity. A similar interpretation was discovered by Choi and Kwon in [8] for the non-symmetric case (1). Here we complete the picture with the truncated and augmented staircase formulas. Our second objective is to use the previous compatibility of the aforementioned map ψ with the generalized Cauchy identities to give the law of some last passage percolation models where constraints are imposed on the locations of nonzero positions in the random matrices considered. These laws will be expressed in terms of Demazure characters and Demazure atoms and thus will have less symmetries than the existing ones which rather use symmetric polynomials. There is nevertheless an interesting intersection in the case $x_i = y_i$ for any $i = 1, \dots, n$. Then, the identity (1) becomes symmetric and can be expanded in terms of Schur functions by using an identity due to Littlewood (see [8]). This case yields a last passage percolation model already studied (see [7]).

The paper is organized as follows. In Section 2, we recall the background on representation theory of $\mathfrak{gl}_n$, the corresponding character theory (its usual and Demazure versions) and its links with the Coxeter monoid and crystal basis theory. Some key results for the purposes of this article are established here for which we did not find references in the literature. We also relate the RSK correspondence with bi-crystal structures and interpret the Cauchy and non-symmetric Cauchy identities in this context. The non-symmetric Cauchy identity is in particular obtained as the restriction of the usual RSK to lower triangular matrices. The goal of Section 3 is to prove that one can also get the truncated staircase Cauchy identity by restriction of RSK to a relevant subset of matrices. To this end, we consider parabolic restrictions of Demazure crystals and show that they admit a simple combinatorial structure. In particular, § 3.3 is devoted to the extended staircase Cauchy identity which is yet obtained by restriction of RSK. The idea here is to use suitable adaptations of Demazure operators (defined on polynomials) acting on crystals. It is also explained in § 3.4 how the extended staircase result allows one to rederive the truncated staircase identity by making more explicit its formulation and connecting it to the approach proposed in [1, 2]. Finally in Section 4, we use the previous combinatorial constructions to get the law of various percolation models in terms of Demazure characters. In the Appendix 5, for the reader convenience, Coxeter monoids and Coxeter-theoretic techniques are given.

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2. Background on representations and characters of $\mathfrak{gl}_n$

In this section, we review some classical results about representation theory of the linear Lie algebra $\mathfrak{gl}_n = \mathfrak{gl}_n(\mathbb{C})$ over the field of complex numbers [14]. Firstly, recall the triangular decomposition $\mathfrak{gl}_n = \mathfrak{gl}_n^+ \oplus \mathfrak{h} \oplus \mathfrak{gl}_n^-$ of $\mathfrak{gl}_n$ into its upper, diagonal and lower parts.

2.1. Representations and characters. Let $\mathcal{P}_n$ be the set of partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ with at most n parts. A partition will be identified with its Young diagram written in French convention (see Example 2.9). The finite-dimensional irreducible polynomial representations of $\mathfrak{gl}_n$ are parametrized by the partitions in $\mathcal{P}_n$. To any $\lambda \in \mathcal{P}_n$, we denote by $V(\lambda)$ the corresponding finite-dimensional representation (or $\mathfrak{gl}_n$-module). By considering only the action of the (commutative) Cartan subalgebra $\mathfrak{h}$ on $V(\lambda)$, one gets the weight space decomposition

$$V(\lambda) = \bigoplus_{\mu \in P} V(\lambda)_\mu$$

where the weight space $P = \mathbb{Z}_{\geq 0}^n = \oplus_{i=1}^n \mathbb{Z}_{\geq 0}\mathbf{e}_i$ is regarded as a subset of $\mathfrak{h}^*$ and for any $\mu \in P$

$$V(\lambda)_\mu = \{v \in V(\lambda) \mid h(v) = \mu(h)v \text{ for any } h \in \mathfrak{h}\}.$$

The symmetric group $\mathfrak{S}_n$ (which is the Weyl group of $\mathfrak{gl}_n$) acts on P by permuting the coordinates of the weights and one then has $\dim V(\lambda)_\mu = \dim V(\lambda)_{\sigma(\mu)}$ for any $\sigma \in \mathfrak{S}_n$ and any $\mu \in P$. The weight space decomposition leads to the notion of character of $V(\lambda)$ which is the polynomial in the indeterminates $x_1, \ldots, x_n$ defined by

$$s_\lambda = \sum_{\mu \in P} \dim V(\lambda)_\mu x^\mu$$

where for any $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$ we use the notation $x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$. By the previous considerations, the polynomial s_λ belongs in fact to the ring $\mathrm{Sym}_{\mathbb{Z}}[x_1, \ldots, x_n]$ of symmetric polynomials in the indeterminates $x_1, \ldots, x_n$ with coefficients in $\mathbb{Z}$. This is the celebrated Schur polynomial which can also be obtained as the quotient of two skew-symmetric polynomials using the formula

$$s_\lambda = \frac{\sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) x^{\sigma(\lambda+\rho)}}{\sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) x^{\sigma(\rho)}},$$

where $\rho = (n-1, n-2, \ldots, 1, 0)$.

Remark 2.1. *Instead of considering the representation theory of $\mathfrak{gl}_n$, we can proceed similarly with the representation theory of its enveloping algebra $U(\mathfrak{gl}_n)$. Simple finite-dimensional $U(\mathfrak{gl}_n)$-modules are still parametrized by the elements of $\mathcal{P}_n$ and we will use the same notation for both representation theories.*

2.2. Bruhat order and Coxeter monoid. Recall that $\mathfrak{S}_n$ is generated by $S = \{s_1, \ldots, s_{n-1}\}$ where for any $i = 1, \ldots, n-1$, s_i is the simple transposition (or simple reflection) flipping i and $i+1$; this yields a realization of $\mathfrak{S}_n$ as a Coxeter group. We denote by $\ell(\sigma)$ the length of a permutation $\sigma \in \mathfrak{S}_n$, defined as the smallest integer $k \geq 0$ such that $\sigma = s_{i_1} \cdots s_{i_k}$, where the s_{i_j}'s are simple reflections. A word of the form $s_{i_1} s_{i_2} \cdots s_{i_k}$ representing $\sigma \in \mathfrak{S}_n$ and such that all the s_{i_j}'s are simple reflections and $\ell(\sigma) = k$ is called a reduced decomposition of σ. We refer the reader to [4] for basic statements on the symmetric group viewed as a Coxeter group.

The (strong) Bruhat order $\leq$ on $\mathfrak{S}_n$ can be defined by $\sigma' \leq \sigma$ in $\mathfrak{S}_n$ if and only if there is a reduced decomposition of σ admitting a subword (not necessarily made of consecutive letters) which is a reduced decomposition of σ', if and only if every reduced decomposition of σ admits a subword which is a reduced decomposition of σ' (see [4, Corollary 2.2.3]). The longest element of $\mathfrak{S}_n$ is denoted by σ_0. Given any partition λ in $\mathcal{P}_n$, we denote by $\mathfrak{S}_\lambda$ its stabilizer under the action of $\mathfrak{S}_n$. Each coset in $\mathfrak{S}_n/\mathfrak{S}_\lambda$ contains a unique element of minimal length and the set of elements of minimal length is denoted by $\mathfrak{S}_n^\lambda$. Then each $\sigma \in \mathfrak{S}_n$ admits a unique decomposition of the form $\sigma = uv$ with $v \in \mathfrak{S}_\lambda$,

$u \in \mathfrak{S}_n^\lambda$ and $\ell(\sigma) = \ell(u) + \ell(v)$. One then has a one-to-one correspondence between the elements of $\mathfrak{S}^\lambda$ and the $\mathfrak{S}_n$-orbit of λ which we denote by $\mathfrak{S}_n\lambda$.

The elementary bubble sort operator π_i, $1 \leq i < n$, on the weak composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$, sorts the entries in positions i and $i+1$ by swapping α_i and α_{i+1} if $\alpha_i > \alpha_{i+1}$, and fixing α otherwise, namely,

$$\pi_i(\alpha) = \begin{cases} s_i\alpha \text{ if } \alpha_i > \alpha_{i+1} \\ \alpha \text{ if } \alpha_i \leq \alpha_{i+1}. \end{cases} \tag{2}$$

Thus elementary bubble sort operators π_i, $1 \leq i < n$, satisfy the relations

$$\begin{aligned} &\pi_i^2 = \pi_i \ (i = 1, \ldots, n), \\ &\pi_i\pi_{i+1}\pi_i = \pi_{i+1}\pi_i\pi_{i+1} \ (i = 1, \ldots, n-1), \ \pi_i\pi_j = \pi_j\pi_i, \ (|i-j| > 1). \end{aligned} \tag{3}$$

It follows from Matsumoto's Lemma [30, 6] that for every $w \in \mathfrak{S}_n$, we may write π_w to mean $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$, whenever $s_{i_1}s_{i_2}\cdots s_{i_k}$ is a reduced word of w in $\mathfrak{S}_n$. Later we will see that the above set of relations define the so-called Coxeter monoid $\mathfrak{M}_n$ [32] (see Section 3.1).

Lemma 2.2. *Let $\lambda \in \mathcal{P}_n$, $w \in \mathfrak{S}_n$, and let $\mu = w\lambda$.*

1. *Let $t = (i\ j)$ be a transposition in $\mathfrak{S}_n$ with $i < j$. If $\mu_i < \mu_j$, then $\ell(tw) < \ell(w)$.*
2. *If $s_{i_1}s_{i_2}\cdots s_{i_k}$ is any reduced decomposition of w, then*

$$w\lambda = \pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}(\lambda) = \pi_w(\lambda).$$

Proof: Recall that, given an element $w \in \mathfrak{S}_n$, the set $N(w) = \{t \in T \mid \ell(tw) < \ell(w)\}$ is the set of (left) inversions of w; it satisfies $|N(w)| = \ell(w)$ and for all $u, v \in \mathfrak{S}_n$, we have the equality (see for instance [4, Chapter 1, Exercise 12])

$$N(uv) = N(u)\Delta uN(v)u^{-1}, \tag{4}$$

where Δ denotes the symmetric difference (note that, in particular, the product uv does not need to be reduced).

The proof of the first point is by induction on $\ell(w)$. If $\ell(w) = 0$, then $w = 1$ and the set of transpositions t such that $\ell(tw) < \ell(w)$ is empty. We have $\mu = \lambda$ and $\lambda_i \geq \lambda_j$ for all $j > i$ in this case. Hence assume that $\ell(w) > 0$. Let $s = s_k$ be a simple transposition such that $w = s_k u$ and $\ell(w) = \ell(u) + 1$. If $s_k = t$, then $i = k$, $j = k+1$, and $\ell(tw) < \ell(w)$, hence we are done. We can thus assume that $t \neq s$. Using (4) above we have $N(w) = N(su) = N(s)\Delta sN(u)s = \{s\}\Delta sN(u)s$ and using the fact that $s \neq t$, we deduce the equivalence

$$\ell(tw) < \ell(w) \Leftrightarrow \ell(stsu) < \ell(u),$$

denoting $sts = (i'\ j')$, by induction it suffices to show that $\nu_{i'} < \nu_{j'}$, where $\nu = u\lambda$. We can assume that $sts \neq t$, otherwise the supports of s and t are disjoint, hence $i' = i$, $j' = j$, and $\nu_i = \mu_i$, $\nu_j = \mu_j$. We can thus assume that $s \in \{s_{i-1}, s_i, s_{j-1}, s_j\}$. We treat the case where $s = s_j$, the other cases are similar. We have $sts = (i\ j+1)$, and we have $\nu_j = \mu_{j+1}$, $\nu_{j+1} = \mu_j$. On the other hand, since i is not in the support of $(j\ j+1)$, we have $\nu_i = \mu_i$. Hence $\nu_i = \mu_i < \mu_j = \nu_{j+1}$, which by induction yields $\ell(stsu) < \ell(u)$.

Let us prove the second point. We argue by induction on k. If $k = 0$ then there is nothing to prove. Assume that $k \geq 1$. By induction we have that $s_{i_2} \cdots s_{i_k}\lambda = \pi_{i_2} \cdots \pi_{i_k}(\lambda)$. Now writing $\mu = s_{i_2} \cdots s_{i_k}\lambda$ and $i = i_1$, by the first point we have that $\mu_i \geq \mu_{i+1}$, otherwise we would have $\ell(s_{i_1}s_{i_2} \cdots s_{i_k}) = k-1$, contradicting the fact that $s_{i_1}s_{i_2} \cdots s_{i_k}$ is reduced. It follows that $s_{i_1}\mu = \pi_{i_1}(\mu)$, hence that $w\lambda = \pi_{i_1}\pi_{i_2} \cdots \pi_{i_k}(\lambda)$, as required. ■

Lemma 2.3. *Consider the set $\mathfrak{S}_n\lambda$, which is in bijection with $\mathfrak{S}_n^\lambda$ through $w\lambda \mapsto w^\lambda$, where w^λ is the representative of minimal length of $w\mathfrak{S}_\lambda$. Then the transitive closure of the relations $\mu < t\mu$, if $\mu_i > \mu_j$, $i < j$, t is the transposition $(i\,j) \in \mathfrak{S}_n$ and $\mu = (\mu_1, \ldots, \mu_n) \in \mathfrak{S}_n\lambda$ yields a partial order on $\mathfrak{S}_n\lambda$, which coincides through the aforementioned bijection with the restriction of the strong Bruhat order on $\mathfrak{S}_n$ to $\mathfrak{S}_n^\lambda$.*

Proof: Assume that $\mu < t\mu$, and let $w \in \mathfrak{S}_n$ such that $\mu = w\lambda$. Denoting $\mu' = tw\lambda = t\mu$, we have $\mu'_i < \mu'_j$. By point 1 of Lemma 2.2, we have $\ell(w) = \ell(ttw) < \ell(tw)$, which shows that $w < tw$ in the strong Bruhat order. It follows that $w^\lambda < (tw)^\lambda$.

Conversely, let $u, v \in \mathfrak{S}_n^\lambda$ such that $u \leq v$. By definition of the strong Bruhat order, there is a sequence $t_1, t_2, \ldots, t_k$ of transpositions such that $u < t_1u < t_2t_1u < \cdots < t_kt_{k-1} \cdots t_2t_1u$. Note that the elements in this sequence are in $\mathfrak{S}_n$ but, apart from u and v, not necessarily in $\mathfrak{S}_n^\lambda$. To conclude the proof it therefore suffices to show that, if $u < tu$ with $u \in \mathfrak{S}_n$, $t \in T$, then $u\lambda \leq tu\lambda$. Letting $\mu = u\lambda$, if $\mu_i < \mu_j$, then by the first point of Lemma 2.2 we have $\ell(tu) < \ell(u)$, contradicting $u < tu$. Hence $\mu_i \geq \mu_j$. If $\mu_i > \mu_j$ then we have $\mu < t\mu$. If $\mu_i = \mu_j$, we have $u\lambda = tu\lambda$. This concludes the proof. ■

Lemma 2.4. *Let $\lambda \in \mathcal{P}_n$ and let $\sigma \in \mathfrak{S}_n^\lambda$. Let $\mu = \sigma\lambda$ and s_i a simple reflection of $\mathfrak{S}_n$. We have the equivalences*

$$\mu_i > \mu_{i+1} \text{ iff } \ell(s_i\sigma) = \ell(\sigma) + 1 \text{ and } s_i\sigma \in \mathfrak{S}_n^\lambda,$$
$$\mu_i = \mu_{i+1} \text{ iff } s_i\sigma \notin \mathfrak{S}_n^\lambda \text{ (in which case we must have } \ell(s_i\sigma) = \ell(\sigma) + 1),$$
$$\mu_i < \mu_{i+1} \text{ iff } \ell(s_i\sigma) = \ell(\sigma) - 1 \text{ (in which case we must have } s_i\sigma \in \mathfrak{S}_n^\lambda).$$

Proof: Assume that $w = s_i\sigma \notin \mathfrak{S}_n^\lambda$. Then $\ell(w) = \ell(\sigma) + 1$. Since $w \notin \mathfrak{S}_n^\lambda$, there is a simple reflection $s_j \in \mathfrak{S}_\lambda$ such that $\ell(ws_j) < \ell(w)$. Take any reduced decomposition $s_{n_1} \cdots s_{n_\ell}$ of σ. We have that $s_is_{n_1} \cdots s_{n_\ell}$ is a reduced decomposition of w and since $\ell(ws_j) < \ell(w)$, by the exchange lemma there is a reduced decomposition of ws_j obtained from $s_is_{n_1} \cdots s_{n_\ell}$ obtained by just removing a letter. If this letter is not s_i, we get that $\ell(\sigma s_j) = \ell(\sigma) - 1$, in contradiction with $\sigma \in \mathfrak{S}_n^\lambda$, since $s_j \in \mathfrak{S}_\lambda$. We thus have that $ws_j = s_{n_1} \cdots s_{n_\ell} = \sigma = s_iw$. It follows that $\sigma^{-1}s_i\sigma = s_j \in \mathfrak{S}_\lambda$. This yields $\sigma^{-1}s_i\sigma\lambda = \lambda$, hence $s_i\mu = \mu$, hence $\mu_i = \mu_{i+1}$. Conversely, assume that $\mu_i = \mu_{i+1}$. We thus have $s_i\sigma\lambda = \sigma\lambda = \mu$. Since $\sigma \in \mathfrak{S}_n^\lambda$, by uniqueness of the element of the element $w \in \mathfrak{S}_n^\lambda$ such that $\mu = w\lambda$, we cannot have $s_i\sigma \in \mathfrak{S}_n^\lambda$. Hence the two statements in the middle line are equivalent.

Assume that $\mu_i > \mu_{i+1}$. Then, since the middle equivalence is already shown, we know that $s_i\sigma \in \mathfrak{S}_n^\lambda$. By Lemma 2.2 (1), we must have $\ell(s_is_i\sigma) < \ell(s_i\sigma)$, forcing $\ell(s_i\sigma) = \ell(\sigma) + 1$. Also by Lemma 2.2 (1), if $\mu_i < \mu_{i+1}$, then $\ell(s_i\sigma) < \ell(\sigma)$, yielding $\ell(s_i\sigma) = \ell(\sigma) - 1$.

We thus have shown that, in each line, the left condition implies the right one (we have even shown that we have equivalence in the middle line). Since the three conditions on the right are disjoint, we must have equivalence in each line. ∎

Lemma 2.5. *Let $\sigma \in \mathfrak{S}_n$ and $\alpha = \sigma\lambda$. We can obtain the minimal representative $\hat{\sigma} \in \mathfrak{S}_n^\lambda$ of σ from any $\pi_\sigma = \pi_{j_1}\pi_{j_2} \cdots \pi_{j_l} \in \mathfrak{M}_n$ such that $\pi_{j_1}\pi_{j_2} \cdots \pi_{j_l}\lambda = \alpha$ with $s_{j_1}s_{j_2} \cdots s_{j_l}$ a (not necessarily reduced) word of an element of $\mathfrak{S}_n$ as follows: for $r = l, \dots, 1$, delete π_{j_r} in $\pi_{j_1}\pi_{j_2} \cdots \pi_{j_l}$ whenever $\mu_{j_r} \leq \mu_{j_r+1}$ in $(\mu_1, \dots, \mu_n) = \pi_{j_{r+1}} \cdots \pi_{j_l}(\lambda)$. The resulting decomposition obtained in this way is a reduced decomposition $\pi_{\hat{\sigma}}$ in $\mathfrak{M}_n$ and gives $\hat{\sigma} \in \mathfrak{S}_n^\lambda$.*

Proof: The fact that the resulting decomposition $\pi_{i_1}\pi_{i_2} \cdots \pi_{i_k}$ satisfies

$$\pi_{i_1}\pi_{i_2} \cdots \pi_{i_k}(\lambda) = \pi_\sigma(\lambda)$$

is clear since a letter π_{j_r} is removed whenever its action on $\pi_{j_{r+1}} \cdots \pi_{j_l}(\lambda)$ is trivial. We show by decreasing induction on k that $s_{i_j} s_{i_{j+1}} \cdots s_{i_k}(\lambda) = \pi_{i_j}\pi_{i_{j+1}} \cdots \pi_{i_k}(\lambda)$ for all j, and that $s_{i_j} \cdots s_{i_k}$ is reduced and lies in $\mathfrak{S}_n^\lambda$. If $k = 0$ then the result is trivially true. Hence let $j \leq k$ and assume that $\mu := s_{i_{j+1}} \cdots s_{i_k}(\lambda) = \pi_{i_{j+1}} \cdots \pi_{i_k}(\lambda)$, and that $s_{i_{j+1}} \cdots s_{i_k}$ is reduced and lies in $\mathfrak{S}_n^\lambda$. We must have $\mu_{i_j} > \mu_{i_j+1}$, otherwise the letter π_{i_j} would have been removed. Hence by definition of the action of the bubble sort operator π_{i_j}, we have $\pi_{i_j}(\mu) = s_{i_j}(\mu)$, which yields $s_{i_j} s_{i_{j+1}} \cdots s_{i_k}(\lambda) = \pi_{i_j}\pi_{i_{j+1}} \cdots \pi_{i_k}(\lambda)$. Setting $w = s_{i_{j+1}} \cdots s_k$ we obtain using Lemma 2.4 (1) that $\ell(s_{i_j} w) = \ell(w) + 1$ and $s_{i_j} w \in \mathfrak{S}_n^\lambda$, hence $s_{i_j} s_{i_{j+1}} \cdots s_{i_k}$ is still reduced, and defines an element of $\mathfrak{S}_n^\lambda$.

It only remains to show that $\tau := s_{i_1} s_{i_2} \cdots s_{i_k}$ is equal to $\hat{\sigma}$. But we have $\pi_\tau(\lambda) = \pi_\sigma(\lambda) = \pi_{\hat{\sigma}}(\lambda)$, which by Lemma 2.2 (2) yields $\tau(\lambda) = \hat{\sigma}(\lambda)$. Since both $\tau, \hat{\sigma}$ lie in $\mathfrak{S}_n^\lambda$, this forces $\tau = \hat{\sigma}$, which concludes the proof. ■

Example 2.6. *Let $n = 4$. We have*

$$\pi_2\pi_2\pi_1\pi_2(3,2,2,1) = \pi_2\pi_2\pi_1(3,2,2,1) = \pi_2\pi_2(2,3,2,1) = \\ = \pi_2(2,2,3,1) = (2,2,3,1).$$

Applying the algorithm described in Lemma 2.5 to the word $\pi_2\pi_2\pi_1\pi_2$ and the weight $\lambda = (3,2,2,1)$ yields $\widehat{\pi_2}\pi_2\pi_1\widehat{\pi_2} = \pi_2\pi_1 = \pi_{s_2 s_1}$, where the hat over the bubble sort operator denotes omission. We indeed have $\pi_2\pi_2\pi_1\pi_2 = \pi_\sigma$ with $\sigma = s_2 s_1 s_2$ and $\hat{\sigma} = s_2 s_1$ (here $\mathfrak{S}_\lambda = \{1, s_2\}$).

2.3. Crystals.

2.3.1. *Abstract crystals.* To each partition $\lambda \in \mathcal{P}_n$ corresponds a crystal graph $B(\lambda)$ which can be regarded as the combinatorial skeleton of the simple module $V(\lambda)$. In particular, its vertices label a distinguished basis of $V(\lambda)$. Its general structure can be defined using the canonical bases introduced by Lusztig [27] and subsequently studied by Kashiwara under the name of global bases (see [19] and [20]). It also admits various combinatorial realizations (*i.e.*, vertex labelings) in terms semistandard tableaux, Littelmann's paths (see [25]) or semi-skyline (see [29], [1]). We will recall the tableau realization below. The (abstract) crystal $B(\lambda)$ is a graph whose set of vertices is endowed with a weight function $\mathrm{wt} : B(\lambda) \to P$ and with the structure of a colored and oriented graph given by the action of the crystal operators $\tilde{f}_i$ and $\tilde{e}_i$ with $i \in I = \{1, \ldots, n-1\}$. More precisely, we have an oriented arrow $b \xrightarrow{i} b'$ between two vertices b and b'

in $B(\lambda)$ if and only if $b' = \tilde{f}_i(b)$ or equivalently $b = \tilde{e}_i(b')$. We have $\tilde{f}_i(b) = 0$ (resp. $\tilde{e}_i(b) = 0$) when no arrow i starts from b (resp. ends at b). Here the symbol 0 should be understood as a sink vertex not lying in $B(\lambda)$. For any $i \in I$, the crystal $B(\lambda)$ can be decomposed into its i-chains which are obtained just by keeping the i-arrows. For such a chain C, we denote by $s(C)$ and $e(C)$ its source and target vertices, respectively. There is a unique vertex b_λ in $B(\lambda)$ such that $\tilde{e}_i(b_\lambda) = 0$ for any $i \in I$ (that is, b_λ is the source vertex of each i-chain containing b_λ) called the highest weight vertex of $B(\lambda)$ and we have $\mathrm{wt}(b_\lambda) = \lambda$. For any $b \in B(\lambda)$, there is a path $b = \tilde{f}_{i_1} \cdots \tilde{f}_{i_r}(b_\lambda)$ from b_λ to b. Let us denote by $S = \{\alpha_1, \dots, \alpha_{n-1}\}$ the set of simple roots of $\mathfrak{gl}_n$ where $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$ for $1 \leq i < n$. The weight function wt satisfies

$$\mathrm{wt}(b) = \lambda - \sum_{k=1}^{r} \alpha_{i_k}.$$

For any $i \in I$, the crystal $B(\lambda)$ decomposes into i-chains. Thus, for any vertex $b \in B(\lambda)$, we can define $\varphi_i(b) = \max\{k \mid \tilde{f}_i^k(b) \neq 0\}$ and $\varepsilon_i(b) = \max\{k \mid \tilde{e}_i^k(b) \neq 0\}$. We then have

$$s_\lambda = \sum_{b \in B(\lambda)} x^{\mathrm{wt}(b)}.$$

The Weyl group W also acts on the vertices of $B(\lambda)$: the action of the simple reflection s_i on $B(\lambda)$ sends each vertex b on the unique vertex b' in the i-chain of b such that $\varphi_i(b') = \varepsilon_i(b)$ and $\varepsilon_i(b') = \varphi_i(b)$. This simply means that b and b' correspond by the reflection with respect to the center of the i-chain containing b. We shall write

$$O(\lambda) = \{\sigma \cdot b_\lambda = b_{\sigma\lambda} \mid \sigma \in \mathfrak{S}_n^\lambda\}$$

for the orbit of the highest weight vertex b_λ of $B(\lambda)$. Observe that $b_{\sigma\lambda}$ is then the unique vertex in $B(\lambda)$ of weight $\sigma\lambda$. The elements of $O(\lambda)$, called the keys of $B(\lambda)$, are those vertices of $B(\lambda)$ which are completely characterized by their weight. Thereby, one has a direct correspondence between the keys and the elements of $\mathfrak{S}_n^\lambda$. For convenience, we often abuse notation and identify the key $b_{\sigma\lambda}$ with $\sigma \in \mathfrak{S}_n^\lambda$.

In fact, one can associate a crystal to any finite-dimensional $\mathfrak{gl}_n$-module by considering its decomposition into irreducible components. This $\mathfrak{gl}_n$-crystal is a disjoint union of connected components, each being isomorphic to a highest weight crystal $B(\lambda)$, $\lambda \in \mathcal{P}_n$. Given two partitions λ and μ in $\mathcal{P}_n$, the crystal associated to the representation $V(\lambda)\otimes V(\mu)$ is the crystal $B(\lambda)\otimes B(\mu)$ whose set of vertices is the direct product of the sets of vertices of $B(\lambda)$ and $B(\mu)$ and whose crystal structure is given by $\mathrm{wt}(a\otimes b)=\mathrm{wt}(a)+\mathrm{wt}(b)$ and by the following rules

$$\tilde{e}_i(u\otimes v)=\begin{cases} u\otimes \tilde{e}_i(v) \text{ if } \varepsilon_i(u)>\varphi_i(v)\\ \tilde{e}_i(u)\otimes v \text{ if } \varepsilon_i(u)\leq\varphi_i(v)\end{cases}$$

and

$$\tilde{f}_i(u\otimes v)=\begin{cases} \tilde{f}_i(u)\otimes v \text{ if } \varphi_i(v)>\varepsilon_i(u)\\ u\otimes \tilde{f}_i(v) \text{ if } \varphi_i(v)\leq\varepsilon_i(u).\end{cases} \tag{5}$$

We adopt the convention that $u\otimes 0=0\otimes v=0$. A key result in crystal theory shows that for any partition $\nu\in\mathcal{P}_n$, the tensor multiplicity $c_{\lambda,\mu}^{\nu}$ of $V(\nu)$ in $V(\lambda)\otimes V(\mu)$ (which is a Littlewood-Richardson coefficient) is equal to the number of connected components in $B(\lambda)\otimes B(\mu)$ with highest weight vertex of weight ν.

2.3.2. *Keys and dilatation of crystals.* Consider k a positive integer and λ a partition. There exists a unique embedding of crystals $\psi_k : B(\lambda)\hookrightarrow B(k\lambda)$ such that for any vertex $b\in B(\lambda)$ and any path $b=\tilde{f}_{i_1}\cdots\tilde{f}_{i_l}(b_\lambda)$ in $B(\lambda)$, we have

$$\psi_k(b)=\tilde{f}_{i_1}^k\cdots\tilde{f}_{i_l}^k(b_{k\lambda}).$$

Since the vertex $b_\lambda^{\otimes k}$ is of highest weight $k\lambda$ in $B(\lambda)^{\otimes k}$, one gets a particular realization $B(b_\lambda^{\otimes k})$ of $B(k\lambda)$ in $B(\lambda)^{\otimes k}$ with highest weight vertex $b_\lambda^{\otimes k}$. This thus gives a canonical embedding

$$\theta_k:\begin{cases} B(b_\lambda)\hookrightarrow B(b_\lambda^{\otimes k})\subset B(b_\lambda)^{\otimes k}\\ b\longmapsto b_1\otimes\cdots\otimes b_k\end{cases} \tag{6}$$

with important properties given in the following theorem and illustrated in Example 2.12.

Theorem 2.7. *(see [20])*

(1) *Let $\sigma\in\mathfrak{S}_n^\lambda$. We have $\theta_k(b_{\sigma\lambda})=b_{\sigma\lambda}^{\otimes k}$.*

(2) *Let $b\in B(\lambda)$. When k has sufficiently many factors, there exist elements $\sigma_1,\dots,\sigma_k$ in $\mathfrak{S}_n^\lambda$ such that $\theta_k(b)=b_{\sigma_1\lambda}\otimes\cdots\otimes b_{\sigma_k\lambda}$. Moreover, in this case*

 (a) *the elements $b_{\sigma_1\lambda}$ and $b_{\sigma_k\lambda}$ in $\theta_k(b)$ do not then depend on k,*

(b) *up to repetition, the sequence* $(\sigma_1\lambda, \ldots, \sigma_k\lambda)$ *in* $\theta_k(b)$ *does not depend on the realization of the crystal* $B(\lambda)$ *and we have* $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k$.

From Assertion 2 of the above theorem, we can define the keys of an element in $B(\lambda)$[‡].

Definition 2.8. *Let* $b \in B(\lambda)$*, then the keys* $K_+(b)$ *and* $K^-(b)$ *of* b *are defined as follows:*

$$K_+(b) = b_{\sigma_1\lambda} \text{ and } K^-(b) = b_{\sigma_k\lambda}.$$

In particular, $K_+(b_{\sigma\lambda}) = K^-(b_{\sigma\lambda}) = b_{\sigma\lambda}$ *for any* $\sigma \in \mathfrak{S}_n^\lambda$*. The orbit* $O(\lambda)$ *is simultaneously the set of left and right keys of* $B(\lambda)$*.*

2.3.3. *Tableau realization.* Recall that each partition λ in $\mathcal{P}_n$ can be identified with its Young diagram. A semistandard tableau T of shape λ is then a filling of λ by letters in the ordered alphabet $\mathcal{A}_n = \{1 < \cdots < n\}$ whose rows weakly increase from left to right and columns strictly increase from bottom to top. The row reading of T is the word $w(T)$ of $\mathcal{A}_n^*$ obtained by reading each row from right to left starting with the bottom row and ending with the top row. The weight of T is the vector $\mathrm{wt}(T) \in \mathbb{Z}_{\geq 0}^n$ whose i-th entry records the number of i's in the filling of T, for $i = 1, \ldots, n$.

Example 2.9. *For* $n = 4$ *the tableau*

$$T = \begin{array}{|c|c|c} \hline 3 & 4 & \multicolumn{1}{c}{} \\ \hline 2 & 2 & \multicolumn{1}{c|}{4} \\ \hline 1 & 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}$$

is a semistandard tableau of shape $\lambda = (3, 3, 2, 0)$ *with row reading* $w(T) = 21142243$ *and weight* $\mathrm{wt}(T) = (2, 3, 1, 2)$*.*

One can realize $B(\lambda)$ using the semistandard tableaux of shape λ just by describing the action of the crystals operators $\tilde{f}_i$ and $\tilde{e}_i, i = 1, \ldots, n-1$ on each such tableau. Assume that i is fixed in $\{1, \ldots, n-1\}$ and T is a semistandard tableau of shape λ. Let $w_i(T)$ be the subword of $w(T)$ obtained by keeping

[‡]We dot use the terminology "left" and "right" keys as in the original definition [22] based on the tableaux model since it does not fit with the positions of $b_{\sigma_1\lambda}$ and $b_{\sigma_k\lambda}$ in $\theta_k(b)$ with the convention of this paper.

only the letters i and $i+1$ in $w(T)$. Now delete recursively all the *factors* $i(i+1)$ in $w_i(T)$. This eventually yields a subword $\tilde{w}_i(T)$ of $w(T)$ of the form

$$\tilde{w}_i(T) = (i+1)^a (i)^b.$$

When $b > 0$ (resp. $a > 0$), $\tilde{f}_i(T)$ (resp. $\tilde{e}_i(T)$) is obtained by replacing in T the letter of $w(T)$ corresponding to the leftmost letter i (resp. to the rightmost $i+1$) surviving in $\tilde{w}_i(T)$ by $i+1$ (resp. by i). When $b = 0$ (resp. $a = 0$), we set $\tilde{f}_i(T) = 0$ (resp. $\tilde{e}_i(T) = 0$) where 0 is understood as a sink vertex as before. This just means that in this case, there is no arrow i starting at T (resp. no arrow i ending at T). Observe that with the notation of the previous paragraph one gets

$$\varepsilon_i(T) = a \text{ and } \varphi_i(T) = b.$$

Also, it is easy to compute the action of $s_i = (i, i+1) \in \mathfrak{S}_n$ on T : the tableau $s_i.T$ is obtained by replacing in T the $a-b$ rightmost letters $i+1$ (resp. the $b-a$ leftmost letters i) of $\tilde{w}_i(T)$ by i (resp. by $i+1$) when $a \geq b$ (resp. $a < b$).

Example 2.10. *By resuming Example 2.9, one gets*

$$\tilde{f}_1(T) = 0 \textit{ and } \tilde{e}_1(T) = \begin{array}{|c|c|c}\hline 3 & 4 & \multicolumn{1}{c}{} \\ \hline 2 & 2 & \multicolumn{1}{c|}{4} \\ \hline 1 & 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array}$$

$$\tilde{f}_2(T) = \begin{array}{|c|c|c}\hline 3 & 4 & \multicolumn{1}{c}{} \\ \hline 2 & 2 & \multicolumn{1}{c|}{4} \\ \hline 1 & 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \textit{ and } \tilde{e}_2(T) = 0 \textit{ with } s_2.T = \begin{array}{|c|c|c}\hline 3 & 4 & \multicolumn{1}{c}{} \\ \hline 2 & 3 & \multicolumn{1}{c|}{4} \\ \hline 1 & 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

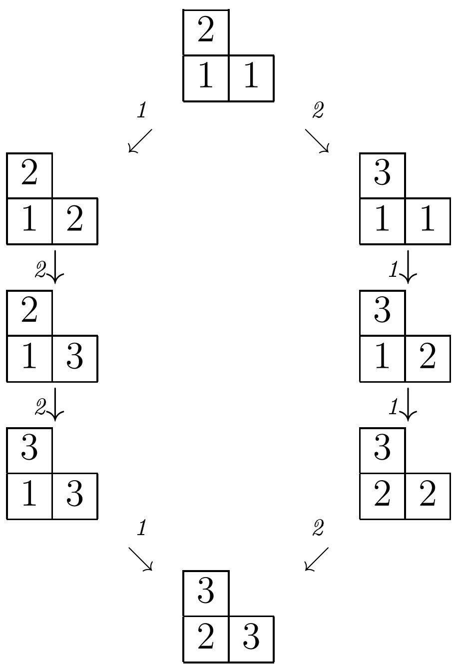
$$\tilde{f}_3(T) = \begin{array}{|c|c|c}\hline 4 & 4 & \multicolumn{1}{c}{} \\ \hline 2 & 2 & \multicolumn{1}{c|}{4} \\ \hline 1 & 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \textit{ and } \tilde{e}_3(T) = \begin{array}{|c|c|c}\hline 3 & 3 & \multicolumn{1}{c}{} \\ \hline 2 & 2 & \multicolumn{1}{c|}{4} \\ \hline 1 & 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}$$

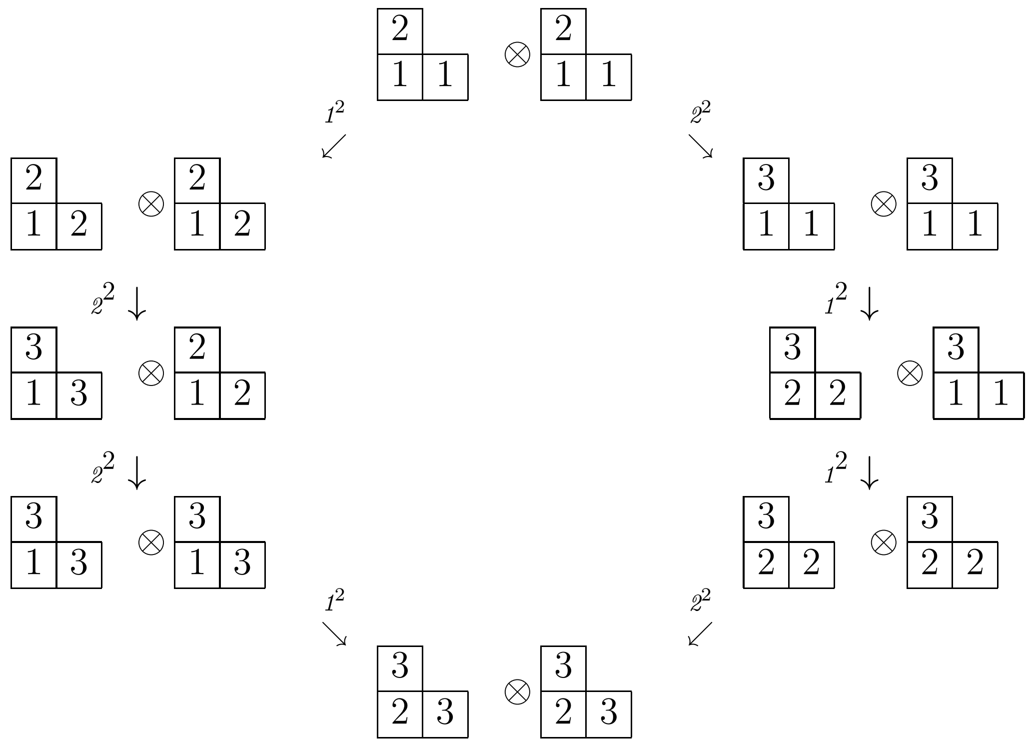
With the above definition of the crystal operators, it is easy to check that the set of semistandard tableaux of shape λ admits the structure of an oriented and connected graph isomorphic to the abstract crystal $B(\lambda)$ (see [19]). In particular its unique highest weight vertex is the Yamanouchi tableau T_λ whose i-th row only contains letters i for any $i = 1, \ldots, n$. In fact the orbit $O(\lambda)$ is also easy to describe in this model: it exactly contains the so-called key tableaux of shape λ which are the semistandard tableaux in which each column is contained in the column located immediately at its left. Their weights correspond to the orbit of $\lambda \in \mathbb{Z}^n$ under the action of $\mathfrak{S}_n$.

Example 2.11. *For $n = 3$, the six key tableaux (or simply keys) of shape $\lambda = (2, 1, 0)$ are*

$$\begin{array}{|c|c}\hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array}\ , \quad \begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array}\ , \quad \begin{array}{|c|c}\hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}\ ,$$
$$\begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline\end{array}\ , \quad \begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array}\ , \quad \begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline\end{array}$$

Example 2.12. *For $n = 3$, $\lambda = (2, 1, 0)$ and $k = 2$, the crystal $B(\lambda)$ and its dilatation $B(\lambda)^{\otimes 2}$ are as follows:*





$$K_+\left(\begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline \end{array}\right) = \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline \end{array},\ K^-\left(\begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline \end{array}\right) = \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline \end{array},\ K_+\left(\begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline \end{array}\right) = \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 2 & 2 \\ \hline \end{array},$$
$$K^-\left(\begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline \end{array}\right) = \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline \end{array}.$$

Remark 2.13.

(1) *In the previous example, the dilatation of the crystal with $k = 2$ suffices to obtain the left and right keys. In general, we need to compute the dilatation with k given by the maximum of the lengths of the i-chains with $i \in \{1, \dots, n-1\}$ in $B(\lambda)$.*

(2) *The left and right keys associated to a semistandard tableau can be computed in a more efficient way than the one obtained from Definition 2.8 by using the Jeu de Taquin procedure* [22, 13]*. This was in fact the initial definition from* [22]*. One can also use the semi-skyline model* [16, 29] *to realize the crystal $B(\lambda)$ in a way which makes the keys very easy to read off (but the crystal structure becomes then more complicated to describe* [29, 1, 2]*). The advantage of Definition 2.8 is that it is independent of the realization of the crystal $B(\lambda)$ and strongly connected to general properties of $\mathfrak{S}_n$ viewed as a Coxeter group.*

(3) *In the notation of § 2.1, we have* $O(\lambda) = \{u.T_\lambda \mid u \in \mathfrak{S}_n^\lambda\}$. *This gives a direct correspondence between the keys and the elements of* $\mathfrak{S}_n^\lambda$. *If we denote by* K_u *the key* $u.T_\lambda$ *associated to* $u \in \mathfrak{S}_n^\lambda$, *it then becomes easy to read the Bruhat order. Indeed, we have* $u \leq v$ *if and only if for each box of the Young diagram* λ, *the letter obtained in* K_u *is less than or equal to the one obtained in* K_v.

(4) *The character* s_λ *associated to the partition* λ *is the Schur function and the tableau realization of crystals allows one to recover its expression*

$$s_\lambda = \sum_{T \in B(\lambda)} x^{\mathrm{wt}(T)}. \tag{7}$$

2.3.4. *Crystals of Demazure modules.* Let λ be a partition and $\sigma \in \mathfrak{S}_n$. Up to scalar multiplication, there exists a unique vector $v_{\sigma\lambda}$ in $V(\lambda)$ of weight $\sigma(\lambda)$. The Demazure module associated to $v_{\sigma\lambda}$ is the $U(\mathfrak{gl}_n^+)$-module defined by

$$V_\sigma(\lambda) := U(\mathfrak{gl}_n^+) \cdot v_{\sigma\lambda}.$$

Demazure [10] introduced the character $\kappa_{\sigma,\lambda}$ of $V_\sigma(\lambda)$ and showed that it can be computed by applying to x^λ a sequence of divided difference operators given by any reduced decomposition of σ. More precisely, for any $i \in \{1, \dots, n-1\}$, define the linear operator D_i on $\mathbb{Z}[x_1, \dots, x_n]$ by

$$D_i(P) = \frac{x_i P - x_{i+1}(s_i \cdot P)}{x_i - x_{i+1}}.$$

Demazure proved that such operators satisfy the relations

$$\begin{aligned} D_i^2 &= D_i \text{ for any } i = 1, \dots, n-1, \\ D_i D_{i+1} D_i &= D_{i+1} D_i D_{i+1} \text{ for any } i = 1, \dots, n-2, \\ D_i D_j &= D_j D_i \text{ for any } i, j = 1, \dots, n-1 \text{ such that } |i-j| > 1. \end{aligned}$$

Thus, given any reduced decomposition $\sigma = s_{i_1} \cdots s_{i_\ell}$ of σ, by Mastumoto's Lemma the operator $D_\sigma = D_{i_1} \cdots D_{i_\ell}$ only depends on σ and not on the chosen reduced decomposition. He also showed that

$$\kappa_{\sigma,\lambda} = D_\sigma(x^\lambda) \in \mathbb{Z}[x_1, \dots, x_n]$$

is the (Demazure) character of $V_\sigma(\lambda)$. In particular, we have $\kappa_{id,\lambda} = x^\lambda$ and $\kappa_{\sigma_{0,\lambda}} = s_\lambda$ and

$$D_i(\kappa_{\sigma,\lambda}) = \begin{cases} \kappa_{s_i\sigma,\lambda} \text{ if } \ell(s_i\sigma) = \ell(\sigma) + 1, \\ \kappa_{\sigma,\lambda} \text{ otherwise.} \end{cases} \tag{8}$$

Later Kashiwara [19] and Littelmann [25] defined a relevant notion of crystals for the Demazure modules. To this end, for any $\sigma \in \mathfrak{S}_n$, consider the Demazure atom

$$\overline{\mathrm{B}}_\sigma(\lambda) = \{b \in B(\lambda) \mid K_+(b) = b_{\sigma\lambda}\}.$$

In particular, $\overline{\mathrm{B}}_{id}(\lambda) = \{b_\lambda\}$.

By definition we have $\overline{\mathrm{B}}_\sigma(\lambda) = \overline{\mathrm{B}}_{\sigma'}(\lambda)$ whenever σ and σ' belong to the same left coset of $\mathfrak{S}_n/\mathfrak{S}_\lambda$. Writing $\sigma = uv$ with $u \in \mathfrak{S}_n^\lambda$ and $v \in \mathfrak{S}_\lambda$, we get $\overline{\mathrm{B}}_\sigma(\lambda) = \overline{\mathrm{B}}_u(\lambda)$ from the characterization of the strong Bruhat order. Thus we can assume that σ belongs to $\mathfrak{S}_n^\lambda$. We then get $B(\lambda) = \bigsqcup_{\sigma \in \mathfrak{S}_n^\lambda} \overline{\mathrm{B}}_\sigma(\lambda)$. There also exists a notion of opposite Demazure module: for any $\sigma \in \mathfrak{S}_n$, it is defined by $V^\sigma(\lambda) := U_q(\mathfrak{gl}_n^-) \cdot v_{\sigma\lambda}$, for which it is relevant to define the opposite Demazure atom

$$\overline{\mathrm{B}}^\sigma(\lambda) = \{b \in B(\lambda) \mid K^-(b) = b_{\sigma\lambda}\}.$$

In particular we have $\overline{\mathrm{B}}^{\sigma_0}(\lambda) = \{b_{\sigma_0\lambda}\}$.

Given σ and σ' in $\mathfrak{S}_n^\lambda$, we shall write $b_{\sigma\lambda} \leq b_{\sigma'\lambda}$ when $\sigma \leq \sigma'$ (recall that $\leq$ denotes the strong Bruhat order on $\mathfrak{S}_n$).

Definition 2.14. *The Demazure crystal* $\mathrm{B}_\sigma(\lambda)$ *and opposite Demazure crystal* $\mathrm{B}^\sigma(\lambda)$ *are defined by*

$$\mathrm{B}_\sigma(\lambda) = \bigsqcup_{\sigma' \in \mathfrak{S}_n^\lambda, \sigma' \leq \sigma} \overline{\mathrm{B}}_{\sigma'}(\lambda) = \{b \in B(\lambda) \mid K_+(b) \leq b_{\sigma\lambda}\}, \tag{9}$$

$$\mathrm{B}^\sigma(\lambda) = \bigsqcup_{\sigma' \in \mathfrak{S}_n^\lambda, \sigma \leq \sigma'} \overline{\mathrm{B}}^{\sigma'}(\lambda) = \{b \in B(\lambda) \mid K^-(b) \geq b_{\sigma\lambda}\},$$

In particular we have $\mathrm{B}_{id}(\lambda) = \{b_\lambda\}$, $\mathrm{B}^{\sigma_0}(\lambda) = \{b_{\sigma_0\lambda}\}$ *and* $\mathrm{B}_{\sigma_0}(\lambda) = \mathrm{B}(\lambda) = \mathrm{B}^{id}(\lambda)$.

To compute the Demazure crystal $\mathrm{B}_\sigma(\lambda)$, it therefore suffices to

- compute the key map K_+ on $B(\lambda)$.
- compute the strong Bruhat order on $\mathfrak{S}_n^\lambda$, or alternatively on the vertices of $O(\lambda)$.

Example 2.15. *Let us resume Example 2.12 with the tableaux model. For* $n = 3$ *and* $\lambda = (2, 1, 0)$, *consider* $\sigma = s_1 s_2$. *We get*

$$K_+(T_{s_1 s_2}) = \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline \end{array}$$

and $\mathrm{B}_\sigma(\lambda)$ *contains exactly the tableaux* T *such that* $K_+(T) \leq K_+(T_{s_1 s_2})$ *(recall that this means that each entry in* T *is less than or equal to its corresponding entry in* $T_{s_1 s_2}$*). These are all the tableaux in* $B(\lambda)$ *except*

$$T_1 = \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline \end{array}, \; T_2 = \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline \end{array} \; \textit{and } T_3 = \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 2 & 3 \\ \hline \end{array}$$

for which we have

$$K_+(T_1) = T_2, K_+(T_2) = T_2 \textit{ and } K_+(T_3) = T_3.$$

The following theorem gathers results established by Kashiwara and Littelmann (see Assertion 2 of Proposition 9.1.3 and Theorem 9.2.4 in [20]). For convenience, we extend $\tilde{f}_i$ and $\tilde{e}_i$, $i \in \{1, \ldots, n-1\}$, to $B(\lambda) \sqcup \{0\}$ by setting them to map 0 to 0.

Theorem 2.16. *Let* $\lambda \in \mathcal{P}_n$*.*

(1) *We have* $\kappa_{\sigma,\lambda} = \sum_{b \in \mathrm{B}_\sigma(\lambda)} x^{\mathrm{wt}(b)}$.
(2) *For any reduced decomposition* $s_{i_1} \cdots s_{i_\ell}$ *of* σ*, we have*
$$\mathrm{B}_\sigma(\lambda) = \{\tilde{f}_{i_1}^{k_1} \cdots \tilde{f}_{i_\ell}^{k_\ell}(b_\lambda) \mid (k_1, \ldots, k_\ell) \in \mathbb{Z}_{\geq 0}^\ell\} \setminus \{0\}.$$
(3) *For any i-chain* C *in* $B(\lambda)$ *and any* $\sigma \in \mathfrak{S}_n$*, only the three following situations can appear*
$$C \cap \mathrm{B}_\sigma(\lambda) = \emptyset, \quad C \cap \mathrm{B}_\sigma(\lambda) = C \textit{ or } C \cap \mathrm{B}_\sigma(\lambda) = s(C),$$
where we recall that $S(C)$ *denotes the source vertex of the chain* C*.*

Remark 2.17. *By the previous theorem, for any* $\sigma \in \mathfrak{S}_n$ *and* $i \in \{1, \ldots, n-1\}$ *such that* $\ell(s_i\sigma) = \ell(\sigma) + 1$ *and* $s_i\sigma\lambda \neq \sigma\lambda$*, we have* $\mathrm{B}_\sigma(\lambda) \subset \mathrm{B}_{s_i\sigma}(\lambda)$*. Moreover, for any i-string* $C \subseteq \mathrm{B}(\lambda)$*, either* $\mathrm{B}_{s_i\sigma}(\lambda) \cap C = \mathrm{B}_\sigma(\lambda) \cap C = \emptyset$*,* $\mathrm{B}_{s_i\sigma}(\lambda) \cap C = \mathrm{B}_\sigma(\lambda) \cap C = C$*, or* $s(C) = \mathrm{B}_\sigma(\lambda) \cap C$ *in which case* $C \subseteq \mathrm{B}_{s_i\sigma}(\lambda)$*.*

2.3.5. *Additional remarks.*

(1) The computation of the key map on $B(\lambda)$ from the definition by dilatation of crystals becomes quickly untractable when λ is far enough in the interior of the Weyl chamber. But as explained in § 2.3.3 it becomes much easier if we use the tableaux realization of crystals.
(2) One can also define the Demazure atom polynomials $\overline{\kappa}_{\sigma,\lambda} = \sum_{b \in \overline{\mathrm{B}}_\sigma(\lambda)} x^{\mathrm{wt}(b)}$. In fact, they can also be obtained without using the crystal theory directly from the linear operators $D_i' = D_i - id, i = 1, \ldots, n-1$. These operators still satisfy the braid relations, but here $(D_i')^2 = -D_i'$ (see [23]).

Then for any reduced decomposition $\sigma = \sigma_{i_1} \cdots \sigma_{i_\ell}$, we have

$$\overline{\kappa}_{\sigma,\lambda} := D'_{i_1} \cdots D'_{i_\ell}(x^\lambda) = \sum_{b \in \overline{\mathrm{B}}_\sigma(\lambda)} x^{\mathrm{wt}(b)}.$$

(3) Rather than labeling the Demazure crystals and the Demazure characters of $B(\lambda)$ by elements of $\mathfrak{S}_n^\lambda$, it is often convenient to label them directly by the elements of the orbit $\mathfrak{S}_n\lambda$. Given $\mu \in \mathfrak{S}_n\lambda$ such that $\mu = \sigma\lambda$ with $\sigma \in \mathfrak{S}_n^\lambda$, we will write B_μ, B^μ instead of $\mathrm{B}_\sigma(\lambda)$, $\mathrm{B}^\sigma(\lambda)$ and $\kappa_\mu, \overline{\kappa}_\mu$ instead of $\kappa_{\sigma,\lambda}$ and $\overline{\kappa}_{\sigma,\lambda}$[§]. Note that $\kappa_{\sigma_0\lambda} = \mathrm{s}_\lambda$.

(4) Demazure characters $\{\kappa_\mu : \mu \in \mathbb{N}^n\}$ and Demazure atoms $\{\bar{\kappa}_\mu : \mu \in \mathbb{N}^n\}$ both form linear $\mathbb{Z}$-bases for $\mathbb{Z}[x_1, \ldots, x_n]$. The operators D_i act on Demazure characters κ_μ via elementary bubble sort operators π_i on the entries of the weak composition $\mu = (\mu_1, \ldots, \mu_n)$ as follows

$$D_i(\kappa_\mu) = \begin{cases} \kappa_{s_i\mu} & \text{if } \mu_i > \mu_{i+1} \\ \kappa_\mu & \text{if } \mu_i \le \mu_{i+1} \end{cases} \Leftrightarrow D_i(\kappa_\mu) = \kappa_{\pi_i(\mu)}. \tag{10}$$

(5) We will adopt the usual convention of [23], identifying each $\mu \in \mathbb{Z}^n$ such that $\mu_{m+1} = \cdots = \mu_n = 0$ with $(\mu_1, \ldots, \mu_m) \in \mathbb{Z}^m$. This notation is compatible with the definition of the Demazure characters since for any $\mu \in \mathbb{Z}^m$, we have $\mathrm{s}_\mu(x_1, \ldots, x_m) = \mathrm{s}_\mu(x_1, \ldots, x_n)$. It is also compatible with the tableaux realization of the crystals because for any such $\mu \in \mathbb{Z}^m$, the Demazure crystal $\mathrm{B}_\mu(\lambda)$ only contains tableaux with letters in $\{1, \ldots, m\}$.

(6) The Demazure and opposite Demazure crystals and atoms can be connected using the Lusztig-Schützenberger involution on the crystal $B(\lambda)$ defined as follows. Let σ_0 be the longest element of $\mathfrak{S}_n$ (defined by $\sigma_0(i) = n + 1 - i$ for any $i = 1, \ldots, n$). For any $b = \tilde{f}_{i_1} \cdots \tilde{f}_{i_r}(b_\lambda)$, set $\iota(b) = \tilde{e}_{n-i_1} \cdots \tilde{e}_{n-i_r}(b_{\sigma_0\lambda})$ where $\mathrm{wt}(\iota b) = \sigma_0 \mathrm{wt}(b)$. One can prove that the map ι is an involution on $B(\lambda)$ reversing the arrows and flipping the labels i and $n - i$, and reversing the weight. We then have

[§]This notation should not be confused with the subset consisting of those vertices in $B(\lambda)$ with weight μ sometimes also denoted $B(\lambda)_\mu$ in the literature.

$K^-(b) = \sigma_0.K_+(\iota(b))$. This implies that, for any reduced decomposition $\sigma = s_{i_1} \cdots s_{i_\ell} \in \mathfrak{S}_n^\lambda$, we get

$$\mathrm{B}^\sigma(\lambda) = \iota(\mathrm{B}_{\sigma_0\sigma}(\lambda)) = \{\tilde{e}_{n-i_1}^{k_1} \cdots \tilde{e}_{n-i_\ell}^{k_\ell}(b_{\sigma_0\lambda}) \mid (k_1, \ldots, k_\ell) \in \mathbb{Z}_{\geq 0}^\ell\} \setminus \{0\} \text{ and} \tag{11}$$

$$\overline{\mathrm{B}}^\sigma(\lambda) = \iota(\overline{\mathrm{B}}_{\sigma_0\sigma}(\lambda)). \tag{12}$$

(7) There is also a notion of opposite Demazure character κ_λ^σ for the opposite Demazure module $V^\sigma(\lambda)$. It satisfies $\kappa_\lambda^\sigma = \sum_{b \in \mathrm{B}^\sigma(\lambda)} x^{\mathrm{wt}(b)}$ and using the involution ι and (11), we have in fact

$$\kappa_\lambda^\sigma(x_1, \ldots, x_n) = \kappa^\mu(x_1, \ldots, x_n) = \kappa_{\sigma_0\mu}(x_n, \ldots, x_1)$$

where $\mu = \sigma\lambda$. Since $\overline{\mathrm{B}}^\sigma(\lambda) = \iota(\overline{\mathrm{B}}_{\sigma_0\sigma}(\lambda))$ we similarly have

$$\overline{\kappa}_\lambda^\sigma(x_1, \ldots, x_n) = \overline{\kappa}^\mu(x_1, \ldots, x_n) = \overline{\kappa}_{\sigma_0\mu}(x_n, \ldots, x_1) = \sum_{b \in \overline{\mathrm{B}}^\sigma(\lambda)} x^{\mathrm{wt}(b)}.$$

2.4. Bicrystals and RSK correspondence. Let m and n be two positive integers. Denote by $\mathcal{M}_{m,n}$ the set of matrices with m rows and n columns with entries in $\mathbb{Z}_{\geq 0}$. The set $\mathcal{M}_{m,n}$ is endowed with the structure of a $(\mathfrak{gl}_m, \mathfrak{gl}_n)$-bicrystal. This means that we can define on $\mathcal{M}_{m,n}$ two commuting¶ families of crystal operators $\tilde{e}_i, \tilde{f}_i, i = 1, \ldots, m-1$ and $\hat{e}_j, \hat{f}_j, j = 1, \ldots, n-1$ so that $\mathcal{M}_{m,n}$ is a crystal for both $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$. In fact $\mathcal{M}_{m,n}$ is the crystal of the $(\mathfrak{gl}_m, \mathfrak{gl}_n)$-module of the symmetric space $S(\mathbb{C}^m \times \mathbb{C}^n)$ (see [9, 24, 23]).

One can define the crystal operators directly on $\mathcal{M}_{m,n}$ or from the RSK correspondence. This is a bijection

$$\psi : \begin{cases} \mathcal{M}_{m,n} \to \bigsqcup_{\lambda \in \mathcal{P}_{\min(m,n)}} B_m(\lambda) \times B_n(\lambda) \\ \qquad A \longmapsto (P(A), Q(A)) \end{cases}$$

where we use the tableaux realization‖ of crystals so that $P(A)$ and $Q(A)$ are semistandard tableaux with the same shape on the alphabets $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$, respectively. We refer to [13] for a complete description of

¶ *i.e.*, two operators chosen in each family commute with each other.

‖ Here we have written $B_m(\lambda)$ and $B_n(\lambda)$ to make apparent the fact that we have a $\mathfrak{gl}_m \times \mathfrak{gl}_n$-crystal.

the combinatorial procedure (illustrated in the example below) based on the Schensted column insertion procedure**.

Example 2.18. *Assume $m = 4$ and $n = 3$ and consider the matrix*

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

It can first be encoded as a tensor product of $n = 3$ row tableaux on the alphabet $\{1, 2, 3, 4\}$ where $m_{i,j}$ gives the number of letters i in the j-th component of the tensor product:

$$L_A = \boxed{1\,|\,1\,|\,2\,|\,3\,|\,3} \otimes \boxed{1\,|\,1\,|\,3\,|\,4} \otimes \boxed{2\,|\,3\,|\,4}\ .$$

One then applies the column insertion procedure from left to right. This means that we begin by reading the second column (this gives 4311 *with the convention of § 2.3.3) and then compute the column insertions*

$$1 \to 1 \to 3 \to 4 \to \boxed{1\,|\,1\,|\,2\,|\,3\,|\,3}\ .$$

We thus get the tableau

$$\begin{array}{|c|c|cccccc} \hline 3 & 4 \\ \hline 1 & 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

in which we successively insert the letters corresponding to the reading 432 *of the third row. This gives the tableau*

$$P(A) = \begin{array}{|c|c|c|c|c|c|c|} \cline{1-1} 4 \\ \cline{1-4} 2 & 3 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 2 & 3 & 3 \\ \hline \end{array}\ .$$

The so-called "recording tableau" $Q(A)$ is obtained by filling with letters i the new boxes appearing during the insertion of row i (the first row being considered as inserted in the empty tableau at the beginning of the procedure). We thus get

$$Q(A) = \begin{array}{|c|c|c|c|c|c|c|} \cline{1-1} 3 \\ \cline{1-4} 2 & 2 & 3 & 3 \\ \hline 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ \hline \end{array}\ .$$

**The convention that we use agrees with that of [21] to which we refer for another description of the RSK procedure and the connection with biwords.

Finally

$$\psi(A) = (P(A), Q(A)).$$

Observe that we also have

$$\psi({}^tA) = (Q(A), P(A))$$

where ${}^tA$ is the transpose of the matrix A.

In any matrix A in $\mathcal{M}_{m,n}$, one can consider all the paths π starting at position $(i,j) = (1,n)$ (northeast corner of A) and ending at position $(i,j) = (m,1)$ (southwest corner of A) where the authorized steps have the form $(i,j) \to (i+1,j)$ or $(i,j) \to (i,j-1)$. To any path π, we associate its time $t(\pi)$, given by the sum of the entries along the path. Here, one can imagine that the path stops for a duration $a_{i,j}$ at position (i,j). We set

$$p(A) = \max_{\pi \text{ path in } A} t(\pi).$$

The following theorem gathers a few results about the RSK correspondence that we shall use later.

Theorem 2.19.

(1) *The map ψ is bijective.*
(2) *For any matrix A in $\mathcal{M}_{m,n}$ we have $P({}^tA) = Q(A)$ and $Q({}^tA) = P(A)$.*
(3) *$\mathcal{M}_{m,n}$ has the structure of a bicrystal: given $A \in \mathcal{M}_{m,n}$, the action of the operators $\widetilde{o} = \tilde{e}_i, \tilde{f}_i, i = 1, \dots, m-1$ and $\widehat{o} = \hat{e}_j, \hat{f}_j, j = 1, \dots, n-1$ satisfies*

$$\widetilde{o}(A) = \psi^{-1}(\widetilde{o}P(A), Q(A)) \text{ and } \widehat{o}(A) = \psi^{-1}(P(A), \widehat{o}Q(A)).$$

(4) *For any matrix A, the integer $p(A)$ is equal to the length of the longest row of the tableau $P(A)$ (or $Q(A)$). It also equals the length of a longest decreasing sequence of the word read off from L_A.*

Example 2.20. *Resuming Example 2.18, one checks that $p(A) = 7$. A longest decreasing word of the word $w(L_A) = 332114311432$ read off from L_A is given by 3321111, which has length 7. This subword corresponds in the matrix A to the path*

$$a_{1,3} = 0 \to a_{1,2} = 2 \to a_{1,1} = 2 \to a_{2,1} = 1 \to a_{3,1} = 2 \to a_{4,1} = 0.$$

The weight of the matrix A is the monomial in the set of variables $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$ defined by

$$(xy)^A = \prod_{1 \leq i \leq m, 1 \leq j \leq n} (x_i y_j)^{a_{i,j}}.$$

On the one hand, using that $\frac{1}{1-x_i y_j} = \sum_{a_{i,j}=0}^{+\infty} (x_i y_j)^{a_{i,j}}$, we can write

$$\prod_{1 \leq i \leq m, 1 \leq j \leq n} \frac{1}{1 - x_i y_j} = \sum_{A \in \mathcal{M}_{m,n}} (xy)^A.$$

On the other hand, observing that, from RSK, we have $(xy)^A = x^{\mathrm{wt}(P(A))} y^{\mathrm{wt}(Q(A))}$, we obtain a Cauchy-like identity using the bijection ψ and (7):

$$\prod_{1 \leq i \leq m, 1 \leq j \leq n} \frac{1}{1 - x_i y_j} = \sum_{A \in \mathcal{M}_{m,n}} x^{\mathrm{wt}(P(A))} y^{\mathrm{wt}(Q(A))} =$$

$$= \sum_{\lambda \in \mathcal{P}_{\min(m,n)}} s_\lambda(x_1, \ldots, x_m) s_\lambda(y_1, \ldots, y_n).$$

Remark 2.21. *Recall the rule given in* (5) *for the action of $\tilde{e}_i, \tilde{f}_i$ on a tensor product of crystals. The action of any operator $\tilde{f}_i, \tilde{e}_i, i = 1, \ldots, m-1$ on a matrix A can be computed from $P(A)$ but also from the product of row tableaux L_A appearing in Example 2.18 just by concatenating their reading words. In particular when $\tilde{f}_i$ (resp. $\tilde{e}_i$) acts on the j-th component of L_A, the matrix $\tilde{f}_i(A)$ (resp. $\tilde{e}_i(A)$) is obtained from A just by changing $a_{i,j}$ into $a_{i,j} - 1$ and $a_{i+1,j}$ into $a_{i+1,j} + 1$ (resp. $a_{i,j}$ into $a_{i,j} + 1$ and $a_{i+1,j}$ into $a_{i+1,j} - 1$). Similarly, when $\hat{f}_j$ (resp. $\hat{e}_j$) acts on A, there is an integer $i \in \{1, \ldots, m\}$ such that $\hat{f}_j(A)$ (resp. $\hat{e}_j(A)$) is obtained from A by changing $a_{i,j}$ into $a_{i,j} - 1$ and $a_{i,j+1}$ into $a_{i,j+1} + 1$ (resp. $a_{i,j}$ into $a_{i,j} + 1$ and $a_{i,j+1}$ into $a_{i,j+1} - 1$).*

2.5. Restriction of the RSK correspondence. Let D be any subset of $\{1, \ldots, m\} \times \{1, \ldots, n\}$ and write $\mathcal{M}_{m,n}^D$ for the subset of $\mathcal{M}_{m,n}$ containing the matrices A such that $a_{i,j} \neq 0$ only if $(i,j) \in D$. In general, the set $\psi(\mathcal{M}_{m,n}^D)$ is not stable by the $\mathfrak{gl}_m \times \mathfrak{gl}_n$-crystals operators. Nevertheless, when D corresponds to the Young diagram of a fixed partition Λ, it follows from Remark 2.21 that $D = D_\Lambda$ is stable under the action of the operators $\tilde{f}_i, i = 1, \ldots, m-1$ and $\widehat{e}_j, j = 1, \ldots, n-1$. The case where $m = n$ and $\varrho = (n, n-1, \ldots, 1)$ is particularly interesting. In matrix coordinates, we indeed get that

$$D_\varrho = \{(i,j) \mid 1 \leq j \leq i \leq n\}.$$

The following theorem, initially established in [23] using the combinatorics of tableaux, has been reproved in [21] using Littelmann paths and in [1] using semi-skyline diagrams combinatorics. In these different versions, the convention for the crystals is not the same and we here follow the one from [21] which is compatible with Kashiwara and Littelmann convention for the tensor products of crystals, which is the most usual one. Later Fu and Lascoux [15] reproved this theorem using properties of divided differences.

Theorem 2.22. *The restriction of the RSK correspondence ψ to $\mathcal{M}_{n,n}^{D_\varrho}$ gives a one-to-one correspondence*

$$\psi : \mathcal{M}_{n,n}^{D_\varrho} \to \bigsqcup_{\lambda\in\mathcal{P}_n}\bigsqcup_{\sigma\in\mathfrak{S}_n^\lambda} \overline{\mathrm{B}}^\sigma(\lambda)\times \mathrm{B}_\sigma(\lambda).$$

Then by considering the weights of the elements in both sides, we get the Cauchy-like identity

$$\prod_{1\le j\le i\le n}\frac{1}{1-x_iy_j} = \sum_{\lambda\in\mathcal{P}_n}\sum_{\sigma\in\mathfrak{S}_n^\lambda}\overline{\kappa}_\lambda^\sigma(x)\kappa_{\sigma,\lambda}(y). \tag{13}$$

Remark 2.23. *Observe that, using Remark 2.3.5, namely* (12) *and (9), we have*

$$\begin{aligned}\bigsqcup_{\lambda\in\mathcal{P}_n}\bigsqcup_{\sigma\in\mathfrak{S}_n^\lambda}\overline{\mathrm{B}}^\sigma(\lambda)\times\mathrm{B}_\sigma(\lambda) &= \bigsqcup_{\lambda\in\mathcal{P}_n}\bigsqcup_{\sigma\in\mathfrak{S}_n^\lambda}\iota\left(\overline{\mathrm{B}}_{\sigma_0\sigma}(\lambda)\right)\times\mathrm{B}_\sigma(\lambda)\\ &= \bigsqcup_{\mu=(\mu_1,\dots,\mu_n)\in\mathbb{Z}^n}\iota\left(\overline{\mathrm{B}}_{\sigma_0\mu}\right)\times\mathrm{B}_\mu \end{aligned} \tag{14}$$

where $\{\lambda\}=\mathfrak{S}_n\mu\cap\mathcal{P}_n$ in each product set of the disjoint union. This gives

$$\prod_{1\le j\le i\le n}\frac{1}{1-x_iy_j} = \sum_{\mu\in\mathbb{Z}_{\ge0}^n}\overline{\kappa}^\mu(x)\kappa_\mu(y) = \sum_{\mu\in\mathbb{Z}_{\ge0}^n}\overline{\kappa}_{\sigma_0\mu}(x_n,\dots,x_1)\kappa_\mu(y_1,\dots,y_n).$$

Note that in [23] *the rows of the Young diagram ϱ are counted from bottom to top, from* 1 *to n, whereas here they are counted from n to* 1 *according to the matrix notation. Replacing x_i with x_{n-i+1} in (13), one recovers Lascoux's non-symmetric Cauchy identity from* [23]

$$\begin{aligned}\prod_{i+j\le n+1}\frac{1}{1-x_iy_j} &= \sum_{\mu\in\mathbb{Z}_{\ge0}^n}\overline{\kappa}^\mu(x_n,\dots,x_1)\kappa_\mu(y)\\ &= \sum_{\mu\in\mathbb{Z}_{\ge0}^n}\overline{\kappa}_{\sigma_0\mu}(x_1,\dots,x_n)\kappa_\mu(y_1,\dots,y_n).\end{aligned}$$

3. Operations on Demazure crystals and refined RSK

3.1. Parabolic restriction in Demazure crystals and truncated staircases. Let p, n be two integers with $1 \leq p \leq n$. The subset $I_p = \{1, \ldots, p-1\} \subseteq \{1, \ldots, n-1\}$ with $I := I_n$, defines a Levi subalgebra $\mathfrak{g}_{I_p}$ of $\mathfrak{gl}_n$ isomorphic to $\mathfrak{gl}_p$ obtained by considering the matrices with zero entries in positions (i, j) with $i > p$ or $j > p$. We set $\mathfrak{g}_{I_n} := \mathfrak{gl}_n$. The algebra $\mathfrak{g}_{I_p}$ has Weyl group $\mathfrak{S}_p = \langle s_i \mid i \in I_p \rangle$ and root system $R_{I_p} = R \cap \operatorname{span}\langle \alpha_i \mid i \in I_p \rangle$, where R denotes the root system of the Weyl group $\mathfrak{S}_n$ of $\mathfrak{gl}_n$. Its cone of dominant weights can be identified with $\mathcal{P}_p$. Given $\lambda \in \mathcal{P}_p = \bigoplus_{i=1}^{p} \mathbb{Z}\mathbf{e}_i$, let us denote by $B_p(\lambda)$ the subcrystal of the $\mathfrak{gl}_n$-crystal $B_n(\lambda) := B(\lambda) = B(\lambda, 0^{n-p})$ obtained by keeping only the vertices connected to its highest weight vertex b_λ by i-arrows with $i \in I_p$. It follows from the general theory of crystals that $B_p(\lambda)$ is a realization of the $\mathfrak{gl}_p$-crystal associated to λ. In terms of characters, this corresponds to the specialization $x_{p+1} = \cdots = x_n = 0$ in the character $s_\lambda(x)$ of $B(\lambda)$. For the tableaux realization of crystals, we recover with $B_p(\lambda)$ the crystal realization of $\mathfrak{gl}_p$-crystals by tableaux of shape λ with entries in the alphabet $[p]$ as a subcrystal of the crystal $B(\lambda)$ of tableaux of shape λ in the alphabet $[n]$. Given $u \in \mathfrak{S}_p$, we will denote by $\mathrm{B}_{p,u}(\lambda), \mathrm{B}_p^u(\lambda), \overline{\mathrm{B}}_{p,u}(\lambda)$ and $\overline{\mathrm{B}}_p^u(\lambda)$ the Demazure, opposite Demazure and atoms associated to u in the $\mathfrak{gl}_p$-crystal $B_p(\lambda)$.

The Coxeter monoid associated to the symmetric group $\mathfrak{S}_n$ is the monoid $\mathfrak{M}_n$ with generators $\boldsymbol{s}_i, i = 1, \ldots, n-1$ and relations

$$\begin{aligned} \boldsymbol{s}_i \boldsymbol{s}_j &= \boldsymbol{s}_j \boldsymbol{s}_i \text{ for any } i, j = 1, \ldots, n-1 \text{ such that } |i-j| > 1 \\ \boldsymbol{s}_i \boldsymbol{s}_{i+1} \boldsymbol{s}_i &= \boldsymbol{s}_{i+1} \boldsymbol{s}_i \boldsymbol{s}_{i+1} \text{ for any } i = 1, \ldots, n-2, \\ \boldsymbol{s}_i^2 &= \boldsymbol{s}_i \text{ for any } i = 1, \ldots, n-1. \end{aligned}$$

Observe that this is exactly the same relations as those satisfied by the Demazure operators and the map $\boldsymbol{s}_i \longmapsto D_i$ yields a faithful representation of the monoid $\mathfrak{M}_n$ on $\mathbb{Z}[x_1, \ldots, x_n]$. There is a canonical bijection between $\mathfrak{S}_n$ and $\mathfrak{M}_n$ sending any reduced decomposition of $\sigma \in \mathfrak{S}_n$ to the same (still reduced) decomposition $\boldsymbol{\sigma} \in \mathfrak{M}_n$. Given any $\sigma \in \mathfrak{S}_n$ and a reduced decomposition $\sigma = s_{i_1} \cdots s_{i_\ell}$, we write σ^{I_p} for the element of $\mathfrak{S}_p$ obtained by the following procedure.

Algorithm 3.1.

(1) *Remove all the $\boldsymbol{s}_{i_a}$ in $\boldsymbol{\sigma}$ such that $i_a \notin I_p$. This yields a word in the generators of $\mathfrak{M}_n$, which may not be reduced.*

(2) *Calculate the element of* $\mathfrak{M}_n$ *represented by the word obtained in* (1), *and denote it* $\boldsymbol{\sigma}^{I_p} \in \mathfrak{M}_n$.

(3) *The element* σ^{I_p}†† *is the element of* $\mathfrak{S}_p$ *associated to* $\boldsymbol{\sigma}^{I_p}$ *through the canonical bijection* $W \longrightarrow \mathfrak{M}_n$.

Lemma 3.2. *The element* σ^{I_p} *obtained by Algorithm 3.1 does not depend on the initial reduced decomposition chosen for* σ.

Proof: See the appendix (Lemma 5.1) for a general proof in arbitrary Coxeter groups. ■

We give an example of this algorithm in Example 3.3 below.

For $\sigma \neq 1$, $\sigma^{I_p} = 1$ if and only if $\sigma \in \mathfrak{S}_{[p,n]}$. Note that $\sigma_0^{I_p}$ is the longest element of $\mathfrak{S}_p$; indeed, writing $\sigma_0^{[p]}$ for the longest element of $\mathfrak{S}_p$ viewed inside $\mathfrak{S}_n$, we have $\ell(\sigma_0) = \ell(\sigma_0^{[p]}) + \ell(\sigma_0^{[p]}\sigma_0)$, since σ_0 has every element of $\mathfrak{S}_n$ appearing as a prefix. Chosing a reduced decomposition of σ_0 beginning by a reduced decomposition of $\sigma_0^{[p]}$ and applying Algorithm 3.1 to this decomposition yields an element in $\mathfrak{M}_n$ which is of the form $\boldsymbol{\sigma_0^{[p]} x}$ for some $x \in \mathfrak{S}_p$. Using the fact that $\ell(\boldsymbol{wv}) \geq \ell(\boldsymbol{w})$ for all $\boldsymbol{w}, \boldsymbol{v} \in \mathfrak{M}_n$, we see that we must have $\boldsymbol{\sigma_0^{[p]} x} = \boldsymbol{\sigma_0^{[p]}}$.

Example 3.3. *Consider the reduced decomposition* $\sigma = s_1 s_2 s_3 s_1 s_2$ *in* $\mathfrak{S}_4$ *and choose* $p = 3$, *hence* $I_3 = \{1, 2\}$. *We then get in* $\mathfrak{M}_n$

$$\boldsymbol{\sigma}^{I_p} = \boldsymbol{s}_1\boldsymbol{s}_2\widehat{\boldsymbol{s}}_3\boldsymbol{s}_1\boldsymbol{s}_2 = \boldsymbol{s}_1\boldsymbol{s}_2\boldsymbol{s}_1\boldsymbol{s}_2 = \boldsymbol{s}_2\boldsymbol{s}_1\boldsymbol{s}_2\boldsymbol{s}_2 = \boldsymbol{s}_2\boldsymbol{s}_1\boldsymbol{s}_2 = \boldsymbol{s}_1\boldsymbol{s}_2\boldsymbol{s}_1.$$

Therefore, $\sigma^{I_3} = s_2 s_1 s_2 = s_1 s_2 s_1 \in \mathfrak{S}_3$.

Proposition 3.4. *Consider* σ *in* $\mathfrak{S}_n$. *The set* $\mathfrak{S}_p^{\leq\sigma} = \{v \in \mathfrak{S}_p \mid v \leq \sigma\}$ *admits* σ^{I_p} *has unique maximal element for* $\leq$, *that is*

$$\mathfrak{S}_p^{\leq\sigma} = \{v \in \mathfrak{S}_p \mid v \leq \sigma^{I_p}\}.$$

Proof: See the appendix (Lemma 5.2) for a general proof in arbitrary Coxeter groups. ■

Now, set $B^p(\lambda) = \iota(B_p(\lambda))$ where ι is the involution in $B(\lambda)$ defined in Remark 2.3.5. Since

$$B_p(\lambda) = \{\tilde{f}_{i_1}^{k_1} \cdots \tilde{f}_{i_N}^{k_N}(b_\lambda) \mid i_1, \ldots, i_N \in [p-1], N \geq 1, (k_{i_1}, \ldots, k_{i_N}) \in \mathbb{Z}_{\geq 0}^N\} \backslash \{0\},$$

††Note that σ^{I_p} is *not* the minimal length element in $\sigma\mathfrak{S}_p$ in general.

we get

$$B^p(\lambda) = \{\tilde{e}_{i_1}^{k_{i_1}} \cdots \tilde{e}_{i_N}^{k_{i_N}}(b_{\sigma_0(\lambda,0^{n-p})}) \mid \\ i_1,\dots,i_N \in [n-p+1,n-1], N \geq 1, (k_{i_1},\dots,k_{i_N}) \in \mathbb{Z}_{\geq 0}^N\} \setminus \{0\}.$$

One can observe that $B^p(\lambda)$ also has the structure of a $\mathfrak{gl}_p$-crystal but this time for the root system with set of simple roots $\{\alpha_{n-p+1},\dots \alpha_{n-1}\}$. The corresponding character is obtained from the specialization $x_1 = \cdots = x_{n-p} = 0$ in the character $s_\lambda(x)$ of $B(\lambda)$.

Corollary 3.5. *For any $\sigma \in \mathfrak{S}_n$, we have the following equalities of sets*

$$1 : \mathrm{B}_\sigma(\lambda) \cap B_p(\lambda) = \mathrm{B}_{p,\sigma^{I_p}}(\lambda), \quad \overline{\mathrm{B}}^\sigma(\lambda) \cap B_p(\lambda) = \begin{cases} \emptyset \textit{ if } \sigma \notin \mathfrak{S}_p, \\ \overline{\mathrm{B}}_p^\sigma(\lambda) \textit{ else.} \end{cases}$$

$$2 : \mathrm{B}^\sigma(\lambda) \cap B_p(\lambda) = \begin{cases} \emptyset \textit{ if } \sigma \notin \mathfrak{S}_p, \\ \mathrm{B}_p^\sigma(\lambda) \textit{ else,} \end{cases} \quad \overline{\mathrm{B}}_\sigma(\lambda) \cap B_p(\lambda) = \begin{cases} \emptyset \textit{ if } \sigma \notin \mathfrak{S}_p, \\ \overline{\mathrm{B}}_{p,\sigma}(\lambda) \textit{ else.} \end{cases}$$

$$3 : \mathrm{B}_\sigma(\lambda) \cap B^p(\lambda) = \begin{cases} \emptyset \textit{ if } \sigma \notin \mathfrak{S}_p, \\ \iota(\mathrm{B}_{p,\sigma_0\sigma}(\lambda)) \textit{ else,} \end{cases} \quad \overline{\mathrm{B}}^\sigma(\lambda) \cap B^p(\lambda) = \begin{cases} \emptyset \textit{ if } \sigma \notin \sigma_0\mathfrak{S}_p, \\ \iota(\overline{\mathrm{B}}_{p,\sigma_0\sigma}(\lambda)) \textit{ else.} \end{cases}$$

$$4 : \mathrm{B}^\sigma(\lambda) \cap B^p(\lambda) = \iota(\mathrm{B}_{p,\sigma_0\sigma^{I_p}}(\lambda)), \quad \overline{\mathrm{B}}_\sigma(\lambda) \cap B^p(\lambda) = \begin{cases} \emptyset \textit{ if } \sigma \notin \sigma_0\mathfrak{S}_p, \\ \iota(\overline{\mathrm{B}}_p^{\sigma_0\sigma}(\lambda)) \textit{ else.} \end{cases}$$

Proof: For the equalities in the first point, we have

$$\begin{aligned} \mathrm{B}_\sigma(\lambda) \cap B_p(\lambda) &= \{b \in B_p(\lambda) \mid K_+(b) \leq b_{\sigma\lambda}\} \\ &= \{b \in B_p(\lambda) \mid K_+(b) \leq b_{\sigma^{I_p}\lambda}\} = \mathrm{B}_{p,\sigma^{I_p}}(\lambda) \end{aligned}$$

where the second equality follows from Proposition 3.4 since $K_+(b)$ belongs to $\mathfrak{S}_p$ for any $b \in B_p(\lambda)$.

For the second equality of the first point, we also obtain

$$\overline{\mathrm{B}}^\sigma(\lambda) \cap B_p(\lambda) = \{b \in B_p(\lambda) \mid K^-(b) = \sigma\}.$$

But $K^-(b)$ belongs to $\mathfrak{S}_p$ for any $b \in B_p(\lambda)$. Therefore, by definition of the strong Bruhat order, $\sigma \leq K^-(b)$ is only possible when $\sigma \in \mathfrak{S}_p$, whence the result. Similarly, for the set equalities of the second point we have

$$\mathrm{B}^\sigma(\lambda) \cap B_p(\lambda) = \{b \in B_p(\lambda) \mid K^-(b) \geq \sigma\}.$$

We have that $K^-(b)$ belongs to $\mathfrak{S}_p$ and thus if $\sigma \leq K^-(b)$, then σ also belongs to $\mathfrak{S}_p$ by definition of the Bruhat order. Therefore

$$\mathrm{B}^\sigma(\lambda) \cap B_p(\lambda) = \begin{cases} \emptyset \text{ if } \sigma \notin \mathfrak{S}_p \\ \mathrm{B}_p^\sigma(\lambda) \text{ otherwise} \end{cases}$$

as claimed. Finally we can write

$$\overline{\mathrm{B}}_\sigma(\lambda) \cap B_p(\lambda) = \{b \in B_p(\lambda) \mid K_+(b) = \sigma\}$$

and we get the result by using that $K_+(b) \in \mathfrak{S}_p$ for any b in $B_p(\lambda)$. The third and fourth set equalities are easily deduced from the two previous ones by applying the involution ι and using the relation (11). ■

3.2. Truncated staircase. In the following, we fix p and q two nonnegative integers such that $n \geq q \geq p \geq 1$. We consider the Young diagram

$$D_{p,q} = \{(i,j) \mid n-p+1 \leq i \leq n, 1 \leq j \leq q\} \cap D_\varrho$$

defined by using the matrix coordinates (i,j). It is the intersection of D_ϱ with a quarter of plane defined by the lines $i = p$ and $j = q$ (in Cartesian coordinates). When $n-p+1 \leq q$, we get the Young diagram (see Figure 1)

$$D_{p,q} = D_{\Lambda(p,q)} \text{ with } \Lambda(p,q) = (q^{n-q+1}, q-1, \ldots, n-p+1).$$

We have in particular $D_{n,n} = D_{\Lambda(n,n)} = D_\varrho$. Observe that if $n-p+1 > q$, there are also other Young sub-diagrams appearing but they all reduce to a rectangle and thus do not yield anything new.

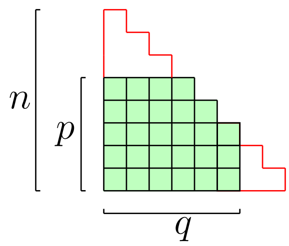


FIGURE 1. The truncated Ferrers shape $\Lambda(p,q)$, in green, fitting the p by q rectangle so that the staircase D_ϱ of size n is the smallest one containing $\Lambda(p,q)$. If $p \leq q$, $(p, p-1, \ldots, 1)$ is the biggest staircase inside $\Lambda(p,q)$.

Definition 3.6. *For any $\mu = (\mu_1, \ldots, \mu_p) \in \mathbb{Z}^p_{\geq 0}$, let $\lambda \in \mathcal{P}_p$ and $\tau \in \mathfrak{S}_p^\lambda$ such that $\mu = \tau\lambda$. By applying $\sigma_0 \in \mathfrak{S}_n$ to μ, one gets $\sigma_0\mu = \sigma_0\tau(\lambda, 0^{n-p})$. We set*

$$\widetilde{\mu} = (\sigma_0\tau)^{I_q}(\lambda, 0^{q-p}, 0^{n-q}).$$

Note that $\tilde{\mu}$ has its last $n-q$ entries equal to zero because $(\sigma_0\tau)^{I_q} \in \mathfrak{S}_q$. We will see in § 3.4 that it also has its first $q-p$ entries equal to zero.

Example 3.7. *Consider* $\mu = (1,3,2)$ *and let* $q = 4$ *and* $n = 5$. *Then letting* $\lambda = (3,2,1)$, *we have*

$$\sigma_0\mu = (0,0,2,3,1) = s_2s_1s_3s_2s_4s_3s_1(3,2,1,0,0).$$

We have

$$(s_2s_1s_3s_2s_4s_3s_1)^{I_4} = s_2s_1s_3s_2s_3s_1$$

which gives $\widetilde{\mu} = s_2s_1s_3s_2s_3s_1(\lambda) = (0,1,2,3,0)$.

Theorem 3.8. *With the above notation, the restriction of the RSK correspondence* ψ *to* $\mathcal{M}_{n,n}^{D_{\Lambda(p,q)}}$ *gives a one-to-one correspondence*

$$\psi : \mathcal{M}_{n,n}^{D_{\Lambda(p,q)}} \to \bigsqcup_{\mu\in\mathbb{Z}_{\geq 0}^p} \iota(\overline{\mathrm{B}}_{p,\mu}) \times \mathrm{B}_{q,\widetilde{\mu}}.$$

In particular, we have

$$\prod_{(i,j)\in D_{\Lambda(p,q)}} \frac{1}{1-x_iy_j} = \sum_{(\mu_1,\ldots,\mu_p)\in\mathbb{Z}_{\geq 0}^p} \overline{\kappa}_{(\mu_p,\ldots,\mu_1)}(x_n,\ldots,x_{n-p+1})\kappa_{\widetilde{\mu}}(y_1,\ldots,y_q).$$

Proof: By Theorem 2.22 together with (14), the restriction of the map ψ from $\mathcal{M}_{n,n}^{D_\varrho}$ to $\mathcal{M}_{n,n}^{D_{\Lambda(p,q)}}$ gives

$$\begin{aligned}\psi(\mathcal{M}_{n,n}^{D_{\Lambda(p,q)}}) &= \bigsqcup_{\lambda\in\mathcal{P}_n}\bigsqcup_{\sigma\in\mathfrak{S}_n^\lambda} \overline{\mathrm{B}}^{\sigma}(\lambda)\cap B^p(\lambda)\times \mathrm{B}_\sigma(\lambda)\cap B_q(\lambda)\\ &= \bigsqcup_{\mu\in\mathbb{Z}_{\geq 0}^n} \overline{\mathrm{B}}^{\mu}\cap B^p(\lambda)\times \mathrm{B}_\mu\cap B_q(\lambda).\end{aligned}$$

By Corollary 3.5, we have $\overline{\mathrm{B}}^{\sigma}(\lambda)\cap B^p(\lambda) = \emptyset$ unless $\sigma\in\sigma_0\mathfrak{S}_p^\lambda$, $\lambda\in\mathcal{P}_p$ and then $\overline{\mathrm{B}}^{\sigma}(\lambda)\cap B^p(\lambda) = \iota(\overline{\mathrm{B}}_{p,\sigma_0\sigma}(\lambda))$. We also obtain in this case that $\mathrm{B}_\sigma(\lambda)\cap B_q(\lambda) = \mathrm{B}_{q,\sigma^{I_q}}(\lambda)$. We thus get

$$\psi(\mathcal{M}_{n,n}^{D_{\Lambda(p,q)}}) = \bigsqcup_{\lambda\in\mathcal{P}_p}\bigsqcup_{\sigma\in\mathfrak{S}_n^\lambda\cap\sigma_0\mathfrak{S}_p} \iota(\overline{\mathrm{B}}_{p,\sigma_0\sigma}(\lambda))\times B_{q,\sigma^{I_q}}(\lambda).$$

As usual, one can replace the two disjoint unions on $\mathcal{P}_p\times\mathfrak{S}_n^\lambda\cap\sigma_0\mathfrak{S}_p^\lambda$ by a simple disjoint union on $\mathbb{Z}_{\geq 0}^p$ by setting $\mu = \sigma_0\sigma\lambda$ with $\sigma\in\mathfrak{S}_p^\lambda$. To determine $\sigma^{I_q}(\lambda)$ from μ, we can compute λ by reordering its coordinates; one then gets $\widehat{\mu} = \sigma_0\mu$, and $\sigma\in\mathfrak{S}_n^\lambda\cap\sigma_0\mathfrak{S}_p$ is determined by the equality $\widehat{\mu} = \sigma\lambda$. Finally, one computes σ^{I_q} by applying Algorithm 3.1 to σ. In particular $\widehat{\mu}$ has its first $n-p$ coordinates equal to zero and can be written $\widehat{\mu} = (0^{n-p},\mu_p,\ldots,\mu_1)$ with

$(\mu, 0^{n-p}) = \sigma_0.\widehat{\mu} = (\mu_1, \ldots, \mu_p, 0^{n-p})$. Therefore, $\sigma^{I_q}(\lambda) = \widetilde{\mu}$ as introduced in Definition 3.6. By considering all the partitions λ in $\mathcal{P}_p$, we thus obtain

$$\psi(\mathcal{M}_{n,n}^{D_{\Lambda(p,q)}}) = \bigsqcup_{\mu \in \mathbb{Z}_{\geq 0}^p} \iota(\overline{\mathrm{B}}_{p,\mu}) \times \mathrm{B}_{q,\widetilde{\mu}}$$

with $\widetilde{\mu} = \sigma^{I_q}\lambda$ in each set of the disjoint union. Finally, we get the Cauchy-like identity by considering the characters of both sides of the set equality. ■

3.3. Demazure operators on crystals and augmented staircases. Consider a partition λ in $\mathcal{P}_n$ and any subset Ω of $B(\lambda)$. We define the character of Ω by setting

$$\mathrm{char}(\Omega) = \mathrm{char}(\Omega)(x_1, \ldots, x_n) = \sum_{b \in \Omega} x^{\mathrm{wt}(b)}.$$

Observe that

$$\mathrm{char}(\iota(\Omega)) = \sum_{b \in \Omega} x^{\sigma_0 \mathrm{wt}(b)} = \mathrm{char}(\Omega)(x_n, \ldots, x_1). \tag{15}$$

For any $i = 1, \ldots, n-1$, denote by $\Delta_i(\Omega)$ the subset of $B(\lambda)$ obtained from Ω by applying operators $\tilde{f}_i^k, k \geq 0$ to the vertices in Ω, that is

$$\Delta_i(\Omega) = \{b \in B(\lambda) \mid \exists k \in \mathbb{Z}_{\geq 0}, \tilde{e}_i^k(b) \in \Omega\}.$$

By Remark 2.17, for any $\sigma \in \mathfrak{S}_n$ and any $i = 1, \ldots, n-1$, we have

$$\Delta_i(\mathrm{B}_\sigma(\lambda)) = \mathrm{B}_{\pi_i(\sigma\lambda)} = \begin{cases} \mathrm{B}_{s_i\sigma}(\lambda) \text{ if } \ell(s_i\sigma) = \ell(\sigma) + 1 \text{ and } s_i\sigma\lambda \neq \sigma\lambda, \\ \mathrm{B}_\sigma(\lambda) \text{ if } \ell(s_i\sigma) = \ell(\sigma) - 1 \text{ or } s_i\sigma\lambda = \sigma\lambda, \end{cases} \tag{16}$$

that is,

$$\Delta_i(\mathrm{B}_\mu) = \mathrm{B}_{\pi_i(\mu)} \text{ with } \mu = \sigma\lambda,$$

where π_i is as defined in (2). In particular, we have $\Delta_i^2(\mathrm{B}_\sigma(\lambda)) = \Delta_i(\mathrm{B}_\sigma(\lambda))$. This thus gives

$$\sum_{b \in \Delta_i(\mathrm{B}_\sigma(\lambda))} x^{\mathrm{wt}(b)} = D_i\left(\sum_{b \in \mathrm{B}_\sigma(\lambda)} x^{\mathrm{wt}(b)}\right) = D_i(\kappa_{\sigma,\lambda}) = \kappa_{\pi_i(\sigma\lambda)} \tag{17}$$

and by using (8) one can interpret Δ_i as an operator on Demazure crystals analogous to the operator D_i on Demazure characters. For the atoms, we get the following lemma.

Lemma 3.9. *For any σ in $\mathfrak{S}_n$ and any s_i such that $\ell(s_i\sigma) = \ell(\sigma) + 1$ and $s_i\sigma\lambda \neq \sigma\lambda$, we have*

$$\Delta_i(\overline{\mathrm{B}}_\sigma(\lambda)) = \overline{\mathrm{B}}_\sigma(\lambda)\bigsqcup\overline{\mathrm{B}}_{s_i\sigma}(\lambda)$$

and

$$\overline{\kappa}_{s_i\sigma,\lambda} + \overline{\kappa}_{\sigma,\lambda} = \sum_{b\in\Delta_i(\overline{\mathrm{B}}_\sigma(\lambda))} x^{\mathrm{wt}(b)} = D_i\left(\sum_{b\in\overline{\mathrm{B}}_\sigma(\lambda)} x^{\mathrm{wt}(b)}\right) = D_i(\overline{\kappa}_{\lambda,\sigma}). \tag{18}$$

Proof: For any b in $B(\lambda)$ and $i = 1, \ldots, n-1$, we have by definition of the key K_+

$$K_+(\tilde{f}_i(b)) \in \{K_+(b), s_iK_+(b)\}.$$

This gives

$$\Delta_i(\overline{\mathrm{B}}_\sigma(\lambda)) \subset \overline{\mathrm{B}}_\sigma(\lambda)\bigsqcup\overline{\mathrm{B}}_{s_i\sigma}(\lambda).$$

Conversely, it is clear that $\overline{\mathrm{B}}_\sigma(\lambda) \subset \Delta_i(\overline{\mathrm{B}}_\sigma(\lambda))$ by definition of Δ_i. Now, if b' belongs to $\overline{\mathrm{B}}_{s_i\sigma}(\lambda)$, we have $K_+(b') = s_i\sigma$ with $\varepsilon_i(b_{K_+(b')}) > 0$ because $\ell(s_i\sigma) = \ell(\sigma) + 1$. By the tensor product rules in crystals (5), there exists an integer k such that $K_+(\tilde{e}_i^k b') = \sigma$, that is such that $\tilde{e}_i^k b' \in \overline{\mathrm{B}}_\sigma(\lambda)$. This shows the inclusion $\overline{\mathrm{B}}_{s_i\sigma}(\lambda) \subset \Delta_i(\overline{\mathrm{B}}_\sigma(\lambda))$. The equality of characters follows from the equality of sets. ■

Remark 3.10.

(1) *Here again, we can reformulate (16) and Lemma 3.9 by setting $\mu = \sigma\lambda$. Using Lemma 2.4, this gives*

$$\Delta_i(\mathrm{B}_\mu) = \begin{cases} \mathrm{B}_{s_i\mu} \text{ if } \mu_i > \mu_{i+1}, \\ \mathrm{B}_\mu \text{ otherwise,} \end{cases}$$

and

$$\Delta_i(\overline{\mathrm{B}}_\mu) = \overline{\mathrm{B}}_\mu\bigsqcup\overline{\mathrm{B}}_{s_i\mu} \text{ if } \mu_i > \mu_{i+1}.$$

(2) *Observe that Lemma 3.9 does not remain true when $\mu_i < \mu_{i+1}$. In this case, we indeed have $\Delta_i(\overline{\mathrm{B}}_\mu) = \overline{\mathrm{B}}_\mu$ whereas $D_i(\overline{\kappa}_\mu) = 0$, as can be seen from (18). Thus, to mimic the action of the operator D_i on $\overline{\kappa}_{\lambda,\sigma}$ at the level of its associated Demazure atoms, we need to replace the action of Δ_i on $\overline{\mathrm{B}}_\mu(\lambda)$ by*

$$\dot{\Delta}_i(\overline{\mathrm{B}}_\mu) = \begin{cases} \Delta_i(\overline{\mathrm{B}}_\mu) = \overline{\mathrm{B}}_\mu\bigsqcup\overline{\mathrm{B}}_{s_i\mu} \text{ if } \mu_i > \mu_{i+1} \\ \Delta_i(\overline{\mathrm{B}}_\mu) = \overline{\mathrm{B}}_\mu \text{ if } \mu_i = \mu_{i+1}, \\ \emptyset \text{ if } \mu_i < \mu_{i+1}. \end{cases} \tag{19}$$

We then always have

$$\mathrm{char}(\dot{\Delta}_i(\overline{\mathrm{B}}_\mu)) = D_i(\overline{\kappa}_\mu).$$

We may linearize the action described in (19) above by defining an action of the monoid of Demazure operators D_i on a free $\mathbb{Z}$-module of rank $|\mathfrak{S}_n^\lambda|$ generated by the formal symbols $\{\bar{c}_{\sigma\lambda} : \sigma \in \mathfrak{S}_n^\lambda\}$, written $\bigoplus_{\sigma\in\mathfrak{S}_n^\lambda} \mathbb{Z}\bar{c}_{\sigma\lambda}$, by setting

$$D_i(\bar{c}_{\sigma\lambda}) = \begin{cases} \bar{c}_{\sigma\lambda} + \bar{c}_{s_i\sigma\lambda} \text{ if } \mu = \sigma\lambda \text{ satisfies } \mu_i > \mu_{i+1} \\ \bar{c}_{\sigma\lambda} \text{ if } \mu = \sigma\lambda \text{ satisfies } \mu_i = \mu_{i+1}, \\ 0 \text{ if } \mu = \sigma\lambda \text{ satisfies } \mu_i < \mu_{i+1}. \end{cases} \tag{20}$$

These operators satisfy the braid relations together with the relations $D_i^2 = D_i$, hence for every $w \in \mathfrak{S}_n$ we can write D_w to mean $D_{i_1}D_{i_2}\cdots D_{i_k}$, where $s_{i_1}s_{i_2}\cdots s_{i_k}$ is a reduced decomposition of w in $\mathfrak{S}_n$. Note that the conditions on the weight μ can be entirely reformulated in terms of the Weyl group $\mathfrak{S}_n$ (Lemma 2.4).

The following lemma establishes crucial properties of the action of the operators D_w on the basis $\{\bar{c}_{\sigma\lambda} : \sigma \in \mathfrak{S}_n^\lambda\}$, which will be used in the proof of Theorem 3.15 below.

Lemma 3.11. *We have*

(1) *Let $A \subseteq \mathfrak{S}_n^\lambda$ and $w \in \mathfrak{S}_n$. Then there exists $B \subseteq \mathfrak{S}_n^\lambda$ such that*

$$D_w\left(\sum_{\sigma\in A} \bar{c}_{\sigma\lambda}\right) = \sum_{\sigma\in B} \bar{c}_{\sigma\lambda}.$$

(2) *Let $\tau, \tau' \in \mathfrak{S}_n^\lambda$ with $\tau \neq \tau', w \in \mathfrak{S}_n$. Then there are $A_1, A_2 \subseteq \mathfrak{S}_n^\lambda$ with $A_1 \cap A_2 = \emptyset$ such that $D_w(\bar{c}_{\tau\lambda}) = \sum_{\sigma\in A_1} \bar{c}_{\sigma\lambda}$ and $D_w(\bar{c}_{\tau'\lambda}) = \sum_{\sigma\in A_2} \bar{c}_{\sigma\lambda}$.*

Proof: Let us first prove the first point. By induction on the length $\ell(w)$ of w, it suffices to prove the result for $w = s_i$, where $i \in \{1, 2, \ldots, n-1\}$. Let $A = A_1 \bigsqcup A_2 \bigsqcup A_3$, where $A_1 = \{\sigma \in A \mid \mu = \sigma\lambda \text{ satisfies } \mu_i > \mu_{i+1}\}$, $A_2 = \{\sigma \in A \mid \mu = \sigma\lambda \text{ satisfies } \mu_i < \mu_{i+1}\}$, and $A_3 = \{\sigma \in A \mid \mu =$

$\sigma\lambda$ satisfies $\mu_i = \mu_{i+1}\}$. By (20) we have

$$D_i\left(\sum_{\sigma\in A}\bar{c}_{\sigma\lambda}\right) = D_i\left(\sum_{\sigma\in A_1}\bar{c}_{\sigma\lambda}\right) + D_i\left(\sum_{\sigma\in A_2}\bar{c}_{\sigma\lambda}\right) + D_i\left(\sum_{\sigma\in A_3}\bar{c}_{\sigma\lambda}\right) =$$
$$= \left(\sum_{\sigma\in A_1}(\bar{c}_{\sigma\lambda} + \bar{c}_{s_i\sigma\lambda})\right) + 0 + \sum_{\sigma\in A_3}\bar{c}_{\sigma\lambda}$$
$$= \sum_{\sigma\in A_1}\bar{c}_{\sigma\lambda} + \sum_{\sigma\in s_iA_1}\bar{c}_{\sigma\lambda} + \sum_{\sigma\in A_3}\bar{c}_{\sigma\lambda}.$$

To conclude the proof, it suffices to notice that $s_iA_1 \subseteq \{\sigma \in \mathfrak{S}_n^\lambda \mid \mu = \sigma\lambda$ satisfies $\mu_i < \mu_{i+1}\}$, hence the union $A_1 \bigcup s_iA_1 \bigcup A_3$ is still disjoint. Therefore setting $B := A_1 \bigsqcup s_iA_1 \bigsqcup A_3$ we get the result.

We now prove the second point. By the first point, there is $B \subseteq \mathfrak{S}_n^\lambda$ such that $D_w(\bar{c}_{\tau\lambda} + \bar{c}_{\tau'\lambda}) = \sum_{\sigma\in B}\bar{c}_{\sigma\lambda}$. But, still by the first point, there are also $A_1, A_2 \subseteq \mathfrak{S}_n^\lambda$ such that $D_w(\bar{c}_{\tau\lambda}) = \sum_{\sigma\in A_1}\bar{c}_{\sigma\lambda}$ and $D_w(\bar{c}_{\tau'\lambda}) = \sum_{\sigma\in A_2}\bar{c}_{\sigma\lambda}$. We thus have

$$\sum_{\sigma\in B}\bar{c}_{\sigma\lambda} = D_w(\bar{c}_{\tau\lambda} + \bar{c}_{\tau'\lambda}) = D_w(\bar{c}_{\tau\lambda}) + D_w(\bar{c}_{\tau'\lambda}) = \sum_{\sigma\in A_1}\bar{c}_{\sigma\lambda} + \sum_{\sigma\in A_2}\bar{c}_{\sigma\lambda},$$

which forces B to be the disjoint union of A_1 and A_2. ∎

In [23] Lascoux gave other non-symmetric Cauchy type identities for any partition $\Lambda \in \mathcal{P}_n$. The idea is to consider the largest staircase $\rho_\Lambda = (m, m-1, \dots, 1)$ contained in the Young diagram of Λ. Then one can choose a box b at position (i_0, j_0), in Cartesian coordinates, in the augmented staircase $(m+1, m, \dots, 1)$ which is not in Λ. The diagonal $L_{i,j} : j - i = j_0 - i_0,$ in Cartesian coordinates, cuts Λ in a northwest part and a southeast part corresponding to the boxes above and below $L_{i,j}$, respectively. Now fill the boxes (i, j), in the $n\times n$ matrix convention, of the NW part of Λ by $i-1$ (*i.e.*, by the $n\times n$ matrix row index minus one), and the boxes (i, j) of the SE part by $j-1$ (*i.e.*, by the index of the column minus one). Let $\sigma(\Lambda, NW) = s_{i_1}\cdots s_{i_a}$ be the element of $\mathfrak{S}_n$ where the word $i_1\cdots i_a$ is obtained from right to left column reading of the NW part of Λ, each column being read from top to bottom. Similarly, let $\sigma(\Lambda, SE) = s_{j_1}\cdots s_{j_b}$ be the element of $\mathfrak{S}_n$ where the word $j_1\cdots j_b$ is obtained from top to bottom row reading of the SE part of Λ, each row being read from right to left.

Example 3.12. *Let $n = 8$ and $\Lambda = (7, 4, 2, 2, 2)$. Take $(i_0, j_0) = (3, 3)$. We have $m = 4$ and $\rho_\Lambda = (4, 3, 2, 1)$,*

$$\Lambda = \begin{array}{|c|c|c|c|c|c|c}
\hline
3 & 3 \\
\cline{1-2}
\blacksquare & 4 \\
\cline{1-2}
\blacksquare & \blacksquare & \blacktriangle \\
\cline{1-4}
\blacksquare & \blacksquare & \blacksquare & 3 \\
\hline
\blacksquare & \blacksquare & \blacksquare & \blacksquare & 4 & 5 & \multicolumn{1}{c|}{6} \\
\hline
\end{array} ,$$

and we have $\sigma(\Lambda, NW) = s_3 s_4 s_3$, and $\sigma(\Lambda, SE) = s_3 s_6 s_5 s_4$.

The following theorem was established in [23] and reproved in [2].

Theorem 3.13. *With the previously introduced notation, we have*

$$\prod_{(i,j)\in\Lambda} \frac{1}{1 - x_i y_j} = \sum_{(\mu_1,\ldots,\mu_m)\in\mathbb{Z}^m} D_{\sigma(\Lambda,NW)} \overline{\kappa}_{(\mu_m,\ldots,\mu_1)}(x_n, \ldots, x_{n-m+1}) \bullet$$
$$\bullet D_{\sigma(\Lambda,SE)} \kappa_{(\mu_1,\ldots,\mu_m)}(y_1, \ldots, y_m),$$

where $D_{\sigma(\Lambda,NW)} = D_{i_1} \cdots D_{i_a}$ and $D_{\sigma(\Lambda,SE)} = D_{j_1} \cdots D_{j_b}$.

Remark 3.14.

(1) *By setting $(\mu_1, \ldots, \mu_m) = \sigma\lambda$ with $\sigma \in \mathfrak{S}_m$ and $\lambda \in \mathcal{P}_m$, we get by (15)*

$$\overline{\kappa}_{(\mu_m,\ldots,\mu_1)}(x_n, \ldots, x_{n-m+1}) = \mathrm{char}\left(\iota\left(\overline{\mathrm{B}}_{(\mu_m,\ldots,\mu_1)}\right)\right) = \mathrm{char}\left(\overline{\mathrm{B}}^{(\mu_1,\ldots,\mu_m)}\right).$$

(2) *Observe that both decompositions $s_{i_1} \cdots s_{i_a}$ and $s_{j_1} \cdots s_{j_b}$ of $\sigma(\Lambda, NW)$ and $\sigma(\Lambda, SE)$ are reduced.*

By using the operators Δ_i on Demazure crystals, one can now deduce from this identity of characters an analogue of Theorem 2.22 for the augmented staircases.

Theorem 3.15. *With the previously introduced notation, the restriction of the RSK correspondence ψ to $\mathcal{M}_{n,n}^{D_\Lambda}$ gives a one-to-one correspondence*

$$\psi : \mathcal{M}_{n,n}^{D_\Lambda} \to \bigsqcup_{(\mu_1,\ldots,\mu_m)\in\mathbb{Z}_{\geq 0}^m} \iota\left(\dot{\Delta}^{\iota}_{\sigma(\Lambda,NW)}(\overline{\mathrm{B}}_{(\mu_m,\ldots,\mu_1)})\right) \times \Delta_{\sigma(\Lambda,SE)}\left(\mathrm{B}_{(\mu_1,\ldots,\mu_m)}\right) \quad (21)$$

where $\Delta_{\sigma(\Lambda,SE)} = \Delta_{j_1} \cdots \Delta_{j_b}$ and $\dot{\Delta}^{\iota}_{\sigma(\Lambda,NW)} = \dot{\Delta}_{n-i_1} \cdots \dot{\Delta}_{n-i_a}$.[‡‡]

[‡‡] It follows from the definition of $\dot{\Delta}$ that product sets of the form $\emptyset \times U$ can appear in the right hand side of (21) and then $\emptyset \times U = \emptyset$ as usual.

Proof: First we need to prove that the right hand side $\mathcal{I}$ of (21) is indeed a disjoint union. To this end, first observe that for any $\nu \in \mathbb{Z}_{\geq 0}^m$, we have

$$\dot{\Delta}^{\iota}_{\sigma(\Lambda,NW)}(\overline{\mathrm{B}}_\nu) = \emptyset \Longleftrightarrow D^{\iota}_{\sigma(\Lambda,NW)}(\overline{\kappa}_\nu) = 0.$$

When $\dot{\Delta}^{\iota}_{\sigma(\Lambda,NW)}(\overline{\mathrm{B}}_\nu) \neq \emptyset$, by point (1) of Lemma 3.11 we get the existence of a set $A_\nu \subset \mathfrak{S}_n\lambda$ such that

$$D^{\iota}_{\sigma(\Lambda,NW)}(\overline{\kappa}_\nu) = \sum_{\delta \in A_\nu} \overline{\kappa}_\delta \text{ and hence } \dot{\Delta}^{\iota}_{\sigma(\Lambda,NW)}(\overline{\mathrm{B}}_\nu) = \bigsqcup_{\delta \in A_\nu} \overline{\mathrm{B}}_\delta.$$

Now by point (2) of Lemma 3.11, we must have $A_\nu \cap A_{\nu'} = \emptyset$ for any $\nu' \in \mathbb{Z}^m$ distinct from ν. Observe also that $\Delta_{\sigma(\Lambda,SE)}\left(\mathrm{B}_{(\mu_1,\dots,\mu_m)}\right)$ is a Demazure crystal by Lemma 3.9. We also get that

$$\bigsqcup_{(\mu_1,\dots,\mu_m) \in \mathbb{Z}_{\geq 0}^m} \dot{\Delta}^{\iota}_{\sigma(\Lambda,NW)}(\overline{\mathrm{B}}_{(\mu_m,\dots,\mu_1)})$$

is a disjoint union of atoms because ι is a crystal involution. This permits to conclude that the set

$$\mathcal{I} \subset \bigsqcup_{\lambda \in \mathcal{P}_n} B(\lambda) \times B(\lambda) = \psi(\mathcal{M}_{n,n}) \tag{22}$$

is indeed a disjoint union composed of Cartesian products sets of an opposite atom and a Demazure crystal which all lie in $\psi(\mathcal{M}_{n,n})$. Indeed, the Cartesian products sets so obtained from distint sequences $(\mu_1, \dots, \mu_m)$ cannot intersect.

Now, by Theorem 2.22 and its alternative formulation (14), the RSK correspondence on $\mathcal{M}_{n,n}$ restricts to a bijection

$$\psi : \mathcal{M}_{n,n}^{D_{\rho_\Lambda}} \to \bigsqcup_{(\mu_1,\dots,\mu_m) \in \mathbb{Z}^m} \iota\left(\overline{\mathrm{B}}_{(\mu_m,\dots,\mu_1)}\right) \times \mathrm{B}_{(\mu_1,\dots,\mu_m)}.$$

Then, the pre-image $\psi^{-1}(\mathcal{I}) \subset \mathcal{M}_{n,n}$ (which is well-defined by (22)) is obtained as the image of $\mathcal{M}_{n,n}^{D_{\rho_\Lambda}}$ under compositions of crystal operators of the form $\hat{f}_{j_1}^{k_1} \cdots \hat{f}_{j_b}^{k_b}$ and $\tilde{e}_{i_1}^{l_1} \cdots \tilde{e}_{i_a}^{l_a}$ (because the involution ι changes each $\tilde{f}_{n-i}$ into $\tilde{e}_i$). By Remark 2.21, this shows that $\psi^{-1}(\mathcal{I})$ is contained in $\mathcal{M}_{n,n}^{D_{\rho_\Lambda}}$. To get the equality $\psi^{-1}(\mathcal{I}) = \mathcal{M}_{n,n}^{D_{\rho_\Lambda}}$, it suffices to consider the characters of both sets which coincide thanks to Theorem 3.13, Equalities (17) and Remark (3.14). ■

3.4. The southeast approach for $\tilde{\mu}$. We now resume the notation of § 3.3 and in particular consider integers p and q such that $1 \le p \le q \le n$ and $n-q+1 \le p$ to perform an augmentation in the SE part of the staircase $\rho = (p, p-1, \ldots, 1)$ as an alternative way to describe the truncated staircase from § 3.2. As illustrated by the figure below, the element $\sigma(\Lambda(p,q), SE) \in \mathfrak{S}_q$ is obtained from top to bottom row reading of the SE part of the augmented staircase, each row being read from right to left. We thus get the following reduced decomposition in $\mathfrak{S}_q$:

$$\sigma(\Lambda(p,q), SE) = \prod_{i=1}^{p-(n-q)-1} (s_{i+n-p-1} \ldots s_i) \prod_{i=0}^{n-q} (s_{q-1} \ldots s_{p-(n-q)+i}). \tag{23}$$

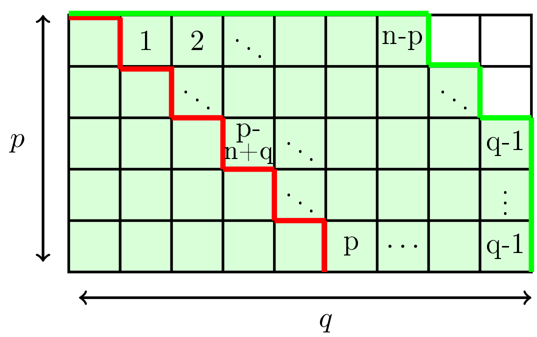


FIGURE 2. The labels in $\Lambda(p,q)/\rho$, $\Lambda(p,q) = (q^{n-q+1}, q-1, \ldots, , n-p+1)$, $\rho = (p, \ldots, 1)$ the maximal staircase contained in Λ, indicate the column index of Λ minus one. The reading word, from right to left and from the top to bottom, defines the reduced word $\sigma(\Lambda(p,q), SE)$.

Resuming the notation of Definition 3.6, let $\mu = (\mu_1, \ldots, \mu_p) \in \mathbb{Z}^p_{\ge 0}$ and $\lambda \in \mathcal{P}_p$ such that $\mu = \tau\lambda$, $\tau \in \mathfrak{S}_p^\lambda$, with $1 \le p \le q \le n$ and $p-(n-q) \ge 1 \Leftrightarrow q \ge n-p+1$. Let $\widehat{\sigma_0\tau} \in \mathfrak{S}_p^\lambda$ such that $\widehat{\sigma_0\tau}\lambda = \sigma_0^{[p]}\mu$ with $\sigma_0^{[p]}$ the longest element of $\mathfrak{S}_p$ (also recall that σ_0 is the longest element of $\mathfrak{S}_n$). We build on [1, Proposition 3] to show the following proposition.

Proposition 3.16. *The element $\tilde{\mu}$ introduced in Definition 3.6 satisfies*

$$\tilde{\mu} = (\sigma_0\tau)^{I_q}(\lambda, 0^{n-p}) = \pi_{\sigma(\Lambda(p,q),SE)}\pi_{\widehat{\sigma_0\tau}}(\lambda, 0^{n-p}) = \pi_{\sigma(\Lambda(p,q),SE)}(\sigma_0^{[p]}\mu, 0^{n-p})$$

where $\sigma(\Lambda(p,q),SE) \in \mathfrak{S}_q$ is defined as in (23). *Equivalently,* $\pi_{(\sigma_0\tau)^{I_q}}$ *and* $\pi_{\sigma(\Lambda(p,q),SE)}\pi_{\widehat{\sigma_0\tau}}$ *have the same action on* $(\lambda, 0^{n-p})$ *and therefore by Lemma 2.2 (2), and Lemma 2.5 they correspond to the same minimal representative in* $\mathfrak{S}_n^{(\lambda,0^{n-p})}$.

Proof: On the one hand we have

$$\sigma_0\tau(\lambda, 0^{n-p}) = \sigma_0(\mu, 0^{n-p}) = (0^{n-p}, \mu_p, \ldots, \mu_2, \mu_1) \tag{24}$$

We will show that the product

$$\prod_{i=1}^{p}(\pi_{i+n-p-1}\cdots\pi_i)\pi_{\widehat{\sigma_0\tau}}$$

of bubble sort operators has the same action on $(\lambda, 0^{n-p})$. We have

$$\prod_{i=1}^{p}(\pi_{i+n-p-1}\cdots\pi_i)\pi_{\widehat{\sigma_0\tau}}(\lambda, 0^{q-p}, 0^{n-q}) =$$

$$= \prod_{i=1}^{p}(\pi_{i+n-p-1}\cdots\pi_i)(\sigma_0^{[p]}\mu, 0^{q-p}, 0^{n-q}) \tag{25}$$

$$= \prod_{i=1}^{p-(n-q)-1}(\pi_{i+n-p-1}\cdots\pi_i)\cdot \tag{26}$$

$$\cdot\prod_{i=0}^{n-q}(\pi_{q-1+i}\cdots\pi_{p-(n-q)+i})(\mu_p, \ldots, \mu_{n-q+1}, \ldots, \mu_1, 0^{q-p}, 0^{n-q}) \tag{27}$$

The bubble sort operators in (25) act on the weak composition

$$(\sigma_0^{[p]}\mu, 0^{n-p}) = (\mu_p, \ldots, \mu_1, 0^{n-p}),$$

shifting $n-p$ times to the right each of the p entries of $\sigma_0^{[p]}\mu$. This is done by shifting $n-p$ times in $\sigma_0^{[p]}\mu$, first in (27), the last $n-q+1$ entries and then, in (26), the remaining first $p-(n-q)-1\geq 0$ entries. That is,

$$(26),(27) =$$

$$= \prod_{i=1}^{p-(n-q)-1} (\pi_{i+n-p-1}\cdots\pi_i)\cdot\prod_{i=0}^{n-q}(\pi_{q-1+i}\cdots\pi_{p-(n-q)+i})(\mu_p,\ldots,\mu_{n-q+1},\ldots,\mu_1,0^{q-p},0^{n-q})$$

$$= \prod_{i=1}^{p-(n-q)-1} (\pi_{i+n-p-1}\cdots\pi_i)\cdot$$

$$\cdot(\pi_{q-1}\cdots\pi_{p-(n-q)})\cdots(\pi_{n-2}\cdots\pi_{p-1})(\pi_{n-1}\cdots\pi_p)(\mu_p,\ldots,\mu_{n-q+1},\ldots,\mu_1,0^{q-p},0^{n-q}) \tag{28}$$

$$= \prod_{i=1}^{p-(n-q)-1} (\pi_{i+n-p-1}\cdots\pi_i)(\mu_p,\ldots,\mu_{n-q+2},0^{n-p},\mu_{n-q+1},\ldots,\mu_1)$$

$$= (\pi_{n-p}\cdots\pi_1)\cdots(\pi_{q-2}\cdots\pi_{p-(n-q)-1})(\mu_p,\ldots,\mu_{n-q+2},0^{n-p},\mu_{n-q+1},\ldots,\mu_1)$$

$$= (0^{n-p},\mu_p,\ldots,\mu_{n-q+2},\mu_{n-q+1},\ldots,\mu_1).$$

The product

$$\prod_{i=1}^{p-(n-q)-1} (\pi_{i+n-p-1}\cdots\pi_i)\cdot\prod_{i=0}^{n-q}(\pi_{q-1+i}\cdots\pi_{p-(n-q)+i})\pi_{\widehat{\sigma_0\tau}} \tag{29}$$

is a reduced decomposition in $\mathfrak{M}_n$ of an element from $\mathfrak{S}_n^{(\lambda,0^{n-p})}$ which acts on $(\lambda,0^{n-p})$ in the same way as $\sigma_0\tau$.

Therefore the minimal representative of $\sigma_0\tau$ in $\mathfrak{S}_n^{(\lambda,0^{n-p})}$ is the minimal representative of the element u with reduced decomposition in $\mathfrak{S}_n$

$$\prod_{i=1}^{p-(n-q)-1} (s_{i+n-p-1}\cdots s_i)\cdot\prod_{i=0}^{n-q}(s_{q-1+i}\cdots s_{p-(n-q)+i})\widehat{\sigma_0\tau}$$

and hence

$$u^{I_q} = \left(\prod_{i=1}^{p-(n-q)-1} (s_{i+n-p-1}\cdots s_i)\cdot\prod_{i=0}^{n-q}(s_{q-1+i}\cdots s_{p-(n-q)+i})\widehat{\sigma_0\tau}\right)^{I_q},$$

which can be calculated using Algorithm 3.1. Note that u^{I_q} and $(\sigma_0\tau)^{I_q}$ may not be equal in $\mathfrak{M}_n$, but they have the same action on $(\lambda,0^{n-p})$: indeed, if u_0 is the common minimal representative in $\mathfrak{S}_{(\lambda,0^{n-p})}$, the elements $\sigma_0\tau$ and u can be written in the form u_0u_λ and u_0x_λ with $u_\lambda, x_\lambda \in \mathfrak{S}_\lambda$ and $\ell(\sigma_0\tau) = \ell(u_0)+\ell(u_\lambda)$, $\ell(u) = \ell(u_0) + \ell(x_\lambda)$ respectively. By definition of Algorithm 3.1, we then have $\boldsymbol{(\sigma_0\tau)}^{I_q} = \boldsymbol{u_0}^{I_q}\boldsymbol{u_\lambda}^{I_q}$ and $\boldsymbol{u}^{I_q} = \boldsymbol{u_0}^{I_q}\boldsymbol{x_\lambda}^{I_q}$ (where the product is in $\mathfrak{M}_n$; see also Remark 5.5 from the Appendix). It follows that $(\sigma_0\tau)^{I_q}$ and u^{I_q} are (now in $\mathfrak{S}_n$) of the form $u_0^{I_q}v$ and $u_0^{I_q}z$ for some $v, z \in \mathfrak{S}_{(\lambda,0^{n-p})}$ respectively, and the second factors v and z thus have trivial action on $(\lambda, 0^{n-p})$.

Passing to $\mathfrak{M}_n$ we have a reduced decomposition

$$\boldsymbol{u} = \left(\prod_{i=1}^{p-(n-q)-1}(\pi_{i+n-p-1}\cdots\pi_i)\cdot\prod_{i=0}^{n-q}(\pi_{q-1+i}\cdots\pi_{p-(n-q)+i})\pi_{\widehat{\sigma_0\tau}}\right),$$

hence the first step of Algorithm 3.1 yields the word

$$\prod_{i=1}^{p-(n-q)-1}(\pi_{i+n-p-1}\cdots\pi_i)\cdot\left(\prod_{i=0}^{n-q}(\pi_{q-1+i}\cdots\pi_{p-(n-q)+i})\right)^{I_q}\pi_{\widehat{\sigma_0\tau}},$$

hence

$$\boldsymbol{u}^{I_q} = \prod_{i=1}^{p-(n-q)-1}(\pi_{i+n-p-1}\cdots\pi_i)\cdot\prod_{i=0}^{n-q}(\pi_{q-1}\cdots\pi_{p-(n-q)+i})\pi_{\widehat{\sigma_0\tau}} = \pi_{\sigma(\Lambda(p,q),SE)}\pi_{\widehat{\sigma_0\tau}}. \tag{30}$$

Note that we omitted in (29) the operators with indices $\geq q$, to obtain $\pi_{\sigma(\Lambda(p,q),SE)}$ with $\sigma(\Lambda(p,q), SE)$ the reduced decomposition in $\mathfrak{S}_q$ given in (23). Hence $(\sigma_0\tau)^{I_q}$ and $\pi_{\sigma(\Lambda(p,q),SE)}\pi_{\widehat{\sigma_0\tau}}$ have the same action on $(\lambda, 0^{n-p})$ and the reduced decomposition of the latter explicitly provides $(\sigma_0\tau)^{I_q}$ in $\mathfrak{S}_n^{(\lambda,0^{n-p})}$. This gives the desired result. ■

We now give a simple algorithm for computing $\tilde{\mu} = (\sigma_0\tau)^{I_q}(\lambda, 0^{n-p})$. Recall that $n - q + 1 \leq p$.

Theorem 3.17. *With the previous notation, we have*

$$\tilde{\mu} = \pi_{\sigma(\Lambda(p,q),SE)}(\sigma_0^{[p]}\mu, 0^{n-p}) = (0^{q-p}, \alpha_1, \ldots, \alpha_p, 0^{n-q})$$

where $\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbb{Z}^p_{\geq 0}$ is computed by the following algorithm: for i running from p to 1

- *for $j = i + 1, \ldots, p$, successively ignore in $\sigma_0^{[p]}\mu = (\mu_p, \ldots, \mu_1)$ the rightmost entry equal to α_j,*
- *set $k_i = \min\{i, n - q + 1\}$,*

- *then* α_i *is the maximum element among the remaining rightmost* k_i *entries of* $(\mu_p, \dots, \mu_1)$.

Example 3.18. *Let* $n = 6$, $p = 4$, $q = 5$, $n - q + 1 = 2$,

(*a*) *If* $\mu = (2, 1, 2, 3)$ *and* $\sigma_0^{[4]}\mu = (3, 2, 1, 2) = \pi_3(3, 2, 2, 1)$ *then* $\alpha = (1, 3, 2, 2)$ *is obtained as follows:* $\alpha_4 = 2$ *is the maximum among the rightmost* $\min\{4, 2\} = 2$ *entries of* $(3, 2, 1, 2)$, $\alpha_3 = 2$ *is the maximum among the rightmost* $\min\{3, 2\} = 2$ *entries of* $(3, 2, 1)$, $\alpha_2 = 3$ *is the maximum among the rightmost* $\min\{2, 2\} = 2$ *entries of* $(3, 1)$, $\alpha_1 = 1$ *is the maximum among the rightmost* $\min\{1, 2\} = 1$ *entries of* (1).

(*b*) *If* $\mu = (1, 2, 3, 2)$ *and* $\sigma_0^{[4]}\mu = (2, 3, 2, 1) = \pi_1(3, 2, 2, 1)$ *then* $\alpha = (1, 2, 3, 2)$ *is given by* $\alpha_4 = 2$ *is the maximum among the rightmost* $\min\{4, 2\} = 2$ *entries of* $(2, 3, 2, 1)$, $\alpha_3 = 3$ *is the maximum among the rightmost* $\min\{3, 2\} = 2$ *entries of* $(2, 3, 1)$, $\alpha_2 = 2$ *is the maximum among the rightmost* $\min\{2, 2\} = 2$ *entries of* $(2, 1)$, $\alpha_1 = 1$ *is the maximum among the rightmost* $\min\{1, 2\} = 1$ *entries of* (1).

Proof: The bubble sort operators in

$$\begin{aligned}
\pi_{\sigma(\Lambda(p,q),SE)} &= \prod_{i=1}^{p-(n-q)-1} (\pi_{i+n-p-1} \cdots \pi_i) \cdot \prod_{i=0}^{n-q} (\pi_{q-1} \cdots \pi_{p-(n-q)+i}) \\
&= (\pi_{n-p} \cdots \pi_1) \cdots (\pi_{q-1} \cdots \pi_{p-(n-q)-2})(\pi_{q-2} \cdots \pi_{p-(n-q)-1}) \cdot \qquad (31) \\
&\cdot (\pi_{q-1} \cdots \pi_{p-(n-q)}) \cdots (\pi_{q-1} \cdots \pi_{p-1})(\pi_{q-1} \cdots \pi_p) \qquad (32)
\end{aligned}$$

act on the weak composition

$$(\mu_p, \dots, \mu_1, 0^{q-p}, 0^{n-q}),$$

first in (32), shifting $q - p$ times to the right the last $n - q + 1$ entries, $\mu_{n-q+1}, \dots, \mu_1$, of $(\mu_p, \dots, \mu_{n-q+1}, \dots, \mu_1)$, and one checks that it sorts them in ascending order $(\mu_{n-q+1}, \dots, \mu_1)_\uparrow$, to get

$$(\mu_p, \dots, \mu_{n-q+2}, 0^{q-p}, (\mu_{n-q+1}, \dots, \mu_1)_\uparrow, 0^{n-q}). \qquad (33)$$

Let α_p be the entry q of (33). Next, the operators in (31) act similarly on the resulting vector (33), reordering in ascending order

$$\mu_{n-q+2} \text{ and } (\mu_{n-q+1}, \dots, \mu_1) \setminus \{\alpha_p\},$$

that is, ignoring the entry q, in $(\mu_{n-q+1}, \dots, \mu_1)_\uparrow$, (33), to get the vector

$$(\mu_p, \dots, \mu_{n-q+3}, 0^{q-p}, (\mu_{n-q+2}, (\mu_{n-q+1}, \dots, \mu_1) \setminus \{\alpha_p\})_\uparrow, \alpha_p, 0^{n-q}). \qquad (34)$$

Let α_{p-1} be the entry $q-1$ of (34). Then reordering μ_{n-q+3} with the just new previous vector (34), ignoring the entries $q-1$ and q, and so on. Observe that after some point, the number of remaining entries in $\sigma_0^{[p]}\mu$ is less than $n-q+1$ and just the i remaining entries are considered. ∎

Let us give two examples illustrating the notation and the results of Proposition 3.16 and Theorem 3.17:

Example 3.19. *Let $n=6$, $p=4$, $q=5$ and $\Lambda=(5^2,4,3)$ where $n-p+1=6-4+1=3<q$ and $n-q+1=2$,*

$$\Lambda = \begin{array}{ccccc} \square & 1 & 2 & & \\ \square & \square & 2 & 3 & \\ \square & \square & \square & 3 & 4 \\ \square & \square & \square & \square & 4 \end{array} \qquad \sigma(\Lambda(4,5),SE) = s_2s_1\, s_3s_2\, s_4s_3\, s_4,$$

(a) Let $\mu=(2,1,2,3)=\tau\lambda\in\mathbb{Z}^4_{\geq 0}$, and $\lambda=(3,2,2,1)$, $\widehat{\sigma_0\tau}\lambda=s_3\lambda=(3,2,1,2)=\sigma_0^{[4]}\mu$. Then on the one hand we have

$$\sigma_0\tau(3,2,2,1,0,0)=\sigma_0(2,1,2,3,0,0)=\sigma_0(\mu,0^2)=(0,0,3,2,1,2).$$

On the other hand, mimicking the proof of Proposition 3.16, we have $\pi_{\widehat{\sigma_0\tau}}=\pi_3$ and the product of the bubble sort operators

$$\prod_{i=1}^{p}(\pi_{i+n-p-1}\cdots\pi_i)\pi_{\widehat{\sigma_0\tau}}$$

is given in this case by $\pi_2\pi_1\pi_3\pi_2\pi_4\pi_3\pi_5\pi_4\pi_3$ and we have

$$\pi_2\pi_1\pi_3\pi_2\pi_4\pi_3\pi_5\pi_4\pi_3(3,2,2,1,0,0)=(0,0,3,2,1,2)=\sigma_0\tau(3,2,2,1,0,0).$$

The decomposition $s_2s_1s_3s_2s_4s_3s_5s_4s_3$ is reduced and lies in $\mathfrak{S}_6^{\lambda}$. We calculate

$$\begin{aligned}(\pi_2\pi_1\pi_3\pi_2\pi_4\pi_3\pi_5\pi_4\pi_3)^{I_5} &= \pi_2\pi_1\pi_3\pi_2\pi_4\pi_3\widehat{\pi_5}\pi_4\pi_3 = \pi_2\pi_1\pi_3\pi_2\pi_4\pi_3\pi_4\pi_3\\ &= \pi_2\pi_1\pi_3\pi_2\pi_4\pi_4\pi_3\pi_4 = \pi_2\pi_1\pi_3\pi_2\pi_4\pi_3\pi_4.\end{aligned}$$

Note that $(\pi_2\pi_1\pi_3\pi_2\pi_4\pi_3\pi_5\pi_4\pi_3)^{I_5} = \pi_{\sigma(\Lambda(4,5),SE)}$ *in this case. Now we have*

$$\begin{aligned}
\pi_{\sigma(\Lambda(4,5),SE)}\pi_3(3,2,2,1,0,0) &= \pi_{\sigma(\Lambda(4,5),SE)}(3,2,1,2,0,0) = \\
&= \pi_2\pi_1\pi_3\pi_2\pi_4\pi_3\pi_4(3,2,1,2,0,0) \\
&= \pi_2\pi_1\pi_3\pi_2\pi_4\pi_3(3,2,1,0,2,0) \\
&= \pi_2\pi_1\pi_3\pi_2\pi_4(3,2,0,1,2,0) \\
&= \pi_2\pi_1\pi_3\pi_2(3,2,0,1,2,0) \\
&= \pi_2\pi_1\pi_3(3,0,2,1,2,0) = \pi_2\pi_1(3,0,1,2,2,0) \\
&= \pi_2(0,3,1,2,2,0) \\
&= (0,1,3,2,2,0) = (0,\alpha,0).
\end{aligned}$$

Note that α *was also computed in part* (a) *of Example 3.18.*

(b) *Let* $\mu = (1,2,3,2) = \tau\lambda \in \mathbb{Z}^4_{\geq}0$, *and* $\lambda = (3,2,2,1)$, $\widehat{\sigma_0\tau}\lambda = s_1\lambda = (2,3,2,1) = \sigma_0^{[4]}\mu$. *Then on the one hand we have*

$$\sigma_0\tau(3,2,2,1,0,0) = \sigma_0(1,2,3,2,0,0) = \sigma_0(\mu,0^2) = (0,0,2,3,2,1).$$

On the other hand, mimicking the proof of Proposition 3.16, we have $\pi_{\widehat{\sigma_0\tau}} = \pi_1$ *and the product of the bubble sort operators* $\prod_{i=1}^p(\pi_{i+n-p-1}\cdots\pi_i)\pi_{\widehat{\sigma_0\tau}}$ *is given in this case by* $\pi_2\pi_1\pi_3\pi_2\pi_4\pi_3\pi_5\pi_4\pi_1$ *and we have*

$$\pi_2\pi_1\pi_3\pi_2\pi_4\pi_3\pi_5\pi_4\pi_1(3,2,2,1,0,0) = (0,0,2,3,2,1) = \sigma_0\tau(3,2,2,1,0,0).$$

The decomposition $s_2s_1s_3s_2s_4s_3s_5s_4s_1$ *is reduced and lies in* $\mathfrak{S}_6^\lambda$. *We calculate*

$$(\pi_2\pi_1\pi_3\pi_2\pi_4\pi_3\pi_5\pi_4\pi_1)^{I_5} = \pi_2\pi_1\pi_3\pi_2\pi_4\pi_3\pi_4\pi_1 = \pi_{\sigma(\Lambda(4,5),SE)}\pi_1.$$

Now we have

$$\begin{aligned}
\pi_{\sigma(\Lambda(4,5),SE)}\pi_1(3,2,2,1,0,0) &= \pi_2\pi_1\pi_3\pi_2\pi_4\pi_3\pi_4\pi_1(3,2,2,1,0,0) \\
&= (0,1,2,3,2,0) = (0,\alpha,0).
\end{aligned}$$

Note that α *was also computed in part* (b) *of Example 3.18.*

4. Last passage percolation in a Young diagram

4.1. LPP on rectangle Young diagrams. We resume the notation of § 2.4. Let $u_1,\dots,u_m$ and $v_1,\dots,v_m$ be two sets of real numbers in the interval $[0,1[$ and consider a family $w_{i,j}$ of independent random variables, with values in $\mathbb{Z}_{\geq 0}$, and such that

$$\mathbb{P}(w_{i,j} = k) = (1-u_iv_j)(u_iv_j)^k \text{ for any } k \in \mathbb{Z}_{\geq 0}. \tag{35}$$

In other words, each $w_{i,j}$ follows a geometric distribution of parameter u_iv_j. We then obtain a random matrix $\mathcal{W}$ with values in $\mathcal{M}_{m,n}$ whose entry at position (i,j) is defined as $w_{i,j}$ for $1 \leq i \leq m$ and $1 \leq j \leq m$. Since the random variables $w_{i,j}$ are independent, for any $A \in \mathcal{M}_{m,n}$ we get

$$\mathbb{P}(\mathcal{W} = A) = \left(\prod_{1\leq i\leq m, 1\leq j\leq n} (1 - u_iv_j) \right) (uv)^A$$

where $(uv)^A = \prod_{1\leq i\leq m,1\leq j\leq n}(u_iv_j)^{a_{i,j}}$.

Now consider the paths in the matrices in $\mathcal{M}_{m,n}$ starting at entry $(1,n)$ and ending at entry $(m,1)$ with possible steps $\longleftarrow$ or $\downarrow$. The length of such a path is defined as the sum of all the entries that it contains. Let us define de map perc which associates to each matrix A in $\mathcal{M}_{m,n}$ the maximum of the length path of all possible aforementioned paths in the matrix A. By Assertion 4 of Theorem 2.19, the integer $\mathrm{perc}(A)$ coincides with the longest row of the tableaux $P(A)$ and $Q(A)$. This is the last passage percolation associated to A. We then define the random variable $G = \mathrm{perc} \circ \mathcal{W}$. Thanks to the above observation and Theorem 2.19, it becomes easy to give the law of the random variable G. Set $\Delta_{m,n} = \prod_{1\leq i\leq m,1\leq j\leq n}(1 - u_iv_j)$. The following theorem was established in [17].

Theorem 4.1. *For any nonnegative integer k, we have*

$$\mathbb{P}(G = k) = \Delta_{m,n} \sum_{\lambda\in\mathcal{P}_{\min(m,n)}|\lambda_1=k} s_\lambda(u)s_\lambda(v).$$

In fact the results in [17] also give a law of large numbers of the variable G and also a Tracy-Widom renormalization theorem, both of which are outside the scope of this note.

4.2. LPP on staircases and non-symmetric Cauchy Kernel. Thanks to Theorem 2.22, the non-symmetric Cauchy kernel identity also yields an interesting last percolation model. This time, we assume $m = n$ and only consider independent random variables $w_{i,j}$ when $1 \leq j \leq i \leq n$ with geometric distributions as in (35). This defines a lower random square matrix $\mathcal{L}$ with nonnegative integer entries and we get

$$\mathbb{P}(\mathcal{L} = A) = \prod_{1\leq j\leq i\leq n} (1 - u_iv_j)(pq)^A.$$

One can interpret this model as follows. Consider paths from position $(1,n)$ to position $(n,1)$ where only the entries in the lower part of A contribute to the length of the paths. We can then define the random variable $L = \mathrm{perc} \circ \mathcal{L}$ and try to determine its law. Since Theorem 2.22 gives a bijective correspondence obtained as the restriction to lower triangular matrices of the RSK map defined on $\mathcal{M}_{n,n}$, the value of L still corresponds to the length of the largest part of the partitions appearing in the right hand side of (13). By Remark 2.23, this yields the following theorem.

Theorem 4.2. *For any nonnegative integer k we have*

$$\begin{aligned}\mathbb{P}(L = k) &= S_n \sum_{\mu \in \mathbb{Z}^n_{\geq 0} \mid \max(\mu) = k} \overline{\mathrm{s}}^{\mu}(u)\mathrm{s}_{\mu}(v) \\ &= S_n \sum_{\mu \in \mathbb{Z}^n \mid \max(\mu) = k} \overline{\mathrm{s}}_{\sigma_0(\mu)}(u_n, \dots, u_1)\mathrm{s}_{\mu}(v_1, \dots, v_n),\end{aligned}$$

where

$$S_n = \prod_{1 \leq j \leq i \leq n} (1 - u_i v_j).$$

4.3. LPP and parabolic restrictions in non-symmetric Cauchy Kernel. Given p and q as in § 2.1, one can similarly use Theorem 3.8 to study the percolation model on random matrices $\mathcal{T}_{p,q}$ with nonnegative random integer coefficients having zero entries in each position (i, j) such that $i \leq n - p$ and $j > q$. Each random variable $w_{i,j}$ with $i \geq n - p + 1$ and $j \leq q$ follows a geometric distribution of parameter $u_i v_j$. Using the same arguments as in § 3.1, we can obtain the law of the random variable $T_{p,q} = \mathrm{perc} \circ \mathcal{T}_{p,q}$.

Theorem 4.3. *For any nonnegative integer k, we have*

$$\mathbb{P}(T_{p,q} = k) = T_{p,q} \sum_{(\mu_1, \dots, \mu_p) \in \mathbb{Z}^p_{\geq 0} \mid \max(\mu) = k} \overline{\mathrm{s}}_{(\mu_p, \dots, \mu_1)}(u_n, \dots, u_{n-p+1})\mathrm{s}_{\widetilde{\mu}}(v_1, \dots, v_q),$$

where

$$T_{p,q} = \prod_{(i,j) \in D_{\Lambda(p,q)}} (1 - u_i v_j).$$

4.4. LPP and augmented staircases. We now resume the notation of § 3.3. For a fixed partition Λ in $\mathcal{P}_n$, we consider random matrices $\mathcal{A}_{\Lambda}$ with nonnegative random integer coefficients having zero entries in each position (i, j) such that $(i, j) \notin \Lambda$. Here again each random variable $w_{i,j}$ for $(i, j) \in \Lambda$ follows a geometric distribution of parameter $u_i v_j$. Let us define the random variable $A_\Lambda = \mathrm{perc} \circ \mathcal{A}_\Lambda$. Then, by Theorems 3.13 and 3.15, we get the law of A_Λ.

Theorem 4.4. *For any nonnegative integer k, we have*

$$\mathbb{P}(A_\Lambda = k) = T_\Lambda \sum_{(\mu_1,\ldots,\mu_m)\in\mathbb{Z}^m \mid \max(\mu)=k} {}_{\sigma(\Lambda,NW)}\overline{\mathrm{s}}_{(\mu_m,\ldots,\mu_1)}(u_n,\ldots,u_{n-m+1}) \\ D_{\sigma(\Lambda,SE)}\mathrm{s}_{(\mu_1,\ldots,\mu_m)}(v_1,\ldots,v_m),$$

where

$$T_\Lambda = \prod_{(i,j)\in D_\Lambda} (1-u_i v_j).$$

5. Appendix

Let (W,S) be a Coxeter system. Let $\mathfrak{M}_W$ be the attached *Coxeter monoid*, that is, the monoid with generators a copy $\mathbf{S}$ of S, the same braid relations as (W,S), and relations $\mathbf{s}^2 = \mathbf{s}$ for all $s \in S$ replacing the relations $s^2 = 1$ for all $s \in S$. Here by *braid relations* we mean the defining relations $st\cdots ts\cdots$, where $t \neq s$ and both sides are strictly alternating products of s and t with $m_{s,t} = m_{t,s}$ factors, where $m_{s,t}$ is the entry of the Coxeter matrix. These relations first appeared in work of Demazure [11, Section 5.6], and the Coxeter monoid was first investigated by Richardson and Springer [32, Section 3.10]. It is well-known (and a consequence of Matsumoto's Lemma) that there is a canonical set-theoretic bijection between W and $\mathfrak{M}_W$: it just sends any reduced decomposition of an element of W or $\mathfrak{M}_W$ to the same decomposition.

Let $I \subseteq S$, $w \in W$ and $s_1 s_2 \cdots s_k$ a reduced decomposition of w. Consider the subword $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ of $s_1 s_2 \cdots s_k$ consisting of those letters in $s_1 s_2 \cdots s_k$ lying in I. Set

$$M((s_1,s_2,\ldots,s_k),I) := \mathbf{s}_{i_1}\mathbf{s}_{i_2}\cdots\mathbf{s}_{i_\ell} \in \mathfrak{M}_W.$$

Lemma 5.1. *The element $M((s_1,s_2,\ldots,s_k),I)$ is independent of the choice $s_1 s_2 \cdots s_k$ of reduced decomposition for w, and we simply denote it by $M(w,I)$.*

Proof: By Matsumoto's Lemma, we know that any two reduced decompositions of w are related by applying a sequence of braid relations. It therefore suffices to show that applying a braid relation to a reduced word for w does not change the element of $\mathfrak{M}_W$ obtained by keeping only those letters in the words which lie in I.

Let $s_1 s_2 \cdots s_k$ be the first reduced decomposition of w, and $s_1' s_2' \cdots s_k'$ be the one obtained after application of a single braid relation. A braid relation involves only two letters $s, t \in S$ $(s \neq t)$. Then $s_1 s_2 \cdots s_k$ (as a word) is of the form $xsts \cdots y$ while $s'_1 s'_2 \cdots s'_k$ is of the form $xtst \cdots y$. If $s, t \notin I$, then it is clear that the two subwords of $s_1 s_2 \cdots s_k$ and $s'_1 s'_2 \cdots s'_k$ consisting of those letters which are not in I coincide, hence that $M((s_1, s_2, \ldots, s_k), I) = M((s'_1, s'_2, \ldots, s'_k), I)$. If $s, t \in I$, then the two subwords differ by a single braid relation, which holds in $\mathfrak{M}_W$, hence define the same element of $\mathfrak{M}_W$. Finally, if only one letter among s and t, say s, is in I, then since $t \notin I$, it follows that the substring $sts \cdots$ contributes k consecutive copies of s to the subword of $s_1 s_2 \cdots s_k$ obtained by deleting the letter not in S, while $tst \cdots$ contributes k or $k-1$ copies of s to the subword of $s'_1 s'_2 \cdots s'_k$, depending on whether $m_{s,t}$ is odd or even. Moreover, since s appears in both sides of the braid relation $sts \cdots = tst \cdots$, then at least one copy of s is contributed in each word. Thanks to the relation $\mathbf{s}^2 = \mathbf{s}$, these consecutive copies of s get reduced to $\mathbf{s}$ in $\mathfrak{M}_W$, again yielding $M((s_1, s_2, \ldots, s_k), I) = M((s'_1, s'_2, \ldots, s'_k), I)$. ■

We denote by $\leq$ the strong Bruhat order on W (or $\mathfrak{M}_W$). We recall that, for $u, v \in W$, the following three conditions are equivalent (see [4, Corollary 2.2.3])

(1) $u \leq v$,
(2) There is a reduced decomposition of v having a reduced decomposition of u as a subword,
(3) Every reduced decomposition of v has a reduced decomposition of u as a subword.

Lemma 5.2. (1) *Let $\mathbf{s_1 s_2} \cdots \mathbf{s_k}$ be a word in the generators of $\mathfrak{M}_W$ and $1 \leq i_1 < i_2 < \cdots < i_\ell \leq k$ such that $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ is a reduced decomposition of an element w of W. Let $\mathbf{s'_1 s'_2} \cdots \mathbf{s'_m}$ be a word obtained from $\mathbf{s_1 s_2} \cdots \mathbf{s_k}$ by applying a single defining relation of $\mathfrak{M}_W$. Then there is a sequence $1 \leq j_1 < j_2 < \cdots < j_\ell \leq m$ such that $s'_{j_1} s'_{j_2} \cdots s'_{j_\ell}$ is a reduced decomposition of w.*

(2) *Let $\mathbf{s_1 s_2} \cdots \mathbf{s_k}$ and $\mathbf{s'_1 s'_2} \cdots \mathbf{s'_m}$ be two (not necessarily reduced) words for the same element $\mathbf{w}$ of $\mathfrak{M}_W$. Let Ω_1 (resp. Ω_2) be the set of elements of W having a reduced decomposition which is a subword of $\mathbf{s_1 s_2} \cdots \mathbf{s_k}$ (resp. $\mathbf{s'_1 s'_2} \cdots \mathbf{s'_m}$). Then $\Omega_1 = \Omega_2$. In particular, this set $\Omega(\mathbf{w})$ depends only on $\mathbf{w}$, and we have*

$$\Omega(\mathbf{w}) = \{x \in W \mid x \leq w\}.$$

Proof: The second point is an immediate corollary of the first one; the last statement is used by taking as word $\mathbf{s_1 s_2 \cdots s_k}$ any reduced decomposition of $\mathbf{w}$.

Let us show the first point. The result is clear if the relation which is applied to the word $\mathbf{s_1 s_2 \cdots s_k}$ is $\mathbf{s} \to \mathbf{s}^2$ or $\mathbf{s}^2 \to \mathbf{s}$, since in the case where we have to consecutive copies of $\mathbf{s}$ in the first or the last word, then at most one can contribute to a reduced decomposition as ss is not reduced in W. Hence $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ also appears as a reduced word of $\mathbf{s'_1 s'_2 \cdots s'_m}$ in this case. Hence assume that the relation which is applied is a braid relation $\mathbf{w_1} = \mathbf{st} \cdots \to \mathbf{ts} \cdots = \mathbf{w_2}$. That is, we have $k = m$ and (as words) $\mathbf{s_1 s_2 \cdots s_k} = \mathbf{s_1 s_2 \cdots s_i w_1 s_j s_{j+1} \cdots s_k}$ while $\mathbf{s'_1 s'_2 \cdots s'_k} = \mathbf{s_1 s_2 \cdots s_i w_2 s_j s_{j+1} \cdots s_k}$.

Denote by p the number $\ell(\mathbf{w_1})$ of factors in either side of the braid relation. The subword $\mathbf{u}$ of $\mathbf{w_1}$ which contributes to the reduced word $s_{i_1} s_{i_2} \cdots s_{i_k}$ is necessarily and alternating product of s and t, otherwise it is not reduced. Moreover, it contributes a subword u of $s_{i_1} s_{i_2} \cdots s_{i_k}$ which is made of consecutive letters, the letters before that subword (resp. after that subword) coming from $\mathbf{s_1 s_2 \cdots s_i}$ (resp. $\mathbf{s_j s_{j+1} \cdots s_k}$). If $\ell(u) < p$, then $\mathbf{u}$ has a unique reduced decomposition, and $\mathbf{w_2}$ also has $\mathbf{u}$ as a subword. Hence the claim holds true in this case. If $\ell(u) = p$, then the whole left side $\mathbf{w_1}$ of the braid relation is contributed as a consecutive subword of $s_{i_1} s_{i_2} \cdots s_{i_k}$. Replacing that subword by the right side $ts \cdots$ of the braid relation yields the required subword of $\mathbf{s'_1 s'_2 \cdots s'_k}$. It stays reduced as it is just obtained from a reduced decomposition by applying a braid relation. ■

Proposition 5.3. *Let $w \in W$. The set $w_I^{\leq} := \{x \in W \mid x \in W_I \text{ and } x \leq w\}$ is equal to $\{x \in W \mid x \leq M(w, I)\}$. In particular, it has a unique maximal element for $\leq$, given by $M(w, I)$.*

Proof: Let $s_1 s_2 \cdots s_k$ be a reduced decomposition of w.

Let $x \in w_I^{\leq}$. Since $x \leq w$, there is a subword $s_{j_1} s_{j_2} \cdots s_{j_m}$, $1 \leq j_1 < j_2 < \cdots < j_m \leq k$ which is a reduced decomposition of x. Since $x \in W_I$, all the letters of $s_{j_1} s_{j_2} \cdots s_{j_m}$ lie in I. In particular, the reduced decomposition $s_{j_1} s_{j_2} \cdots s_{j_m}$ is a subword of the subword $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ of $s_1 s_2 \cdots s_k$ consisting of those letters which lie in I. Putting Lemmas 5.1 and 5.2 (2) together we get that $x \leq M(w, I)$.

To conclude the proof, it therefore suffices to see that $M(w, I) \leq w$. By Lemma 5.2 (2), we know that any (not necessarily reduced) word for $M(w, I)$ in $\mathfrak{M}_W$ has a subword which is a reduced word for $M(w, I)$, as this property is independent of the chosen word, and it holds if we take any reduced decomposition of $M(w, I)$ in $\mathfrak{M}_W$. But by definition of $M(w, I)$, there is a subword if $s_1 s_2 \cdots s_k$ which is a (not necessarily reduced) decomposition of $M(w, I)$ in $\mathfrak{M}_W$. Hence $s_1 s_2 \cdots s_k$ must have a reduced decomposition of $M(w, I)$ appearing as a subword. ■

Example 5.4. *Let W be of type A_3 and let $w = s_1 s_2 s_3 s_1 s_2$. The list of reduced words for w is given by*

- $s_1 s_2 s_3 s_1 s_2$,
- $s_1 s_2 s_1 s_3 s_2$,
- $s_2 s_1 s_2 s_3 s_2$,
- $s_2 s_1 s_3 s_2 s_3$,
- $s_2 s_3 s_1 s_2 s_3$.

Extracting the subword with letters in I from every such decomposition yields

- $s_1 s_2 s_1 s_2$,
- $s_1 s_2 s_1 s_2$,
- $s_2 s_1 s_2 s_2$,
- $s_2 s_1 s_2$,
- $s_2 s_1 s_2$.

In $\mathfrak{M}_W$ we get

- $\mathbf{s_1 s_2 s_1 s_2} = \mathbf{s_1 s_1 s_2 s_1} = \mathbf{s_1 s_2 s_1}$,
- $\mathbf{s_1 s_2 s_1 s_2} = \mathbf{s_1 s_1 s_2 s_1} = \mathbf{s_1 s_2 s_1}$,
- $\mathbf{s_2 s_1 s_2 s_2} = \mathbf{s_2 s_1 s_2} = \mathbf{s_1 s_2 s_1}$,
- $\mathbf{s_2 s_1 s_2} = \mathbf{s_1 s_2 s_1}$,
- $\mathbf{s_2 s_1 s_2} = \mathbf{s_1 s_2 s_1}$.

Note that the obtained is element is distinct from w_I, the element from the canonical decomposition $w = w^I w_I$, which is given here by $s_1 s_2$.

Remark 5.5. *It is a consequence of the definition of $M(w, I)$ and Lemma 5.1 that if $u, v \in W$ with $\ell(uv) = \ell(u) + \ell(v)$, then $M(uv, I) = M(u, I)M(v, I)$ (where the product is taken in $\mathfrak{M}_W$).*

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Olga Azenhas
CMUC, Department of Mathematics, University of Coimbra
E-mail address: oazenhas@mat.uc.pt

Thomas Gobet
Institut Denis Poisson Tours. Université de Tours Parc de Grandmont, 37200 Tours, France.
E-mail address: thomas.gobet@lmpt.univ-tours.fr

Cédric Lecouvey
Institut Denis Poisson Tours. Université de Tours Parc de Grandmont, 37200 Tours, France.
E-mail address: cedric.lecouvey@lmpt.univ-tours.fr