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MULTIPLICATION OF CLOSED BALLS IN $\mathbb{C}^n$

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ABSTRACT: Motivated by circular complex interval arithmetic, some operations on closed balls in $\mathbb{C}^n$ are considered. Essentially, the properties of possible multiplications for closed balls in $\mathbb{C}^n$, related either to the Hadamard product of vectors or to the 2-fold vector cross product when $n \in \{3, 7\}$, are studied. In addition, certain equations involving the defined multiplications are solved.

KEYWORDS: closed ball, multiplication, 2-fold vector cross product, Hadamard product of vectors.
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1. Introduction

Circular complex interval arithmetic, as can be seen in the books [2], due to Alefeld and Herzberger, and [19], by Petković and Petković, deals with closed balls in $\mathbb{C}$. Over the years, research related to interval mathematics, namely [13], [17] and [18], has been produced. In reference [13], Gargantini and Henrici apply circular complex interval arithmetic to the determination of polynomial zeros. Johansson, in [17], exhibits the advantages of ball arithmetic for rigorous algebraic computation with real numbers. Reference [18], whose editor is Pedrycz, compiles works in the context of granular computing. More recently, in [6], Beites, Nicolás and Vitória presented an arithmetic for closed balls in $\mathbb{R}^n$; the particular case $n = 2$ can be identified with $\mathbb{C}$.

In the present work, some operations on closed balls in $\mathbb{C}^n$ are considered. In particular, known results for closed balls in $\mathbb{R}^n$ are extended to closed balls in $\mathbb{C}^n$. To start with, in section 2, we recall definitions and results related to the complex vector space $\mathbb{C}^n$ endowed with a 2-fold vector cross product when $n \in \{3, 7\}$, closed balls and the Hadamard product of vectors. Vector cross products, as referred in [4] and other works cited therein, appear in control theory and in the description of spacecraft attitude control. The latter product, as also mentioned in reference [4], can be found in applications to machine learning and lossy compression algorithms for JPEG images.

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Recall that 2-fold vector cross products exist only for d-dimensional vector spaces with $d \in \{1, 3, 7\}$ ($d = 1$ is the trivial case), [10]. This fact is a consequence of the generalized Hurwitz Theorem: over a field of characteristic different from 2, if A is a finite dimensional composition algebra with identity (or Hurwitz algebra, [3]), then A is isomorphic either to the base field, a separable quadratic extension of the base field, a generalized quaternion algebra or a generalized octonion algebra, [16]. For other aspects connected with vector cross products, Hadamard products and composition algebras, see for instance, respectively, references: [4], [5], [7], [8], [9], [10], [14]; [15], [20]; [3], [11], [12].

In section 3, an addition for closed balls in $\mathbb{C}^n$ is examined. In section 4, properties of possible multiplications for these closed balls, related either to the Hadamard product of vectors or to the 2-fold vector cross product when $n \in \{3, 7\}$, are established. (Anti-)Commutativity, (power-)associativity, existence of neutral element and reciprocal of each element, and also its square root(s), are studied. Inclusion monotonicity – the basis for diverse applications of interval arithmetic, [2] – holds for two out of four of the considered multiplications, as well as for the addition. Moreover, the (sub)distributivity of each multiplication relative to the addition is analysed. Finally, certain equations involving the defined multiplications are solved.

2. Preliminaries

Throughout the work, consider the usual complex vector space $\mathbb{C}^n$. In addition, $\mathbb{C}^{n\times n}$ denotes the set of all $n \times n$ complex matrices, and we identify $\mathbb{C}^{n\times 1}$ with $\mathbb{C}^n$.

The complex vector space $\mathbb{C}^n$, together with the standard Hermitian inner product $(\cdot, \cdot)_h : (\mathbb{C}^n)^2 \to \mathbb{C}$, is a complex inner product space. Recall that, for all $x = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$, $y = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T \in \mathbb{C}^n$,

$$(x, y)_h = \sum_{t=1}^{n} x_t \overline{y_t}$$

and, for all $x, y, z \in \mathbb{C}^n$, $\alpha, \beta \in \mathbb{C}$,

$$(\alpha x + \beta y, z)_h = \alpha (x, z)_h + \beta (y, z)_h \text{ (linearity in the first coordinate)}, \quad (1)$$

$$(x, y)_h = \overline{(y, x)_h} \text{ (conjugate or Hermitian symmetry)}, \quad (2)$$

$$(x, x)_h \in \mathbb{R}_0^+ \text{ and } (x, x)_h = 0 \Leftrightarrow x = 0 \text{ (positive definiteness)}. \quad (3)$$

Also, (1) and (2) imply conjugate or Hermitian linearity in the second coordinate, that is,

$$(x, \alpha y + \beta z)_h = \overline{\alpha}(x, y)_h + \overline{\beta}(x, z)_h. \tag{4}$$

The complex vector space $\mathbb{C}^n$, together with the norm $\|\cdot\| : \mathbb{C}^n \to \mathbb{R}$ induced by $(\cdot, \cdot)_h$, is also a normed linear space. Recall that, for all $x \in \mathbb{C}^n$,

$$\|x\| = \sqrt{(x, x)_h},$$

where $\sqrt{\cdot}$ stands for the real, positive or null root, and, for all $x, y \in \mathbb{C}^n$, $\alpha \in \mathbb{C}$,

$$\|x\| \in \mathbb{R}_0^+ \text{ and } \|x\| = 0 \Leftrightarrow x = 0, \tag{5}$$

$$\|\alpha x\| = |\alpha|\|x\|, \tag{6}$$

$$\|x + y\| \leq \|x\| + \|y\| \text{ (triangle inequality)}, \tag{7}$$

where $|\cdot|$ stands for the modulus of a complex number.

The closed ball A in $\mathbb{C}^n$ with center $a \in \mathbb{C}^n$ and radius $r \in \mathbb{R}_0^+$ is defined by

$$A = \langle a; r \rangle = \{x \in \mathbb{C}^n : \|x - a\| \leq r\}.$$

The set of closed balls in $\mathbb{C}^n$ is denoted by $\mathfrak{B}$, and by $\mathfrak{B}^+$ or $\mathfrak{B}^0$ if, respectively, $r \in \mathbb{R}^+$ or $r = 0$.

Let $A = \langle a; r_1 \rangle, B = \langle b; r_2 \rangle \in \mathfrak{B}$. The closed balls A and B are equal ($A = B$) if set-theoretic equality holds, that is, $a = b$ and $r_1 = r_2$. A is contained in B ($A \subseteq B$) if set-theoretic inclusion is valid.

Let $x = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T \in \mathbb{C}^n$. The ∞-norm $\|\cdot\|_\infty$ of x is defined by $\|x\|_\infty = \max\limits_{i \in \{1,\dots,n\}} |x_i| \in \mathbb{R}_0^+$, where $|\cdot|$ stands for the modulus of a complex number.

Let $x = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T, y = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T \in \mathbb{C}^n$. The Hadamard (componentwise) product $\circ$ of x and y is $x \circ y \in \mathbb{C}^n$ with $i, 1$ entry, $i \in \{1, \dots, n\}$, given by $x_i y_i$.

Endow the complex vector space $\mathbb{C}^n$ with the nondegenerate symmetric bilinear form $(\cdot, \cdot)$ defined by

$$(x, y) = (x, \overline{y})_h.$$

Now consider $n \in \{3, 7\}$ and equip $\mathbb{C}^n$ also with the 2-fold vector cross product $\times : (\mathbb{C}^n)^2 \to \mathbb{C}^n$. Recall that $\times$ is the bilinear map that, for any $x, y \in \mathbb{C}^n$,

$$(x \times y, x) = (x \times y, y) = 0, \tag{8}$$

$$(x \times y, x \times y) = \begin{vmatrix} (x,x) & (x,y) \\ (y,x) & (y,y) \end{vmatrix}. \tag{9}$$

The trilinear map $(\cdot \times \cdot, \cdot)$ is skew-symmetric due to (8), and so $\times$ is anti-commutative, [10].

The 2-fold vector cross product in $\mathbb{C}^n$, $n \in \{3,7\}$, can be approached from a matrix point of view, [7, 14]. Let $a = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix}^T \in \mathbb{C}^n$. Consider the linear mapping

$$\begin{array}{rccl} a_\times : & \mathbb{C}^n & \to & \mathbb{C}^n \\ & x & \mapsto & a_\times(x) = a \times x. \end{array}$$

For each $a \in \mathbb{C}^n$, there exists a unique matrix $S_a \in \mathbb{C}^{n \times n}$ such that

$$a \times x = S_a x, \tag{10}$$

where, for $n = 3$,

$$S_a = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{11}$$

and, for $n = 7$,

$$S_a = \begin{bmatrix} 0 & -a_3 & a_2 & -a_5 & a_4 & -a_7 & a_6 \\ a_3 & 0 & -a_1 & -a_6 & a_7 & a_4 & -a_5 \\ -a_2 & a_1 & 0 & a_7 & a_6 & -a_5 & -a_4 \\ a_5 & a_6 & -a_7 & 0 & -a_1 & -a_2 & a_3 \\ -a_4 & -a_7 & -a_6 & a_1 & 0 & a_3 & a_2 \\ a_7 & -a_4 & a_5 & a_2 & -a_3 & 0 & -a_1 \\ -a_6 & a_5 & a_4 & -a_3 & -a_2 & a_1 & 0 \end{bmatrix}. \tag{12}$$

These skew-symmetric matrices, and other related matrices, were studied by Beites, Nicolás and Vitória in [7] for $n = 7$, namely regarding invertibility, index and nullspace. An earlier study for $n = 3$ can be found in [14], article due to Gross, Trenkler and Troschke.

3. Addition

Throughout this section, consider the usual complex vector space $\mathbb{C}^n$. Consider also the binary operation $+_{\mathfrak{B}} : \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$, hereinafter called addition $+_{\mathfrak{B}}$, defined by

$$A +_{\mathfrak{B}} B = \langle a; r_1 \rangle +_{\mathfrak{B}} \langle b; r_2 \rangle := \langle a + b; r_1 + r_2 \rangle.$$

The subsequent results establish several properties related to $+_{\mathfrak{B}}$.

Theorem 3.1. *The addition $+_{\mathfrak{B}}$ is commutative and associative. Moreover, $\langle 0; 0\rangle$ is the neutral element relative to $+_{\mathfrak{B}}$.*

Proof: Owing to the commutativity and to the associativity of the addition in $\mathbb{C}^n$, as well as to the commutativity and to the associativity of the addition in $\mathbb{C}$, it is straightforward to prove that, for all $A, B, C \in \mathfrak{B}$, $A +_{\mathfrak{B}} B = B +_{\mathfrak{B}} A$ and $(A +_{\mathfrak{B}} B) +_{\mathfrak{B}} C = A +_{\mathfrak{B}} (B +_{\mathfrak{B}} C)$. Taking into account the neutral elements of $\mathbb{C}^n$ and $\mathbb{C}$ relative to the respective additions, it is also direct to prove that $\langle 0; 0\rangle$ is the neutral element relative to $+_{\mathfrak{B}}$. ∎

Corollary 3.2. *The set of elements of $\mathfrak{B}$ which possess reciprocal relative to the addition $+_{\mathfrak{B}}$ is $\mathfrak{B}^0$. Furthermore, the reciprocal of $\langle a; 0\rangle \in \mathfrak{B}^0$ relative to $+_{\mathfrak{B}}$ is $\langle -a; 0\rangle$.*

Proof: Let $E = \langle 0; 0\rangle$. Let $A = \langle a; r_1\rangle \in \mathfrak{B}$. Suppose that $A' = \langle a'; r_1'\rangle \in \mathfrak{B}$ is the reciprocal of A relative to $+_{\mathfrak{B}}$. We have

$$A +_{\mathfrak{B}} A' = E \Leftrightarrow \langle a + a'; r_1 + r_1'\rangle = \langle 0; 0\rangle.$$

Thus, $a' = -a$ and $r_1' = -r_1$. ∎

Lemma 3.3. *Let $A, B \in \mathfrak{B}$. Then $A +_{\mathfrak{B}} B = \{x + y : x \in A \wedge y \in B\}$.*

Proof: ($\subseteq$) Let $A = \langle a; r_1\rangle$, $B = \langle b; r_2\rangle \in \mathfrak{B}$. Let $u \in A +_{\mathfrak{B}} B$. Then $\|u - (a + b)\| \leq r_1 + r_2$. If $r_1 + r_2 = 0$ then $u = a + b$ and the inclusion holds. If $r_1 + r_2 \neq 0$, then consider $u = v + (u - v)$ with $v = \alpha u + (1 - \alpha)(a + b) - b$, where $\alpha = \frac{r_1}{r_1 + r_2}$. Then we obtain

$$\|v - a\| = \alpha\|u - (a + b)\| \leq r_1$$

and

$$\|u - v - b\| = (1 - \alpha)\|u - (a + b)\| \leq r_2.$$

Consequently, $v \in A$ and $u - v \in B$, and, once again, the inclusion holds.

($\supseteq$) Let $x \in A = \langle a; r_1\rangle$ and $y \in B = \langle b; r_2\rangle$. Then $\|x - a\| \leq r_1$, $\|y - b\| \leq r_2$ and $\|x + y - (a + b)\| \leq \|x - a\| + \|y - b\| \leq r_1 + r_2$. Therefore, $x + y \in A +_{\mathfrak{B}} B = \langle a + b; r_1 + r_2\rangle$. ∎

Theorem 3.4. *The addition $+_{\mathfrak{B}}$ is inclusion monotonic.*

Proof: Let $A_m, B_m \in \mathfrak{B}$ such that $A_m \subseteq B_m$, $m \in \{1,2\}$. Hence, $A_1 +_{\mathfrak{B}} A_2 \subseteq B_1 +_{\mathfrak{B}} B_2$ since, applying Lemma 3.3 twice, we have

$$\begin{aligned} A_1 +_{\mathfrak{B}} A_2 &= \{x + y : x \in A_1 \wedge y \in A_2\} \\ &\subseteq \{x + y : x \in B_1 \wedge y \in B_2\} \\ &= B_1 +_{\mathfrak{B}} B_2. \end{aligned}$$

■

4. Multiplications

Throughout this section, unless stated otherwise, consider the usual complex vector space $\mathbb{C}^n$. We start with an auxiliary result for the following subsections, each devoted to a possible multiplication for closed balls in $\mathbb{C}^n$.

Lemma 4.1. *Let $A = \langle a; r_1 \rangle, B = \langle b; r_2 \rangle \in \mathfrak{B}$. Then $A \subseteq B$ if and only if $\|a - b\| \leq r_2 - r_1$. In particular, if A and B are concentric then $A \subseteq B$ if and only if $r_1 \leq r_2$.*

Proof: ($\Rightarrow$) Suppose that $A \subseteq B$. Assume that $\|a - b\| > r_2 - r_1$. Consider the line passing through a and b. This line intersects the border of A at a point x such that $\|x - b\| = \|a - b\| + \|x - a\| > r_2 - r_1 + r_1 = r_2$, which leads to the contradiction $x \notin B$.

($\Leftarrow$) Let $x \in A$. Then $\|x - a\| \leq r_1$. Hence, $x \in B$ since

$$\|x - b\| = \|x - a + a - b\| \leq \|x - a\| + \|a - b\| \leq r_2.$$

The particular result for concentric balls is immediate. ■

4.1. Multiplication $\circ_{\mathfrak{B},r}$. Consider the binary operation $\circ_{\mathfrak{B},r} : \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$, hereinafter called multiplication $\circ_{\mathfrak{B},r}$, defined by

$$A \circ_{\mathfrak{B},r} B = \langle a; r_1 \rangle \circ_{\mathfrak{B},r} \langle b; r_2 \rangle := \langle a \circ b + r_2 a + r_1 b; r_1 r_2 \rangle.$$

Even though $\circ_{\mathfrak{B},r}$ is not inclusion monotonic, the following properties hold for $\circ_{\mathfrak{B},r}$.

Theorem 4.2. *The multiplication $\circ_{\mathfrak{B},r}$ is commutative and associative. Moreover, $\langle 0; 1 \rangle$ is the neutral element relative to $\circ_{\mathfrak{B},r}$.*

Proof: As the Hadamard product $\circ$ of vectors is commutative and associative on $\mathbb{C}^n$, so is the multiplication $\circ_{\mathfrak{B},r}$. It is straightforward that, for all $\langle a; r_1 \rangle \in \mathfrak{B}$, $\langle a; r_1 \rangle = \langle a; r_1 \rangle \circ_{\mathfrak{B},r} \langle 0; 1 \rangle$. ■

Theorem 4.3. *The set of elements of $\mathfrak{B}$ which possess reciprocal relative to the multiplication $\circ_{\mathfrak{B},r}$ is $\mathfrak{R} = \{A = \langle a; r_1 \rangle \in \mathfrak{B}^+ : a = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix}^T \in$*

$\mathbb{C}^n \wedge a_l \neq -r_1, l \in \{1,\ldots,n\}\}$. *Furthermore, the reciprocal of* $\langle a; r_1\rangle \in \mathfrak{R}$ *relative to* $\circ_{\mathfrak{B},r}$ *is* $\langle b; \frac{1}{r_1}\rangle$ *with* $b = \begin{bmatrix} b_1 & \ldots & b_n \end{bmatrix}^T \in \mathbb{C}^n$ *such that* $b_l = -\frac{a_l}{r_1(r_1+a_l)}$, $l \in \{1,\ldots,n\}$.

Proof: Let A=$\langle a; r_1\rangle \in \mathfrak{B}^+$. Let $b = \begin{bmatrix} b_1 & \ldots & b_n \end{bmatrix}^T \in \mathbb{C}^n$ such that $\langle a; r_1\rangle \circ_{\mathfrak{B},r} \langle b; 1/r_1\rangle = \langle 0; 1\rangle$. As $a \circ b + \frac{1}{r_1}a + r_1 b = 0$, we get

$$a_l b_l + \frac{1}{r_1}a_l + r_1 b_l = 0, l \in \{1,\ldots,n\}. \qquad \blacksquare$$

Let $A \in \mathfrak{B}$. We define the powers of A relative to $\circ_{\mathfrak{B},r}$ by

$$A^0 = \langle 0; 1\rangle \text{ and } A^k = A^{k-1} \circ_{\mathfrak{B},r} A \text{ for } k \in \mathbb{N}.$$

Denote $\begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}^T$ by $a^{\circ 0}$ and $a^{\circ(k-1)} \circ a$ by $a^{\circ k}$ for $k \in \mathbb{N}$.

Theorem 4.4. *The multiplication* $\circ_{\mathfrak{B},r}$ *is power-associative.*

Proof: Due to Theorem 4.2, for all $A \in \mathfrak{B}$, $A^2 \circ_{\mathfrak{B},r} A = A \circ_{\mathfrak{B},r} A^2$ and $(A^2 \circ_{\mathfrak{B},r} A) \circ_{\mathfrak{B},r} A = A^2 \circ_{\mathfrak{B},r} A^2$ are valid. The result follows from [1]. $\blacksquare$

Theorem 4.5. *Let* $A = \langle a; r_1\rangle \in \mathfrak{B}$. *Relative to the multiplication* $\circ_{\mathfrak{B},r}$, *for all* $k \in \mathbb{N}$, $A^k = \langle \sum_{i=1}^k \binom{k}{i} r_1^{k-i} a^{\circ i}; r_1^k\rangle$.

Proof: We use induction on k. The equality obviously holds for $k = 1$. Suppose that it is true for k. Then we have

$$\begin{aligned} A^{k+1} &= A^k \circ_{\mathfrak{B},r} A \\ &= \langle \textstyle\sum_{l=1}^k \binom{k}{l} r_1^{k-l} a^{\circ l}; r_1^k\rangle \circ_{\mathfrak{B},r} \langle a; r_1\rangle \\ &= \langle \textstyle\sum_{l=1}^k \binom{k}{l} r_1^{k-l} a^{\circ(l+1)} + \sum_{l=1}^k \binom{k}{l} r_1^{k+1-l} a^{\circ l} + r_1^k a; r_1^{k+1}\rangle \\ &= \langle a^{\circ(k+1)} + \textstyle\sum_{l=2}^k \left[\binom{k}{l-1} + \binom{k}{l}\right] r_1^{k+1-l} a^{\circ l} + (k+1) r_1^k a; r_1^{k+1}\rangle \\ &= \langle \textstyle\sum_{l=1}^{k+1} \binom{k+1}{l} r_1^{k+1-l} a^{\circ l}; r_1^{k+1}\rangle. \quad \blacksquare \end{aligned}$$

Theorem 4.6. *Let* $A = \langle a; r_1\rangle \in \mathfrak{B}$ *with* $a = \begin{bmatrix} a_1 & \ldots & a_n \end{bmatrix}^T \in \mathbb{C}^n$. *The square roots of* A *relative to the multiplication* $\circ_{\mathfrak{B},r}$ *are given by* $A^{1/2} = \langle b; \sqrt{r_1}\rangle$, *with* $b = \begin{bmatrix} b_1 & \ldots & b_n \end{bmatrix}^T \in \mathbb{C}^n$ *such that* $b_l = -\sqrt{r_1} \pm \sqrt{r_1 + a_l}$ *for* $l \in \{1,\ldots,n\}$, *where* $\sqrt{\cdot}$ *stands, accordingly, for the real, positive or null root and for the complex roots.*

Proof: Let $B = \langle b; s\rangle \in \mathfrak{B}$ such that $A = B^2$. As $\langle a; r_1\rangle = \langle b^{\circ 2} + 2sb; s^2\rangle$, we have $s^2 = r_1$ and $b_l^2 + 2sb_l - a_l = 0$ for $l \in \{1,\ldots,n\}$. Thus, $b_l = -s \pm \sqrt{s^2 + a_l}$. $\blacksquare$

Theorem 4.7. *The multiplication $\circ_{\mathfrak{B},r}$ is distributive with respect to the addition $+_{\mathfrak{B}}$.*

Proof: Let $A = \langle a; r_1 \rangle$, $B = \langle b; r_2 \rangle$ and $C = \langle c; r_3 \rangle \in \mathfrak{B}$. Then we have

$$\begin{aligned} A \circ_{\mathfrak{B},r} (B +_{\mathfrak{B}} C) &= \langle a; r_1 \rangle \circ_{\mathfrak{B},r} (\langle b; r_2 \rangle +_{\mathfrak{B}} \langle c; r_3 \rangle) \\ &= \langle a \circ (b + c) + (r_2 + r_3)a + r_1(b + c); r_1(r_2 + r_3) \rangle \\ &= \langle a \circ b + a \circ c + r_2 a + r_3 a + r_1 b + r_1 c; r_1 r_2 + r_1 r_3 \rangle \\ &= \langle a \circ b + r_2 a + r_1 b; r_1 r_2 \rangle +_{\mathfrak{B}} \langle a \circ c + r_3 a + r_1 c; r_1 r_3 \rangle \\ &= (A \circ_{\mathfrak{B},r} B) +_{\mathfrak{B}} (A \circ_{\mathfrak{B},r} C). \quad \blacksquare \end{aligned}$$

Theorem 4.8. *Let $A = \langle a; r_1 \rangle \in \mathfrak{B}^+$ such that $a = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix}^T \in \mathbb{C}^n$ with $a_k \neq -r_1$, $k \in \{1, \dots, n\}$. Let $B = \langle b; r_2 \rangle \in \mathfrak{B}$. Then the unique solution of the equation $A \circ_{\mathfrak{B},r} X = B$ is given by $X = \langle x; r_3 \rangle \in \mathfrak{B}$, where $x = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T \in \mathbb{C}^n$, with*

$$x_k = (a_k + r_1)^{-1}(b_k - r_3 a_k), k \in \{1, \dots, n\}$$

and

$$r_3 = r_1^{-1} r_2.$$

Proof: From the definition of $\circ_{\mathfrak{B},r}$, the equation $A \circ_{\mathfrak{B},r} X = B$ assumes the form

$$\langle a \circ x + r_3 a + r_1 x; r_1 r_3 \rangle = \langle b; r_2 \rangle,$$

which leads to $(a_k + r_1)x_k = b_k - r_3 a_k$, $k \in \{1, \dots, n\}$, and $r_1 r_3 = r_2$. $\blacksquare$

Theorem 4.9. *Let $B = \langle b; r_2 \rangle$, $C = \langle c; r_1 \rangle \in \mathfrak{B}$. Then, the solutions of the equation $X^2 = B \circ_{\mathfrak{B},r} X + C$ are given by $X = \langle x; r_3 \rangle \in \mathfrak{B}$, where $x = [x_1 \dots x_n]^T \in \mathbb{C}^n$, with*

$$x_k = 2^{-1} \left(b_k - \sqrt{r_2^2 + 4r_1} \pm \sqrt{(b_k + r_2)^2 + 4(r_1 + c_k)} \right), \; k \in \{i, \dots, n\},$$

$$r_3 = 2^{-1} \left(r_2 + \sqrt{r_2^2 + 4r_1} \right) \textit{ and } r_3 = 0 \textit{ if } r_1 = 0.$$

Proof: From the definition of $\circ_{\mathfrak{B},r}$, the equation $X^2 = B \circ_{\mathfrak{B},r} X + C$ takes the form

$$\langle x \circ x + 2r_3 x; r_3^2 \rangle = \langle b \circ x + r_3 b + r_2 x + c; r_3 r_2 + r_1 \rangle.$$

So, $r_3^2 - r_2 r_3 - r_1 = 0$ and $r_3 = 2^{-1} \left(r_2 + \sqrt{r_2^2 + 4r_1} \right)$. Also, since $r_2 - \sqrt{r_2^2 + 4r_1} \in \mathbb{R}_0^+$ if and only if $r_1 = 0$, we have $r_3 = 0$ if $r_1 = 0$. On the other hand, for each $k \in \{1, \ldots, n\}$, we have $x_k^2 + (2r_3 - b_k - r_2)x_k - (r_3 b_k + c_k) = 0$, which leads to

$$x_k = 2^{-1}\left(b_k - \sqrt{r_2^2 + 4r_1} \pm \sqrt{(b_k + r_2)^2 + 4(r_1 + c_k)}\right). \qquad \blacksquare$$

Corollary 4.10. *Let $E = \langle 0; 1\rangle$. Then the solutions of the equation $X^2 = X + E$ are given by the golden balls $X = \langle x; \frac{1+\sqrt{5}}{2}\rangle$, with $x = \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix}^T \in \mathbb{R}^n$ such that $x_k \in \{-\sqrt{5}, 0\}$ for $k \in \{1, \ldots, n\}$ and where $\sqrt{\cdot}$ stands for the real, positive root.*

Proof: By Theorem 4.2, the equation $X^2 = X + E$ can be rewritten as $X^2 = E \circ_{\mathfrak{B},r} X + E$. Then, the result follows from Theorem 4.9. $\blacksquare$

4.2. Multiplication $\circ_{\mathfrak{B},c}$.

Consider the binary operation $\circ_{\mathfrak{B},c} : \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$, hereinafter called multiplication $\circ_{\mathfrak{B},c}$, defined by

$$A \circ_{\mathfrak{B},c} B = \langle a; r_1\rangle \circ_{\mathfrak{B},c} \langle b; r_2\rangle := \langle a \circ b; r_1\|b\|_\infty + r_2\|a\|_\infty + r_1 r_2\rangle.$$

Although $\circ_{\mathfrak{B},c}$ is not associative, as presented below, $\circ_{\mathfrak{B},c}$ possesses diverse properties.

Theorem 4.11. *The multiplication $\circ_{\mathfrak{B},c}$ is commutative. Moreover, $\langle 1; 0\rangle$ is the neutral element relative to $\circ_{\mathfrak{B},c}$.*

Proof: As the Hadamard product $\circ$ of vectors is commutative on $\mathbb{C}^n$, it is clear that $\circ_{\mathfrak{B},c}$ is commutative. Denote $\begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}^T \in \mathbb{C}^n$ by 1. Let $A = \langle a; r\rangle \in \mathfrak{B}$. Then we get $A \circ_{\mathfrak{B},c} \langle 1; 0\rangle = \langle a \circ 1; r\rangle = A$. $\blacksquare$

Theorem 4.12. *The set of elements of $\mathfrak{B}$ which possess reciprocal relative to the multiplication $\circ_{\mathfrak{B},c}$ is $\mathfrak{R} = \{A = \langle a; 0\rangle \in \mathfrak{B}^0 : a = \begin{bmatrix} a_1 & \ldots & a_n \end{bmatrix}^T \in \mathbb{C}^n \wedge a_l \neq 0, l \in \{1, \ldots, n\}\}$. Furthermore, the reciprocal of $\langle a; 0\rangle \in \mathfrak{R}$ relative to $\circ_{\mathfrak{B},c}$ is $\langle b; 0\rangle$ with $b_l = a_l^{-1}$, $l \in \{1, \ldots, n\}$.*

Proof: Let $A = \langle a; r\rangle \in \mathfrak{B}$. Suppose that $B = \langle b; s\rangle$ is the reciprocal of A relative to $\circ_{\mathfrak{B},c}$. Then we have

$$A \circ_{\mathfrak{B},c} B = \langle a; r\rangle \circ_{\mathfrak{B},c} \langle b; s\rangle = \langle a \circ b; r\|b\|_\infty + s\|a\|_\infty + rs\rangle = \langle 1; 0\rangle.$$

Hence, $b_l = a_l^{-1}$, $l \in \{1, \ldots, n\}$, whenever $a_l \neq 0$. In addition, $r\|b\|_\infty + s\|a\|_\infty + rs = 0$, which allows to arrive at $r = s = 0$. $\blacksquare$

Let $A \in \mathfrak{B}$. We define the powers of A relative to $\circ_{\mathfrak{B},c}$ by

$$A^0 = \langle 1; 0\rangle \text{ and } A^k = A^{k-1} \circ_{\mathfrak{B},c} A \text{ for } k \in \mathbb{N}.$$

Denote $\begin{bmatrix} 1 & \dots & 1 \end{bmatrix}^T$ by $a^{\circ 0}$ and $a^{\circ(k-1)} \circ a$ by $a^{\circ k}$ for $k \in \mathbb{N}$.

Theorem 4.13. *The multiplication $\circ_{\mathfrak{B},c}$ is power-associative.*

Proof: To prove that, for all $A \in \mathfrak{B}$ and for all $m, s \in \mathbb{N}$, $A^s \circ_{\mathfrak{B},c} A^m = A^{s+m}$, invoking [1], it suffices to show that $A^2 \circ_{\mathfrak{B},c} A = A \circ_{\mathfrak{B},c} A^2$ and $(A^2 \circ_{\mathfrak{B},c} A) \circ_{\mathfrak{B},c} A = A^2 \circ_{\mathfrak{B},c} A^2$. By Theorem 4.11, the former equality holds. As for the latter equality, let $A = \langle a; r\rangle \in \mathfrak{B}$. On the one hand, we obtain

$$\begin{aligned} A^2 \circ_{\mathfrak{B},c} A &= \langle a^{\circ 2}; 2r\|a\|_\infty + r^2\rangle \circ_{\mathfrak{B},c} \langle a; r\rangle \\ &= \langle a^{\circ 3}; r\|a^{\circ 2}\|_\infty + \|a\|_\infty(2r\|a\|_\infty + r^2) + r(2r\|a\|_\infty + r^2)\rangle \\ &= \langle a^{\circ 3}; 3r\|a\|_\infty^2 + 3r^2\|a\|_\infty + r^3\rangle \end{aligned}$$

and

$$\begin{aligned} (A^2 \circ_{\mathfrak{B},c} A) \circ_{\mathfrak{B},c} A &= \langle a^{\circ 3}; 3r\|a\|_\infty^2 + 3r^2\|a\|_\infty + r^3\rangle \circ_{\mathfrak{B},c} \langle a; r\rangle \\ &= \langle a^{\circ 4}; 4r\|a\|_\infty^3 + 6r^2\|a\|_\infty^2 + 4r^3\|a\|_\infty + r^4\rangle. \end{aligned}$$

On the other hand, we get

$$\begin{aligned} A^2 \circ_{\mathfrak{B},c} A^2 &= \langle a^{\circ 2}; 2r\|a\|_\infty + r^2\rangle \circ_{\mathfrak{B},c} \langle a^{\circ 2}; 2r\|a\|_\infty + r^2\rangle \\ &= \langle a^{\circ 4}; 4r\|a\|_\infty^3 + 6r^2\|a\|_\infty^2 + 4r^3\|a\|_\infty + r^4\rangle. \quad \blacksquare \end{aligned}$$

Theorem 4.14. *Let $A = \langle a; r\rangle \in \mathfrak{B}$. Relative to the multiplication $\circ_{\mathfrak{B},c}$, for all $k \in \mathbb{N}$, $A^k = \langle a^{\circ k}; (\|a\|_\infty + r)^k - \|a\|_\infty^k\rangle$.*

Proof: We proceed by induction on k. The equality clearly holds for $k = 1$. Suppose that it is also valid for k. Then we have

$$\begin{aligned} A^{k+1} &= A^k \circ_{\mathfrak{B},c} A \\ &= \langle a^{\circ k}; \textstyle\sum_{l=1}^k \binom{k}{l} r^l \|a\|_\infty^{k-l}\rangle \circ_{\mathfrak{B},c} \langle a; r\rangle \\ &= \langle a^{\circ(k+1)}; r\|a^{\circ k}\|_\infty + \textstyle\sum_{l=1}^k \binom{k}{l} r^l \|a\|_\infty^{k+1-l} + \sum_{l=1}^k \binom{k}{l} r^{l+1}\|a\|_\infty^{k-l}\rangle \\ &= \langle a^{\circ(k+1)}; (k+1)r\|a\|_\infty^k + \textstyle\sum_{l=2}^k \left[\binom{k}{l} + \binom{k}{l-1}\right] r^l \|a\|_\infty^{k+1-l} + r^{k+1}\rangle \\ &= \langle a^{\circ(k+1)}; \textstyle\sum_{l=1}^{k+1} \binom{k+1}{l} r^l \|a\|_\infty^{k+1-l}\rangle \\ &= \langle a^{\circ(k+1)}; (\|a\|_\infty + r)^{k+1} - \|a\|_\infty^{k+1}\rangle. \quad \blacksquare \end{aligned}$$

Theorem 4.15. *Let $A = \langle a; r\rangle \in \mathfrak{B}$ with $a = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix}^T \in \mathbb{C}^n$. The square roots of A relative to the multiplication $\circ_{\mathfrak{B},c}$ are given by $A^{1/2} = \langle a^{\circ 1/2}; \sqrt{r + \|a\|_\infty} - \sqrt{\|a\|_\infty}\rangle$, with $a^{\circ 1/2} = (\sqrt{a_1}, \dots, \sqrt{a_n})$, where $\sqrt{\cdot}$ stands, accordingly, for the real, positive or null root and for the complex roots.*

Proof: Let $B = \langle b; s\rangle \in \mathfrak{B}$ such that $A = B^2$. From $\langle a; r\rangle = \langle b^{\circ 2}; s^2 + 2s\|b\|_\infty\rangle$ we have $b = a^{\circ 1/2}$ and $s^2 + 2\sqrt{\|a\|_\infty} s - r = 0$. ■

Theorem 4.16. *The multiplication $\circ_{\mathfrak{B},c}$ is inclusion monotonic.*

Proof: Let $A_m = \langle a_m; r_m\rangle$, $B_m = \langle b_m; s_m\rangle \in \mathfrak{B}$ such that $A_m \subseteq B_m$, $m \in \{1, 2\}$. We aim to prove that $A_1 \circ_{\mathfrak{B},c} A_2 \subseteq B_1 \circ_{\mathfrak{B},c} B_2$. By Lemma 4.1, $\|b_m - a_m\| \leq s_m - r_m$, $m \in \{1, 2\}$, and it is enough to prove that

$$\|b_1 \circ b_2 - a_1 \circ a_2\| \leq s_1\|b_2\|_\infty + s_2\|b_1\|_\infty + s_1 s_2 - r_1\|a_2\|_\infty - r_2\|a_1\|_\infty - r_1 r_2.$$

Observe that

$$\begin{aligned}
\|b_1 \circ b_2 - a_1 \circ a_2\| &= \|b_1 \circ b_2 - b_1 \circ a_2 + b_1 \circ a_2 - a_1 \circ a_2\| \\
&\leq \|b_1 \circ (b_2 - a_2)\| + \|(b_1 - a_1) \circ a_2\| \\
&\leq \|b_1\|_\infty\|b_2 - a_2\| + \|a_2\|_\infty\|b_1 - a_1\| \\
&\leq \|b_1\|_\infty(s_2 - r_2) + \|a_2\|_\infty(s_1 - r_1).
\end{aligned}$$

In addition, we have

$$\begin{aligned}
s_1\|a_2\|_\infty &\leq s_1\|b_2\|_\infty + s_1\|a_2 - b_2\|_\infty \leq s_1\|b_2\|_\infty + s_1(s_2 - r_2), \\
-r_2\|b_1\|_\infty &\leq -r_2\|a_1\|_\infty + r_2\|a_1 - b_1\|_\infty \leq -r_2\|a_1\|_\infty + r_2(s_1 - r_1).
\end{aligned}$$

The former and the latter inequalities lead to the result. ■

Theorem 4.17. *The multiplication $\circ_{\mathfrak{B},c}$ is subdistributive with respect to the addition $+_{\mathfrak{B}}$.*

Proof: Let $A = \langle a; r_1\rangle$, $B = \langle b; r_2\rangle, C = \langle c; r_3\rangle \in \mathfrak{B}$. Applying Lemma 4.1,

$$\begin{aligned}
A \circ_{\mathfrak{B},c} & (B +_{\mathfrak{B}} C) = \langle a; r_1\rangle \circ_{\mathfrak{B},c} (\langle b; r_2\rangle +_{\mathfrak{B}} \langle c; r_3\rangle) \\
&= \langle a \circ (b + c); r_1\|b + c\|_\infty + (r_2 + r_3)\|a\|_\infty + r_1(r_2 + r_3)\rangle \\
&= \langle a \circ b + a \circ c; r_2\|a\|_\infty + r_1 r_2 + r_3\|a\|_\infty + r_1 r_3 + r_1\|b + c\|_\infty\rangle \\
&\subseteq \langle a \circ b + a \circ c; r_1\|b\|_\infty + r_2\|a\|_\infty + r_1 r_2 + r_1\|c\|_\infty + r_3\|a\|_\infty + r_1 r_3\rangle \\
&= (A \circ_{\mathfrak{B},c} B) +_{\mathfrak{B}} (A \circ_{\mathfrak{B},c} C). \quad ■
\end{aligned}$$

Theorem 4.18. *Let $A = \langle a; r_1\rangle \in \mathfrak{B}$ such that $a = \begin{bmatrix} a_1 & \ldots & a_n \end{bmatrix}^T \in \mathbb{C}^n$ with $a_k \neq 0$, $k \in \{1, \ldots, n\}$. Let $B = \langle b; r_2\rangle \in \mathfrak{B}$. Then the unique solution of the equation $A \circ_{\mathfrak{B},c} X = B$ is given by $X = \langle x; r_3\rangle \in \mathfrak{B}$, where $x = \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix}^T \in \mathbb{C}^n$, with*

$$x_k = a_k^{-1} b_k, k \in \{1, \ldots, n\},$$

and

$$r_3 = (\|a\|_\infty + r_1)^{-1}(r_2 - r_1\|x\|_\infty).$$

Proof: The rewriting of the equation $A \circ_{\mathfrak{B},c} X = B$ leads to

$$\langle a \circ x; r_1\|x\|_\infty + r_3\|a\|_\infty + r_1 r_3\rangle = \langle b; r_2\rangle$$

From here, we have $a_k x_k = b_k$, $k \in \{1, \ldots, n\}$, and $(\|a\|_\infty + r_1)r_3 = r_2 - r_1\|x\|_\infty$. ∎

Theorem 4.19. *Let $B = \langle b; r_2\rangle$, $C = \langle c; r_1\rangle \in \mathfrak{B}$. Then, the solutions of the equation $X^2 = B \circ_{\mathfrak{B},c} X + C$ are given by $X = \langle x; r_3\rangle \in \mathfrak{B}$, where $x = \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix}^T \in \mathbb{C}^n$, with*

$$x_k = 2^{-1}\left(b_k \pm \sqrt{b_k^2 + 4c_k}\right),\ k \in \{1, \ldots, n\},$$

$$r_3 = 2^{-1}\left(r_2 + \|b\|_\infty - 2\|x\|_\infty + \sqrt{(r_2 + \|b\|_\infty - 2\|x\|_\infty)^2 + 4(r_1 + r_2\|x\|_\infty)}\right),$$

where $\sqrt{\cdot}$ stands for the real, positive root, and

$$r_3 = 0 \text{ if } r_1 = 0 \text{ and } r_2\|x\|_\infty = 0.$$

Proof: From the definition of $\circ_{\mathfrak{B},c}$, the equation $X^2 = B \circ_{\mathfrak{B},c} X + C$ takes the form

$$\langle x \circ x; 2r_3\|x\|_\infty + r_3^2\rangle = \langle b \circ x + c; r_2\|x\|_\infty + r_3\|b\|_\infty + r_2 r_3 + r_1\rangle.$$

So, $r_3^2 + (2\|x\|_\infty - \|b\|_\infty - r_2)\, r_3 - \|x\|_\infty r_2 - r_1 = 0$. Solving this equation, we have

$$r_3 = 2^{-1}\left(\|b\|_\infty + r_2 - 2\|x\|_\infty + \sqrt{(\|b\|_\infty + r_2 - 2\|x\|_\infty)^2 + 4(r_1 + r_2\|x\|_\infty)}\right).$$

Notice that $\|b\|_\infty + r_2 - 2\|x\|_\infty - \sqrt{(\|b\|_\infty + r_2 - 2\|x\|_\infty)^2 + 4(r_1 + r_2\|x\|_\infty)} \in \mathbb{R}_0^+$ if and only if $r_1 + r_2\|x\|_\infty = 0$, in which case $r_3 = 0$. On the other hand, for each $k \in \{1, \ldots, n\}$ we must have that $x_k^2 = b_k x_k + c_k$, which leads to $x_k = 2^{-1}(b_k \pm \sqrt{b_k^2 + 4c_k})$. ∎

Corollary 4.20. *Let $E = \langle 1; 0\rangle$. Then the solutions of the equation $X^2 = X + E$ are given by the balls $X = \langle x; 0\rangle$, with $x = \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix}^T \in \mathbb{R}^n$ such that $x_k = 2^{-1}(1 \pm \sqrt{5})$ for $k \in \{1, \ldots, n\}$, where $\sqrt{\cdot}$ stands for the real, positive root.*

Proof: By Theorem 4.11, the equation $X^2 = X + E$ can be rewritten as $X^2 = E \circ_{\mathfrak{B},c} X + E$. Then, the result follows from Theorem 4.19. ∎

4.3. Multiplication $\times_{\mathfrak{B},r}$. Throughout this subsection, consider the usual complex vector space $\mathbb{C}^n$ with $n \in \{3, 7\}$. Consider also the binary operation $\times_{\mathfrak{B},r} : \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$, hereinafter called multiplication $\times_{\mathfrak{B},r}$, defined by

$$A \times_{\mathfrak{B},r} B = \langle a; r_1 \rangle \times_{\mathfrak{B},r} \langle b; r_2 \rangle := \langle a \times b + r_2 a + r_1 b; r_1 r_2 \rangle.$$

Despite the fact that commutativity, anti-commutativity, associativity and inclusion monotonicity do not hold, $\times_{\mathfrak{B},r}$ satisfies the subsequent properties.

Theorem 4.21. *The neutral element relative to the multiplication $\times_{\mathfrak{B},r}$ is $\langle 0; 1 \rangle$.*

Proof: Let $A = \langle a; r_1 \rangle \in \mathfrak{B}$. Then we have

$$\langle a; r_1 \rangle \times_{\mathfrak{B},r} \langle 0; 1 \rangle = \langle a; r_1 \rangle = \langle 0; 1 \rangle \times_{\mathfrak{B},r} \langle a; r_1 \rangle. \qquad \blacksquare$$

Corollary 4.22. *The set of elements of $\mathfrak{B}$ which possess reciprocal relative to the multiplication $\times_{\mathfrak{B},r}$ is $\mathfrak{B}^+$. Furthermore, the reciprocal of $\langle a; r_1 \rangle \in \mathfrak{B}^+$ relative to $\times_{\mathfrak{B},r}$ is $\langle -\frac{1}{r_1^2} a; \frac{1}{r_1} \rangle$.*

Proof: Let $A = \langle a; r_1 \rangle \in \mathfrak{B}^+$. Then we obtain

$$\langle a; r_1 \rangle \times_{\mathfrak{B},r} \left\langle -\frac{1}{r_1^2} a; \frac{1}{r_1} \right\rangle = \langle 0; 1 \rangle = \left\langle -\frac{1}{r_1^2} a; \frac{1}{r_1} \right\rangle \times_{\mathfrak{B},r} \langle a; r_1 \rangle. \qquad \blacksquare$$

Let $A \in \mathfrak{B}$. We define the powers of A relative to $\times_{\mathfrak{B},r}$ by

$$A^0 = \langle 0; 1 \rangle \text{ and } A^k = A^{k-1} \times_{\mathfrak{B},r} A \text{ for } k \in \mathbb{N}.$$

Theorem 4.23. *The multiplication $\times_{\mathfrak{B},r}$ is power-associative.*

Proof: To prove that, for all $A \in \mathfrak{B}$ and for all $m, s \in \mathbb{N}$, $A^s \times_{\mathfrak{B},r} A^m = A^{s+m}$, invoking [1], it suffices to show that $A^2 \times_{\mathfrak{B},r} A = A \times_{\mathfrak{B},r} A^2$ and $(A^2 \times_{\mathfrak{B},r} A) \times_{\mathfrak{B},r} A = A^2 \times_{\mathfrak{B},r} A^2$.

Let $A = \langle a; r_1 \rangle \in \mathfrak{B}$. Then we get

$$\begin{aligned} A^2 \times_{\mathfrak{B},r} A &= \langle 2 r_1 a; r_1^2 \rangle \times_{\mathfrak{B},r} \langle a; r_1 \rangle \\ &= \langle 3 r_1^2 a; r_1^3 \rangle \\ &= \langle a; r_1 \rangle \times_{\mathfrak{B},r} \langle 2 r_1 a; r_1^2 \rangle \\ &= A \times_{\mathfrak{B},r} A^2 \end{aligned}$$

and

$$\begin{aligned}(A^2 \times_{\mathfrak{B},r} A) \times_{\mathfrak{B},r} A &= \langle 3r_1^2 a; r_1^3\rangle \times_{\mathfrak{B},r} \langle a; r_1\rangle \\ &= \langle 4r_1^3 a; r_1^4\rangle \\ &= \langle 2r_1 a; r_1^2\rangle \times_{\mathfrak{B},r} \langle 2r_1 a; r_1^2\rangle \\ &= A^2 \times_{\mathfrak{B},r} A^2. \qquad \blacksquare\end{aligned}$$

Theorem 4.24. *Let $A = \langle a; r_1\rangle \in \mathfrak{B}$. Relative to the multiplication $\times_{\mathfrak{B},r}$, for all $k \in \mathbb{N}$, $A^k = \langle k r_1^{k-1} a; r_1^k\rangle$.*

Proof: Let $A = \langle a; r_1\rangle \in \mathfrak{B}$. It is easy to see that the base case holds. Concerning the induction step, we have

$$A^k = A^{k-1} \times_{\mathfrak{B},r} A = \langle (k-1) r_1^{k-2} a; r_1^{k-1}\rangle \times_{\mathfrak{B},r} \langle a; r_1\rangle = \langle k r_1^{k-1} a; r_1^k\rangle. \qquad \blacksquare$$

Theorem 4.25. *Let $A = \langle a; r\rangle \in \mathfrak{B}^+$. The square root of A relative to the multiplication $\times_{\mathfrak{B},r}$ is $A^{1/2} = \left\langle \frac{1}{2\sqrt{r}} a; \sqrt{r}\right\rangle$, where $\sqrt{\cdot}$ stands for the real, positive root.*

Proof: Let $A = \langle a; r\rangle \in \mathfrak{B}^+$. Let $B = \langle b; s\rangle \in \mathfrak{B}$ such that $B^2 = A$. Thus, $s^2 = r$ and $S_b b + 2sb = a$, which leads to the result by [7, Proposition 4, Property 6] and [14, Property (A)]. $\blacksquare$

Theorem 4.26. *The multiplication $\times_{\mathfrak{B},r}$ is distributive relative to the addition $+_{\mathfrak{B}}$.*

Proof: Let $A = \langle a; r_1\rangle, B = \langle b; r_2\rangle, C = \langle c; r_3\rangle \in \mathfrak{B}$. Then we obtain

$$\begin{aligned}A \times_{\mathfrak{B},r} (B +_{\mathfrak{B}} C) &= \langle a; r_1\rangle \times_{\mathfrak{B},r} \langle b + c; r_2 + r_3\rangle \\ &= \langle a \times (b + c) + (r_2 + r_3) a + r_1 (b + c); r_1 (r_2 + r_3)\rangle \\ &= \langle a \times b + r_2 a + r_1 b + a \times c + r_3 a + r_1 c; r_1 r_2 + r_1 r_3\rangle \\ &= A \times_{\mathfrak{B},r} B +_{\mathfrak{B}} A \times_{\mathfrak{B},r} C\end{aligned}$$

A similar reasoning provides the proof of the right distributivity. $\blacksquare$

Lemma 4.27. *Let $a \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}$. The matrix $S_a + \alpha I_n$ is invertible if and only if $\alpha \neq 0$ and α is not a square root of $-(a, \overline{a})_h$.*

Proof: From [7, Lemma 9], the result is valid for $n = 7$. For $n = 3$, a straightforward calculation of $\det(S_a + \alpha I_3)$ leads to $\alpha(\alpha^2 + (a, \overline{a})_h)$. In the stated conditions, $\det(S_a + \alpha I_3) = 0$ if and only if $\alpha = 0$ or $\alpha^2 = -(a, \overline{a})_h$. $\blacksquare$

Theorem 4.28. *Let $a \in \mathbb{C}^n$. Let $\alpha \in \mathbb{C}\backslash\{0\}$ such that α is not a square root of $-(a,\overline{a})_h$. Then*

$$(S_a + \alpha I_n)^{-1} = -(\alpha^2 + (a,\overline{a})_h)^{-1}(S_a - \alpha I_n - \alpha^{-1}aa^T). \tag{13}$$

Proof: By [7, Theorem 10], the result holds for $n = 7$. Now consider $n = 3$. From Lemma 4.27, $S_a + \alpha I_3$ is invertible. Invoking [14, Property (A) and Property (3.1)], we get

$$\begin{aligned}
&(S_a + \alpha I_3)(-(\alpha^2 + (a,\overline{a})_h)^{-1}(S_a - \alpha I_3 - \alpha^{-1}aa^T))\\
&\quad = -(\alpha^2 + (a,\overline{a})_h)^{-1}(S_a^2 - \alpha S_a - \alpha^{-1}S_a aa^T + \alpha S_a - \alpha^2 I_3 - aa^T)\\
&\quad = -(\alpha^2 + (a,\overline{a})_h)^{-1}(-(a,\overline{a})_h I_3 - \alpha^2 I_3)\\
&\quad = I_3.
\end{aligned}$$

■

Theorem 4.29. *Let $A = \langle a; r_1\rangle \in \mathfrak{B}^+$ such that r_1 is not a square root of $-(a,\overline{a})_h$. Let $B = \langle b; r_2\rangle \in \mathfrak{B}$. Then the unique solution of the equation $A \times_{\mathfrak{B},r} X = B$ is given by $X = \langle x; r_3\rangle \in \mathfrak{B}$, with*

$$x = -(r_1^2 + (a,\overline{a})_h)^{-1}(S_a - r_1 I_n - r_1^{-1}aa^t)(b - r_3 a) \text{ and } r_3 = r_1^{-1}r_2.$$

Proof: By (10), the equation $A \times_{\mathfrak{B},r} X = B$ assumes the form

$$\langle S_a x + r_3 a + r_1 x; r_1 r_3\rangle = \langle b; r_2\rangle,$$

where S_a is given by (11)-(12). From here, we arrive at $(S_a + r_1 I_n)x = b - r_3 a$ and $r_1 r_3 = r_2$. As $r_1 \in \mathbb{C}\backslash\{0\}$, since $r_1 \in \mathbb{R}^+$, and r_1 is not a square root of $-(a,\overline{a})_h$, by Theorem 4.28, $S_a + r_1 I_n$ is invertible and $(S_a + r_1 I_n)^{-1} = -(r_1^2 + (a,\overline{a})_h)^{-1}(S_a - r_1 I_n - r_1^{-1}aa^t)$. ■

Theorem 4.30. *Let $B = \langle b; r_2\rangle, C = \langle c; r_1\rangle \in \mathfrak{B}$. Then,*

- *if $r_1 = 0$, $r_2 \in \mathbb{R}^+$ and r_2 is not a square root of $-(b,\overline{b})_h$, then the unique solution of the equation $X^2 = B \times_{\mathfrak{B},r} X + C$ is given by $X = \langle x; r_3\rangle \in \mathfrak{B}$, with*

  $$x = (r_2^2 + (b,\overline{b})_h)^{-1}(S_b - r_2 I_n - r_2^{-1}bb^t)c \text{ and } r_3 = 0;$$

- *if $r_2^2 + 4r_1 \in \mathbb{R}^+$ and $\sqrt{r_2^2 + 4r_1}$ is not a square root of $-(b,\overline{b})_h$, where $\sqrt{\cdot}$ stands for the real, positive root, then the unique solution of the equation $X^2 = B \times_{\mathfrak{B},r} X + C$ is given by $X = \langle x; r_3\rangle \in \mathfrak{B}$, with*

$$x = ((r_2 - 2r_3)^2 + (b,\overline{b})_h)^{-1}(S_b - (r_2 - 2r_3)I_n - (r_2 - 2r_3)^{-1}bb^t)(c + r_3 b)$$

and

$$r_3 = 2^{-1}\left(r_2 + \sqrt{r_2^2 + 4r_1}\right).$$

Proof: From (10), the equation $X^2 = B \times_{\mathfrak{B},r} X + C$ may be written as

$$\langle S_x x + 2r_3 x; r_3^2\rangle = \langle S_b x + r_3 b + r_2 x + c; r_2 r_3 + r_1\rangle,$$

where S_a is given by (11)-(12). On the one hand, we have $r_3^2 = r_2 r_3 + r_1$, which leads to $r_3 = 2^{-1}(r_2 \pm \sqrt{r_2^2 + 4r_1})$, and $r_2 - \sqrt{r_2^2 + 4r_1} \in \mathbb{R}_0^+$ if and only if $r_1 = 0$. On the other hand, taking into account [7, Proposition 4, Property 6] and [14, Property (A)], we have $(S_b + (r_2 - 2r_3)I_n)x = -r_3 b - c$. As $r_2 - 2r_3 \in \mathbb{C}\backslash\{0\}$ and $r_2 - 2r_3$ is not a square root of $-(b, \overline{b})_h$ under the stated assumptions, by Theorem 4.28, $S_b + (r_2 - 2r_3)I_n$ is invertible and $(S_b + (r_2 - 2r_3)I_n)^{-1} = -((r_2 - 2r_3)^2 + (b, \overline{b})_h)^{-1}(S_b - (r_2 - 2r_3)I_n - (r_2 - 2r_3)^{-1}bb^t)$. ■

Corollary 4.31. *Let $E = \langle 0; 1\rangle$. Then the unique solution of the equation $X^2 = X + E$ is given by the golden ball $X = \langle 0; \frac{1+\sqrt{5}}{2}\rangle$, where $\sqrt{\cdot}$ stands for the real, positive root.*

Proof: By Theorem 4.21, the equation $X^2 = X + E$ can be rewritten as $X^2 = E \times_{\mathfrak{B},r} X + E$. The result then follows from Theorem 4.30. ■

4.4. Multiplication $\times_{\mathfrak{B},c}$. Throughout this subsection, consider the usual complex vector space $\mathbb{C}^n$ with $n \in \{3, 7\}$. Consider also the binary operation $\times_{\mathfrak{B},c} : \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$, hereinafter called multiplication $\times_{\mathfrak{B},c}$, defined by

$$A \times_{\mathfrak{B},c} B = \langle a; r_1\rangle \times_{\mathfrak{B},c} \langle b; r_2\rangle := \langle a \times b; r_2\|a\| + r_1\|b\| + r_1 r_2\rangle.$$

Despite the fact that commutativity, anti-commutativity, associativity, existence of neutral element and power-associativity do not hold, $\times_{\mathfrak{B},c}$ satisfies the subsequent properties.

Theorem 4.32. *Let $A = \langle 0; r\rangle \in \mathfrak{B}$. The square roots of A relative to the multiplication $\times_{\mathfrak{B},c}$ are given by $A^{1/2} = \langle b; -\|b\| + \sqrt{\|b\|^2 + r}\rangle$, with $b \in \mathbb{C}^n$, where $\sqrt{\cdot}$ stands for the real, positive or null root.*

Proof: Let $A = \langle 0; r\rangle \in \mathfrak{B}$. Let $B = \langle b; s\rangle \in \mathfrak{B}$ such that $B^2 = A$. From $s^2 + 2\|b\|s - r = 0$, we have $s = -\|b\| + \sqrt{\|b\|^2 + r} \in \mathbb{R}_0^+$. ■

Theorem 4.33. *The multiplication $\times_{\mathfrak{B},c}$ is inclusion monotonic.*

Proof: Let $A_m = \langle a_m; r_m\rangle$, $B_m = \langle b_m; s_m\rangle \in \mathfrak{B}$ such that $A_m \subseteq B_m$, $m \in \{1,2\}$. We aim to prove that $A_1 \times_{\mathfrak{B},c} A_2 \subseteq B_1 \times_{\mathfrak{B},c} B_2$. From Lemma 4.1, $\|a_m - b_m\| \leq s_m - r_m$, $m \in \{1,2\}$. We also have

$$A_1 \times_{\mathfrak{B},c} A_2 = \langle a_1 \times a_2; r_2\|a_1\| + r_1\|a_2\| + r_1r_2\rangle$$

and

$$B_1 \times_{\mathfrak{B},c} B_2 = \langle b_1 \times b_2; s_2\|b_1\| + s_1\|b_2\| + s_1s_2\rangle.$$

As

$$\begin{aligned}
&\|a_1 \times a_2 - b_1 \times b_2\| \\
&= \| - b_2 \times (a_1 - b_1) + b_1 \times (a_2 - b_2) + (a_1 - b_1) \times (a_2 - b_2)\| \\
&\leq \|b_2\|\|a_1 - b_1\| + \|b_1\|\|a_2 - b_2\| + \|a_1 - b_1\|\|a_2 - b_2\| \\
&\leq \|b_2\|(s_1 - r_1) + \|b_1\|(s_2 - r_2) + (s_1 - r_1)(s_2 - r_2)
\end{aligned}$$

and

$$-\|b_m\| \leq -\|a_m\| + \|a_m - b_m\| \leq -\|a_m\| + s_m - r_m, m \in \{1,2\},$$

we obtain $\|a_1 \times a_2 - b_1 \times b_2\| \leq \beta - \alpha$, where $\beta = s_2\|b_1\| + s_1\|b_2\| + s_1s_2$ and $\alpha = r_2\|a_1\| + r_1\|a_2\| + r_1r_2$. Once again by Lemma 4.1, the result follows. ■

Theorem 4.34. *The multiplication $\times_{\mathfrak{B},c}$ is subdistributive with respect to the addition $+_{\mathfrak{B}}$.*

Proof: Let $A = \langle a; r_1\rangle, B = \langle b; r_2\rangle, C = \langle c; r_3\rangle \in \mathfrak{B}$. By Lemma 4.1, we have

$$\begin{aligned}
A \times_{\mathfrak{B},c} (B +_{\mathfrak{B}} C) &= \langle a; r_1\rangle \times_{\mathfrak{B},c} \langle b + c; r_2 + r_3\rangle \\
&= \langle a \times (b + c); (r_2 + r_3)\|a\| + r_1\|b + c\| + r_1(r_2 + r_3)\rangle \\
&\subseteq \langle a \times b + a \times c; r_2\|a\| + r_1\|b\| + r_1r_2 + r_3\|a\| + r_1\|c\| + r_1r_3\rangle \\
&= A \times_{\mathfrak{B},c} B +_{\mathfrak{B}} A \times_{\mathfrak{B},c} C
\end{aligned}$$

Thus, left subdistributivity holds. An analogous reasoning leads to the right subdistributivity. ■

Theorem 4.35. *Let $A = \langle a; r_1\rangle, B = \langle b; r_2\rangle \in \mathfrak{B}$ such that $\|a\|$ and r_1 are not simultaneously null, $(a, \overline{a})_h \neq 0$ and $(a, \overline{b})_h = 0$. Then the solutions of the equation $A \times_{\mathfrak{B},c} X = B$ are given by $X = \langle x; r_3\rangle \in \mathfrak{B}$, with*

$$x = -(a, \overline{a})_h^{-1} S_a b + \lambda a, \lambda \in \mathbb{C},$$

and

$$r_3 = (\|a\| + r_1)^{-1}(r_2 - r_1\|x\|).$$

Proof: By (10), the equation $A \times_{\mathfrak{B},c} X = B$ assumes the form

$$\langle S_a x; r_3\|a\| + r_1\|x\| + r_1 r_3\rangle = \langle b; r_2\rangle,$$

where S_a is given by (11)-(12). Hence, we have $S_a x = b$ and $(\|a\| + r_1)r_3 = r_2 - r_1\|x\|$. The stated solutions, as $(a, \overline{a})_h \neq 0$ and $(a, \overline{b})_h = 0$, are a consequence of [7, Theorem 14] and [14, Theorem 2] . ■

Theorem 4.36. *Let $B = \langle b; r_2\rangle, C = \langle c; r_1\rangle \in \mathfrak{B}$ such that $(b, \overline{b})_h \neq 0$ and $(b, \overline{c})_h = 0$. Then the solutions of the equation $X^2 = B \times_{\mathfrak{B},c} X + C$ are given by $X = \langle x; r_3\rangle \in \mathfrak{B}$, with*

$$x = (b, \overline{b})_h^{-1} S_b c + \lambda b, \lambda \in \mathbb{C},$$

$$r_3 = 2^{-1}\left(r_2 + \|b\| - 2\|x\| + \sqrt{(r_2 + \|b\| - 2\|x\|)^2 + 4(r_1 + r_2\|x\|)}\right),$$

where $\sqrt{\cdot}$ stands for the real, positive or null root, and

$$r_3 = 0 \textit{ if } r_1 = 0 \textit{ and } r_2\|x\| = 0.$$

Proof: From (10), the equation $X^2 = B \times_{\mathfrak{B},c} X + C$ may be written in the form

$$\langle S_x x; 2r_3\|x\| + r_3^2\rangle = \langle S_b x + c; r_2\|x\| + r_3\|b\| + r_2 r_3 + r_1\rangle,$$

where S_b is given by (11)-(12). Observe that, applying [7, Proposition 4, Property 6] and [14, Property (A)], we have $S_b x = -c$, whose solutions follow from [7, Theorem 14] and [14, Theorem 2] . In addition, we arrive at

$$r_3^2 - (r_2 + \|b\| - 2\|x\|)r_3 - r_1 - r_2\|x\| = 0,$$

that is,

$$r_3 = 2^{-1}\left(r_2 + \|b\| - 2\|x\| \pm \sqrt{(r_2 + \|b\| - 2\|x\|)^2 + 4r_1 + 4r_2\|x\|}\right).$$

Observe that

$$2^{-1}\left(r_2 + \|b\| - 2\|x\| - \sqrt{(r_2 + \|b\| - 2\|x\|)^2 + 4r_1 + 4r_2\|x\|}\right) \in \mathbb{R}_0^+$$

if and only if $4r_1 + 4r_2\|x\| = 0$. ■

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