Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 23–12

# NOTES ON THE SPATIAL PART OF A FRAME

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*Dedicated to Themba Dube on the occasion of his 65th birthday*

Abstract: A locale (frame) $L$ has a largest spatial sublocale generated by the primes (spectrum points), the spatial part $\mathsf{Sp}L$. In this paper we discuss some of the properties of the embeddings $\mathsf{Sp}L \subseteq L$. First we analyze the behaviour of the spatial parts in the assembly: the points of $L$ and of $\mathsf{S}(L)^{\mathsf{op}}$ ($\cong$ the congruence frame) are in a natural one-one correspondence while the topologies of $\mathsf{Sp}L$ and $\mathsf{Sp}(\mathsf{S}(L)^{\mathsf{op}})$ differ. Then we concentrate on some special types of embeddings of $\mathsf{Sp}L$ into $L$, namely in the questions when $\mathsf{Sp}L$ is complemented, closed, or open. While in the first part $L$ was general, here we need some restrictions (weak separation axioms) to obtain suitable formulas.

Keywords: Frame, locale, prime element, spectrum, sublocale, Boolean sublocale, supplement, spatial part, largest pointless sublocale.

2020 Mathematics Subject Classification: 06D22, 18F70.

# Introduction

The largest spatial sublocale of a frame $L$, the *spatial part* $\mathsf{Sp}L$, is the sublocale generated by all the primes of $L$. More concretely: the primes $p$ in a frame $L$ lead to one-point[1] sublocales $\{p, 1\}$, and $\mathsf{Sp}L$ is the join of all of them. It can be obtained from the spectrum $\Sigma L$ of $L$ as the image of the $\Omega$-$\Sigma$ adjunction (which gives rise to the obvious spatial reflection). This paper is concerned with some aspects of its behaviour.

In Preliminaries we introduce the basic concepts and notation. In particular, we recall the necessary facts on sublocales of frames viewed as generalized subspaces of generalized spaces.

Received May 6, 2023.

[1] At the first sight it may be slightly confusing that *one-point* subobject (sublocale, generalized subspace) has *two elements*. The top 1 is technical, and has to be in every sublocale. Thus, the *void generalized subspace* is the one-element set $\{1\}$.

In the natural extension of frame to the congruence frame (viewed here as the dual of the co-frame of subspaces, $\mathsf{S}(L)^{\mathsf{op}}$, the first step in the well-known assembly) the $\mathsf{S}(L)^{\mathsf{op}}$ is typically much bigger than $L$. But the system of primes of $\mathsf{S}(L)^{\mathsf{op}}$ remains the same (up to isomorphism) and one might expect that the spatial part will do the same. In Section 3 this is analyzed in detail and shown, among others, that it is not the case: the topology of $\mathsf{Sp}(\mathsf{S}(L)^{\mathsf{op}})$ is in fact (isomorphic to) the front topology of the original one.

While the first part concerned quite general frames, in the sequel, where we deal with the ways $\mathsf{Sp}L$ can be embedded into $L$, we need some (weak) special properties to get suitable technical formulas for the supplement $\mathsf{Sp}L^{\#}$ in $\mathsf{S}(L)$. They are the very weak separation axioms $T_1$ and $T_D$. The next section is devoted to their properties and relations with the more standard ones (subfitness, fitness, Hausdorff type properties). Then, in Section 5 we obtain expedient formulas and use them in Section 6 to characterize the $L$ for which $\mathsf{Sp}L$ are complemented, closed, or open (in particular we learn that the $\mathsf{Sp}(\mathsf{S}(L)^{\mathsf{op}})$ is always open in $\mathsf{S}(L)^{\mathsf{op}}$).

# 1. Preliminaries

**1.1. Notation.** A join (supremum) of a subset $A \subseteq (X, \leq)$, if it exists, will be denoted by $\bigvee A$, and we write $a \vee b$ for $\bigvee\{a, b\}$; similarly we write $\bigwedge A$ and $a \wedge b$ for meets (infima). We write $\uparrow a$ for $\{x \mid x \geq a\}$.

The smallest element of a poset (the supremum $\bigvee \emptyset$), if it exists, will be denoted by 0, and the largest one (the infimum $\bigwedge \emptyset$) will be denoted by 1.

**1.1.1.** An element $b$ is a *pseudocomplement* (resp. *supplement*) of $a$ if

$$a \wedge x = 0 \text{ iff } x \leq b \quad \text{resp.} \quad a \vee x = 1 \text{ iff } x \geq b.$$

We will denote it by $a^*$ (resp. $a^{\#}$). The element $b$ is a *complement* of $a$ if $a \wedge b = 0$ and $a \vee b = 1$.

> *In a distributive lattice, a complement, if it exists, is uniquely determined and it is both a pseudocomplement and a supplement.*

We will denote the complement of an element $a$, when it exists, by $a^{\mathsf{c}}$.

*Exact meets* in a distributive lattice $L$ are the meets $\bigwedge_i a_i$ such that for all $b \in L$, $(\bigwedge_i a_i) \vee b = \bigwedge_i (a_i \vee b)$.

**1.2. Frames.** Recall that a *frame* is a complete lattice $L$ satisfying the distributivity rule

$$(\bigvee A) \wedge b = \bigvee\{a \wedge b \,|\, a \in A\} \tag{frm}$$

for all $A \subseteq L$ and $b \in L$, and that a *frame homomorphism* $h\colon L \to M$ preserves all joins and all finite meets. The resulting category is denoted by **Frm**. A typical frame is $\Omega(X)$, the lattice of open subsets of a space $X$. A frame is *spatial* if it is isomorphic to an $\Omega(X)$.

A *coframe* satisfies (cofrm), the condition (frm) with the roles of joins and meets reversed.

**1.2.1.** Recall that monotone maps $\ell\colon X \to Y$ and $r\colon Y \to X$ are *adjoint* ($\ell$ to the left, $r$ to the right) if one has $\ell(x) \leq y$ iff $x \leq r(y)$ and that if $X, Y$ are complete lattices then left adjoints are precisely the maps preserving suprema, and right adjoints are precisely the maps preserving infima. The equality (frm) states that for every $b \in L$ the mapping $- \wedge b = (x \mapsto x \wedge b)\colon L \to L$ preserves all joins (suprema). Hence each of the mappings $- \wedge b$ has a right Galois adjoint resulting in a *Heyting operation* $\to$ satisfying

$$a \wedge b \leq c \;\Leftrightarrow\; a \leq b \to c,$$

and each frame is a Heyting algebra (note that, however, the frame homomorphisms do not coincide with the Heyting ones so that **Frm** differs from the category of complete Heyting algebras).

Similarly, every coframe is a co-Heyting algebra with the co-Heyting operation $\smallsetminus$ (usually referred to as the *difference*) satisfying

$$a \smallsetminus b \leq c \quad \text{iff} \quad a \leq b \vee c.$$

The operation $\to$ and some of its basic properties (e.g. $a \to a = 1$, $a \to b = 1$ iff $a \leq b$, $1 \to a = a$, and $a \to (b \to c) = (a \wedge b) \to c$) will be used in the sequel.

Note that every frame has pseudocomplements, namely $a^* = a \to 0$ and every coframe has supplements, namely $a^{\#} = 1 \smallsetminus a$.

**1.2.2.** An element $p \neq 1$ is said to be *prime* if $a \wedge b \leq p$ only if $a \leq p$ or $b \leq p$; it is *maximal* if $p < x \leq 1$ implies that $x = 1$. Obviously,

*every maximal element is prime.*

On the other hand, for instance for regular[2] frames each prime is maximal. For more details about this see Section 4.

The set of all prime (resp. maximal) elements of $L$ will be denoted by

$$\mathsf{Prm}(L) \quad \text{resp.} \quad \mathsf{Max}(L).$$

**1.2.3.** An element $a \in L$ is said to be *covered* if there is a $b \in L$ such that $a < x$ iff $b \leq x$ (cf. [10]).

**Observation.** *An element $a$ is covered if and only if for any meet $a = \bigwedge_i x_i$ there is an $i$ with $a = x_i$. Consequently, every covered element is prime.*

(If $a$ is covered and $a = \bigwedge_i x_i$ then we cannot have $x_i > a$ for all $i$; on the other hand, if the other statement holds then $b = \bigwedge\{x \mid a < x\}$ covers $a$.)

A frame is said to be $T_D$ (see [2]) if every prime is covered; more about it in Section 4.

**1.2.4. An easy but important observation.** *Let $p$ be prime in a frame. Then*

$$a \to p = \begin{cases} 1 \ \textit{if}\ a \leq p \\ p \ \textit{otherwise} \end{cases}$$

(if $a \not\leq p$ we have, because $a \wedge (a \to p) \leq p$, $a \to p \leq p \leq a \to p$).

**1.3. Localic maps, the category Loc.** The functor $\Omega\colon \mathbf{Top} \to \mathbf{Frm}$ from the category of topological spaces and continuous maps into that of frames ($\Omega(f)$ sending an open set $U \subseteq Y$ to $f^{-1}[U]$ for a continuous map $f\colon X \to Y$ in **Top**) is a full embedding on an important and substantial part of **Top**, the subcategory of *sober* spaces. This justifies to regard frames as a natural generalization of spaces. Since $\Omega$ is contravariant, one introduces the *category of locales* **Loc** as the dual of the category of frames. Often one just considers the formal $\mathbf{Frm}^{\mathsf{op}}$ but it is of advantage to represent it as a concrete category with specific maps as morphisms. For this purpose one defines a *localic map* $f\colon L \to M$ as the (unique) right Galois adjoint of a frame homomorphism $h = f^*\colon M \to L$. This can be done since frame homomorphisms preserve suprema; but of course not every mapping preserving infima is a localic one. We refer to [14] for more information about the category of locales and details to the sublocales below.

[2] A frame is *regular* if every $a$ can be written as $a = \bigvee\{x \mid x \prec a\}$, where $x \prec a$ stands for the existence of a $u$ such that $x \wedge u = 0$ and $u \vee a = 1$ (for details see e.g [15]). A space is regular in the standard sense iff the frame $\Omega(X)$ is regular.

**1.4. Sublocales.** A *sublocale* of a frame $L$ is a subset $S \subseteq L$ such that

(1) $M \subseteq S$ implies $\bigwedge M \in S$, and
(2) if $a \in L$ and $s \in S$ then $a \to s \in S$

(such subsets $S \subseteq L$ are precisely those for which the embedding $j\colon S \subseteq L$ is a localic map). The system

$$\mathsf{S}(L)$$

of all sublocales of $L$ is a coframe (hence it has supplements $S^{\#}$), with the lattice operations

$$\bigwedge_{i\in I} S_i = \bigcap_{i\in I} S_i \quad \text{and} \quad \bigvee_{i\in I} S_i = \{\bigwedge A \mid A \subseteq \bigcup_{i\in I} S_i\},$$

and the frame $\mathsf{S}(L)^{\mathsf{op}}$ is regular (see the footnote in 1.2.2).

The top element of $\mathsf{S}(L)$ is $L$ and the bottom is the *least sublocale* $\mathsf{O} = \{1\}$.

**1.4.1. Open and closed sublocales.** We have the *open sublocales*

$$\mathfrak{o}(a) = \{a \to x \mid x \in L\} = \{x \mid x = a \to x\}$$

and the *closed sublocales*

$$\mathfrak{c}(a) = \uparrow a = \{x \mid x \leq a\}$$

(they are complements of each other and correspond to open and closed subspaces of classical spaces, and to the Isbell's *open* and *closed parts* from [9]). It is easy to see that the closure $\overline{S}$ of $S$ is equal to $\bigcap\{\uparrow x \mid S \subseteq \uparrow x\} = \uparrow \bigwedge S$. Furthermore, we denote by $\mathsf{int}(S)$ the interior of $S$, namely the join of all the open sublocales contained in $S$. In this context, we point out that the formula

$$\mathsf{int}(S)^{\#} = \overline{S^{\#}}$$

holds for each sublocale $S$ of $L$ (see [8, p. 29]).

We have that

$$\mathfrak{c}(a) \subseteq \mathfrak{o}(b) \text{ iff } a \vee b = 1, \tag{1.4.1}$$

and every sublocale can be written as $S = \bigcap_i (\mathfrak{c}(a_i) \vee \mathfrak{o}(b_i))$.

A sublocale $S$ is *fitted* (Isbell, [9]) if it is a meet of open ones. For a sublocale $S \subseteq L$, the *fitting* of $S$ ([6]) is the sublocale $\bigcap\{\mathfrak{o}(a) \mid S \subseteq \mathfrak{o}(a)\}$.

**1.4.2.** Further we will need the

$$\mathfrak{b}(a) = \{x \to a \mid x \in L\}.$$

$\mathfrak{b}(a)$ is obviously the smallest sublocale containing $a$ and one has that

> *the* $\mathfrak{b}(a)$*'s are precisely the Boolean sublocales of* $L$ (see e.g. [14]).

Finally we will need the *one-point sublocales*

$$\widetilde{p} = \{p, 1\}$$

with $p$ prime element of $L$ (recall 1.2.4). These are precisely the sublocales with exactly two elements (with exactly one non-trivial element). Thus, in particular, for a prime element $p$,

$$\mathfrak{b}(p) = \widetilde{p}.$$

**1.5.** From [16, 14] recall the

$$\mathsf{S}_{\mathsf{c}}(L) = \{\bigvee_i \mathfrak{c}(a_i) \mid a_i \in L\} \subseteq \mathsf{S}(L).$$

It is always a frame.

A sublocale $S$ is said to be *smooth* ([9]) if it is a join of complemented sublocales. Recall from [1, 2] the

$$\mathsf{S}_{\mathsf{b}}(L) = \{S \mid S \text{ smooth in } \mathsf{S}(L)\}.$$

One has that

> $\mathsf{S}_{\mathsf{b}}(L)$ *is the (co)Booleanization of* $\mathsf{S}(L)$*.*

# 2. Spectrum and the spatial sublocale

**2.1.** We have $\mathfrak{o}(0) = \mathsf{O}$, $\mathfrak{o}(1) = L$, $\mathfrak{o}(a \wedge b)$ and $\mathfrak{o}(\bigvee_i a_i) = \bigvee_i \mathfrak{o}(a_i)$. Hence the system of all open sublocales constitutes on $L$ a sort of topology (with the proviso that the joins do not coincide with set unions, and the zero $\mathsf{O} = \{1\}$ is not void as a set; this is sometimes called a *lattice topology*).

Hence we may think of a frame as of a (lattice based) topological space

$$(L, \{\mathfrak{o}(a) \mid a \in L\}).$$

**2.1.1. Note.** In fact, the lattice-topological nature of frames is deeper. Localic maps $f\colon L \to M$ create naturally adjoint image and preimage maps $f[-]\colon \mathsf{S}(L) \to \mathsf{S}(M)$ and $f_{-1}[-]\colon \mathsf{S}(M) \to S(L)$ and the preimage preserves open and closed sublocales (see e.g. [14]). Moreover, localic maps can be identified as (set) maps $L \to M$ respecting closed and open sublocales in preimages ([7]).

**2.2. A well known fact.** *If $f\colon L \to M$ is a localic map then*

$$f[\mathsf{Prm}(L)] \subseteq \mathsf{Prm}(M).$$

(Indeed, if $a \wedge b \le f(p)$ then $f^*(a) \wedge f^*(b) \le p$, hence, say, $f^*(a) \le p$ and $a \le f(p)$.)

**2.3. Spectrum.** Recall that the spectrum of $L$ can be understood as

$$\Sigma L = (\mathsf{Prm}(L), \{\Sigma_a \,|\, a \in L\}) \quad \text{where} \quad \Sigma_a = \{p \in \mathsf{Prm}(L) \,|\, a \nleq p\},$$

and for a localic map $f\colon L \to M$, $\Sigma f$ is simply the restriction of $f$ to a mapping $\mathsf{Prm}(L) \to \mathsf{Prm}(M)$.

By 1.2.4, $a \nleq p$ iff $a \to p = p$, that is, iff $p \in \mathfrak{o}(a)$. Hence

$$\Sigma_a = \mathfrak{o}(a) \cap \mathsf{Prm}(L).$$

Therefore, if we consider $L$ as a topological space endowed with the (lattice) topology as in 2.1 (constituted by the open sublocales), the spectrum is the subspace (now a standard one, the carrier lattice is that of all subsets of $\mathsf{Prm}(L)$) carried by the subset $\mathsf{Prm}(L)$.

**2.4.** Moreover, since a prime $p$ is not in $\Sigma_b$ iff $b \le p$ we have

$$\mathsf{Prm}(L) \smallsetminus \Sigma_b = {\uparrow} b \cap \mathsf{Prm}(L) = \mathfrak{c}(b) \cap \mathsf{Prm}(L)$$

and hence

$$(\mathfrak{o}(a) \cap \mathfrak{c}(b)) \cap \mathsf{Prm}(L) = \Sigma_a \cap (\mathsf{Prm}(L) \smallsetminus \Sigma_b). \tag{2.4.1}$$

By 1.2.3, we have for a $T_D$-frame generally

$$(\bigvee_i S_i) \cap \mathsf{Prm}(L) = (\bigcup_i S_i) \cap \mathsf{Prm}(L) = \bigcup_i (S_i \cap \mathsf{Prm}(L))$$

(since if $p \in \bigvee_i S_i \cap \mathsf{Prm}(L)$ then $p = \bigwedge_i x_i$ with $x_i \in S_i$ and hence for some $j$, $p = x_j \in S_j$).

Recall that the *front topology* $\tau^f$ associated with a topology $\tau$ is the topology generated by all open and all closed subsets (open and closed elements in the lattice topology). Thus we have in particular

**2.4.1. Proposition.** *If $L$ is a $T_D$-frame then the front topology on $\mathsf{Prm}(L)$ is the subspace topology of the front topology on $L$.*

(Indeed, $\bigcup_i(\Sigma_{a_i} \cap (\mathsf{Prm}(L) \smallsetminus \Sigma_{b_i})) = (\bigvee_i(\mathfrak{o}(a_i) \cap \mathfrak{c}(b_i))) \cap \mathsf{Prm}(L)$; use also (2.4.1)).

**2.5. The spatial part of *L*.** The mapping

$$\sigma = (a \mapsto \Sigma_a)\colon L \to \Omega\Sigma L$$

is an onto frame homomorphism. The associated localic map $\gamma\colon \Omega\Sigma L \to L$ is defined by

$$\sigma(a) \subseteq U \quad \text{iff} \quad a \leq \gamma(U)$$

and hence for any $u$ such that $U = \Sigma_u$,

$$a \leq \gamma(U) \quad \text{iff} \quad a \nleq p \Rightarrow u \nleq p \quad \text{iff} \quad u \leq p \Rightarrow a \leq p \quad \text{iff} \quad a \leq \bigwedge\{p \mid u \leq p\}.$$

Consequently we see that

$$\gamma(U) = \bigwedge\{p \mid u \leq p\} \text{ for any } u \text{ such that } U = \Sigma_u.$$

**2.5.1. The largest spatial sublocale of *L*.** The sublocale associated with the surjection $\sigma$ (the imprint of the one-to-one localic map $\gamma$) is the set of all $a \in L$ such that $a = \gamma(\Sigma_a) = \bigwedge\{p \mid a \leq p\}$, that is, it is the sublocale

$$\mathsf{Sp}L = \bigvee\{\{p, 1\} \mid p \in \mathsf{Prm}(L)\}.$$

We will call it the *spatial part* of $L$ (obviously it is the smallest sublocale of $L$ containing all the points).

# 3. Spatial part of $\mathsf{S}(L)^{\mathsf{op}}$

The fact that the primes of $L$ are in a natural correspondence with the (co)primes in the lattice of sublocales is well-known (see e.g. [19, Lemma 11]). We describe the correspondence in detail in 3.2 in the notation we need.

**3.1.** Since $\mathsf{S}(L)^{\mathsf{op}}$ is a regular frame, the primes are the maximal elements, in other words,

*they are the minimal elements of* $\mathsf{S}(L)$.

For each $a \in L$, the smallest sublocale containing $a$ of $\mathsf{S}(L)$ is the

$$\mathfrak{b}(a) = \{x \to a \mid x \in L\}.$$

Thus,

*the minimal elements of* $\mathsf{S}(L)$ *are the* $\mathfrak{b}(a)$ *such that*

$$\forall x \in \mathfrak{b}(a),\ \ \mathfrak{b}(x) = \mathfrak{b}(a).$$

**3.1.1.** Consequently

*For each minimal element* $\mathfrak{b}(a)$ *of* $\mathsf{S}(L)$ *and* $x \nleq a$ *there is a* $y \in L$ *such that* $a = y \to (x \to a)$.

**3.2. Proposition.** *The primes in* $K = \mathsf{S}(L)^{\mathsf{op}}$ *are precisely the* $\mathfrak{b}(p)$ *with* $p \in \mathsf{Prm}(L)$.

*Proof*: Let $\mathfrak{b}(a)$ satisfy 3.1.1. Then for each $x \nleq a$ there exists a $y$ with $a = y \to (x \to a)$. The standard inequalities

$$a = y \to (x \to a) \geq x \to a \geq a$$

yield $x \to a = x$ for any $x \nleq a$, and for $x \leq a$ trivially $x \to a = 1$. Hence $\{a, 1\}$ is a sublocale which is the characterization of primes.

On the other hand, $\mathfrak{b}(p) = \{x \to p \,|\, x \in L\}$ with $p$ prime is $\{p, 1\}$ and thus trivially minimal. ■

**3.2.1.** Hence in the assembly tower

$$L \longrightarrow \overline{\mathsf{S}}(L) \longrightarrow \overline{\mathsf{S}}^2(L) \longrightarrow \cdots \longrightarrow \overline{\mathsf{S}}^\alpha(L) \longrightarrow \cdots$$

(where $\overline{\mathsf{S}}(L) = \mathsf{S}(L)^{\mathsf{op}}$, $\overline{\mathsf{S}}^{\alpha+1} = \overline{\mathsf{S}}\ \overline{\mathsf{S}}^\alpha$ and in the limit case $\overline{\mathsf{S}}^\lambda(L)$ is the directed sum of the previous members, and the arrows represent the natural embeddings $a \mapsto \mathfrak{c}(a)$) the primes are isomorphically copied; for more see 3.3.1 below. By [17, 1.1] there exist spatial $L$ with strictly increasing tower, hence any such spatial $L$ can be epimorphically embedded as a subframe into arbitrarily large frames $M$.

**3.3. The front topology of $\Sigma L$.** The natural correspondence

$$\beta = (p \mapsto \mathfrak{b}(p))\colon \mathsf{Prm}(L) \cong \mathsf{Prm}(\mathsf{S}(L)^{\mathsf{op}})$$

does not make $\Sigma L$ homeomorphic with $\Sigma(\mathsf{S}(L)^{\mathsf{op}})$. One has instead the following ([19, Theorem 12]):

**Proposition.** $\beta$ *carries a homeomorphism*

$$(\mathsf{Prm}(L), \Omega^f(\Sigma L)) \cong \Sigma(\mathsf{S}(L)^{\mathsf{op}})$$

*of the front topology of* $\Sigma L$ *with the spectrum of* $\mathsf{S}(L)^{\mathsf{op}}$.

*Proof*: Since $p \notin \uparrow a$ iff $a \not\leq p$, we have $\beta^{-1}[\Sigma_{\mathfrak{c}(a)}] = \Sigma_a$. Moreover, $p \notin \mathfrak{o}(a)$ iff $a \leq p$ since $(p \vee a) \wedge (a \to p) = p$ implies $p \vee a \leq p$ or $a \to p \leq p$ and the latter would lead to the contradiction $p = a \to p \in \mathfrak{o}(a)$. Hence $\beta^{-1}[\Sigma_{\mathfrak{o}(a)}] = \mathsf{Prm}(L) \smallsetminus \Sigma_a$ and for any $S \in \mathsf{S}(L)^{\mathsf{op}}$, with $S = \bigcap_i (\mathfrak{c}(a_i) \vee \mathfrak{o}(b_i))$,

$$\beta^{-1}[\Sigma_S] = \beta^{-1}[\bigcup_i (\Sigma_{\mathfrak{c}(a_i)} \cap \Sigma_{\mathfrak{o}(b_i)})] = \bigcup_i (\Sigma_{a_i} \cap (\mathsf{Prm}(L) \smallsetminus \Sigma_{b_i})) \in \Omega^f(\Sigma L)).$$

On the other hand,

$$\beta(\Sigma_a) = \{\mathfrak{b}(p) \mid p \in \mathsf{Prm}(L), a \not\leq p\} = \Sigma_{\mathfrak{c}(a)} \in \Omega\Sigma(\mathsf{S}(L)^{\mathsf{op}})$$

and

$$\beta(\mathsf{Prm}(L) \smallsetminus \Sigma_a) = \{\mathfrak{b}(p) \mid p \in \mathsf{Prm}(L), a \leq p\} = \Sigma_{\mathfrak{o}(a)} \in \Omega\Sigma(\mathsf{S}(L)^{\mathsf{op}}). \quad \blacksquare$$

**3.3.1.** Note that, because of the fact that for $T_0$ spaces the second front topology is discrete, in the 3.2.1 the spatial parts starting with the second step are discrete.

**3.4.** For each localic map $f\colon L \to M$, the *cokernel* of $f$ is the sublocale $ff^*[M]$ of $M$. This is indeed a sublocale:
(1) For any $x_i = ff^*(a_i) \in ff^*[M]$,

$$ff^*(\bigwedge_i x_i) \leq \bigwedge_i ff^*(x_i) = \bigwedge_i ff^*ff^*(a_i) = \bigwedge_i ff^*(a_i) = \bigwedge_i x_i$$

hence $\bigwedge_i x_i \in ff^*[M]$.
(2) If $b \in M$ and $x = ff^*(a) \in ff^*[M]$ then

$$b \to x = b \to ff^*(a) = ff^*(b \to ff^*(a))$$

since

$$\begin{aligned} ff^*(b \to ff^*(a)) \wedge b \leq ff^*(b \to ff^*(a)) \wedge ff^*(b) = \\ &= ff^*((b \to ff^*(a)) \wedge b) = ff^*(b \wedge ff^*(a)) \leq \\ &\leq ff^*ff^*(a) = ff^*(a). \end{aligned}$$

**3.4.1.** Let

$$d_L\colon \mathsf{S}(L)^{\mathsf{op}} \to L$$

denote the map given by $S \mapsto \bigwedge S$ (the *dissolution map* ([9, 17]). We will need the following well-known property (it appeared first in [19]), here presented for localic maps:

**Proposition.** *Let $f\colon M \to L$ be a localic map such that $f^*(x)$ is complemented for every $x \in L$. Then there exists a unique $\widetilde{f} \in \mathbf{Loc}$ that makes the following triangle commutative.*

$$\begin{tikzcd} \mathsf{S}(L)^{\mathsf{op}} \arrow[r, "d_L"] & L \\ M \arrow[u, dotted, "\widetilde{f}"] \arrow[ur, "f"'] & \end{tikzcd}$$

*Proof*: For each $y \in M$ take $\widetilde{f}(y)$ as the cokernel of the composite

$$\mathfrak{c}(y) \xhookrightarrow{j_{\mathfrak{c}(y)}} M \xrightarrow{f} L,$$

that is, $\widetilde{f}(y) = f j^*_{\mathfrak{c}(y)} f^*[L]$. The checking is a straightforward exercise. ∎

**3.4.2.** In particular, for any space $X$ and

$$q_X = (U \mapsto \mathrm{int}_{\Omega(X)}(U))\colon \Omega^f(X) \to \Omega(X)$$

there exists a unique $\widetilde{q_X} \in \mathbf{Loc}$ such that the triangle

$$\begin{tikzcd} \mathsf{S}(\Omega X)^{\mathsf{op}} \arrow[r, "d_{\Omega X}"] & \Omega X \\ \Omega^f X \arrow[u, dotted, "\widetilde{q_X}"] \arrow[ur, "q_X"'] & \end{tikzcd} \qquad (3.4.2)$$

commutes. Note that $q_X(U) = \{V \in \Omega(X) \mid \mathrm{int}_{\Omega(X)}(V \cup U) = V\}$ for every $U \in \Omega(X)$. In particular, for $U \in \Omega(X)$,

$$q_X(U) = \{V \in \Omega(X) \mid U \subseteq V\} = \mathfrak{c}_{\Omega(X)}(U)$$

and

$$\begin{aligned} q_X(X \smallsetminus U) &= \{V \in \Omega(X) \mid \mathrm{int}_{\Omega(X)}(V \cup (X \smallsetminus U)) = V\} = \\ &= \{V \in \Omega(X) \mid U \to V = V\} = \mathfrak{o}_{\Omega(X)}(U). \end{aligned}$$

**3.4.3.** It is easy to see that $(d_L)_L$ is a natural transformation from the assembly functor $\mathsf{S}\colon \mathbf{Loc} \to \mathbf{Loc}$ into the identity functor. Putting it together with triangle (3.4.2), for $X = \Sigma L$, and the homeomorphism from 3.3 we get the following commutative diagram with the embeddings $\mathsf{Sp}L \subseteq L$ and $\mathsf{Sp}(\mathsf{S}(L)^{\mathsf{op}}) \subseteq \mathsf{S}(L)^{\mathsf{op}}$:

$$\begin{tikzcd}
{} & \mathsf{S}(L)^{\mathsf{op}} \arrow[r, "d_L"] & L & {} \\
\mathsf{Sp}(\mathsf{S}(L)^{\mathsf{op}}) \arrow[ur, hook, bend left] & & & \mathsf{Sp}L \arrow[ul, hook', bend right] \\
& \mathsf{S}(\Omega\Sigma L)^{\mathsf{op}} \arrow[uu, "\mathsf{S}(\gamma_L)"] \arrow[r, "d_{\Omega\Sigma L}"] & \Omega\Sigma L \arrow[uu, "\gamma_L"] \arrow[ur, bend right, "\gamma_L"'] & \\
\Omega\Sigma(\mathsf{S}(L)^{\mathsf{op}}) \arrow[uu, bend left, "\gamma_{\mathsf{S}(L)^{\mathsf{op}}}"] \arrow[uuur, "\gamma_{\mathsf{S}(L)^{\mathsf{op}}}"] \arrow[r, leftrightarrow, "\cong"] & \Omega^f(\Sigma L) \arrow[u, dotted, "\widetilde{q}_{\Sigma L}"] \arrow[ur, "q_{\Sigma_L}"'] & &
\end{tikzcd}$$

# 4. Some separation axioms, in particular the very weak $T_1$ and $T_D$

The short proofs in this section are either folklore or already in literature (e.g. in [15, 2, 21]). We present them for convenience of the reader.

**4.1. Subfitness.** A frame is *subfit* ([9], *conjunctive* in [20], for history see e.g. [15]) if

$$a \nleq b \;\Rightarrow\; \exists c, a \vee c = 1 \text{ and } b \vee c \neq 1. \tag{sfit}$$

**4.1.1.** Recall 1.5. By [16, 14], for a subfit $L$ the mapping

$$\kappa = (S \mapsto \bigvee\{\uparrow a \mid \uparrow a \subseteq S\}) \colon \mathsf{S}(L) \to \mathsf{S_c}(L)$$

is a coframe homomorphism, $\mathsf{S_c}(L)$ is Boolean, and

$$\text{for any } S \in \mathsf{S}(L), \quad \kappa(S) = S^{\#\#}.$$

**4.1.2.** If $L$ is subfit then $\mathsf{S_b}(L) = \mathsf{S_c}(L)$. Consequently, in the subfit case, $\mathsf{S_c}(L)$ is the (co)Booleanization of $\mathsf{S}(L)$.

**4.2. Fitness.** See [9] (or, e.g. [15]). A frame $L$ is *fit* if

$$a \nleq b \quad \Rightarrow \quad \exists c, a \vee c = 1 \;\&\; c \to b \nleq b. \tag{fit}$$

This is a property stronger than subfitness and weaker than the well known regularity.

**4.3. Hausdorff type axioms.** The *strong Hausdorff* axiom (sH) [9] requires, mimicking the characteristic property in spaces, that the diagonal in the square $L \times L$ in **Loc** is closed. We will get by with the weaker one, usually called simply *Hausdorff*, requiring that for all $a \neq 1$,

$$a = \bigvee \{x \mid x \sqsubset a\} \quad \text{where} \quad x \sqsubset a \text{ means } x \leq a \ \&\ x^* \not\leq a \tag{H}$$

(Johnstone-Sun [12], Paseka-Šmarda [13]), or the even weaker *point Hausdorff* (Rosicky-Šmarda [18]), denoted (pH), requiring that

$$\text{every semiprime element is maximal} \tag{pH}$$

(an element $p$ is *semiprime* if $a \wedge b = 0$ only if $a \leq p$ or $b \leq p$). One has the implications

$$\text{(sH)} \ \Rightarrow \ \text{(H)} \ \Rightarrow \ \text{(pH)}$$

(the first implication is standard, for the second one, if $p$ is semiprime and $p < a \neq 1$ then there is an $x \sqsubset a$ such that $x \not\leq p$ and we have $x^* \leq p < a$, a contradiction) and both (H) and (pH) are for spaces equivalent with the classical Hausdorff property.

**4.4. Axiom $T_1$.** A frame $L$ satisfies $T_1$ if every prime element in $L$ is maximal. Thus, $L$ is $T_1$ if and only if each one-point sublocale $\widetilde{p} = \{p, 1\} = \mathfrak{b}(p)$ ($p$ prime, recall 1.4.2) is closed.

**4.4.1. Proposition.** *Each Hausdorff frame and each fit one is $T_1$.*

*Proof*: The first follows from the implication (H)⇒(pH) above but because the property of fitness is incomparable with (H) it has to be considered separately. By (fit), if $p$ is prime and $p < a$ then there is a $c$ with $a \vee c = 1$ and $c \to p \not\leq p$. Since $c \wedge (c \to p) \leq p$ we have $c \leq p$ and hence $a = a \vee p = 1$. ■

**4.5. Axiom $T_D$.** The axiom $T_D$ appeared in the classical context already in [3] and in a way played from the very beginning a role in delimiting the spaces that can be well investigated without points: $T_D$-topological spaces with isomorphic lattices of open sets are isomorphic ([22]), subspaces of $T_D$ spaces are correctly represented by sublocales ([4] – and probably sooner). Later it appeared as an intrinsic characteristic of frames in the study of the difference between general spatial frames and those that can be represented by a $T_D$-one. Only quite recently this was extended to an axiom concerning a general frame in [2].

**4.5.1.** Recall that a frame is $T_D$ if

$$\textit{every prime element in } L \textit{ is covered.} \qquad (T_D)$$

**4.5.2. Proposition.** *The following are equivalent for a frame $L$:*

(1) *$L$ is $T_D$.*
(2) *For every prime $p$, $\mathfrak{b}(p) = \mathfrak{c}(p) \cap \mathfrak{o}(a)$ for some $a$.*
(3) *For every prime $p$, $\mathfrak{b}(p)$ is complemented.*

*Proof*: (1)⇒(2): Let $a$ cover $p$. By 1.2.4, $p \in \mathfrak{o}(a)$. But if $x \neq p, 1$ and $x \in \mathfrak{o}(a)$, that is, $x = a \to x$ then $x \ngeq a$ and hence $x \ngeq p$.
(2)⇒(3) is trivial.
(3)⇒(1): Let $S$ be the complement of $\mathfrak{b}(p) = \{p, 1\}$. Then $p \notin S$ but for all $x > p$ one has to have $x \in S$ (else it would not be in $S \vee \mathfrak{b}(p)$). Thus, $b = \bigwedge\{x \mid p < x\}$ is in $S$ and has to cover $p$. ■

**4.5.3. Observation.** *Subfit plus $T_D$ implies $T_1$.*
(If $p$ is prime covered by $b$ and if $p < x$ then there is a $c$ such that $b \vee c = 1 \neq p \vee c$; $p < p \vee c$ would give $b \leq p \vee c$ and hence $1 = b \vee p \leq p \vee c$ a contradiction. Hence, $p = p \vee c$ so that $c \leq p$, and $b = b \vee p \geq b \vee c = 1$.)

# 5. $\mathsf{Sp}L$ under low separation axioms

**5.1.** Recall from 4.4 that $L$ is $T_1$ if

$$\mathsf{Prm}(L) = \mathsf{Max}(L), \qquad (T_1)$$

that is, if every one-point sublocale $\widetilde{p} = \{p, 1\}$ is closed.

**Note.** Although the formula for $T_1$ is in simple lattice terms, the concept does not play much of a role just in itself. For instance, in the subfit context it states only that the spectrum is $T_1$ in the classical sense (this is not true quite generally, though: for instance, if $B$ is a locale without points and $L$ is $B$ extended by a new bottom element $\bot$ then $\mathsf{Prm}(L) = \{\bot\}$ so that the spectrum is $T_1$; but $L$ is not a $T_1$-locale because $\bot$ is not maximal). But the concept is of importance as a consequence of essentially point-free conditions as the point-free variants of Hausdorff axiom, or fitness (recall 4.4.1).

Thus, in the $T_1$-case we will work with

$$\mathsf{Sp}L = \bigvee\{\mathfrak{c}(p) = \uparrow p \mid p \in \mathsf{Prm}(L)\}. \qquad (T_1\text{-}\mathsf{Sp})$$

**5.2.** For subfit locales a covered prime is always maximal; hence under subfitness $T_D$-locales and $T_1$-locales are equivalent concepts (recall 4.5.3).

**5.3. The largest pointless sublocale.** (cf. [10]). A locale is *pointless* if it does not contain any prime element. Recall 4.5.2(3). We have

**5.3.1. Proposition.** *Each $T_D$-locale has the largest pointless sublocale, namely*

$$\Phi L = \bigcap\{\mathfrak{b}(p)_{,}| \, p \in \mathsf{Prm}(L)\}.^3$$

*In particular, for $T_1$-locales we have*

$$\Phi L = \bigcap\{\mathfrak{o}(p) \,|\, p \in \mathsf{Prm}(L)\}.$$

*Proof*: Since $\mathfrak{b}(p) \cap \mathfrak{b}(p)^{\mathsf{c}} = \mathsf{O}$, $p$ is not in $\mathfrak{b}(p)^{\mathsf{c}}$ and hence $\Phi L \cap \mathsf{Prm}(L) = \emptyset$, that is, $\Phi L$ is pointless. Now let $S$ contain no prime element. Then by 1.2.4, $S \cap \mathfrak{b}(p) = \mathsf{O}$ for all $p \in \mathsf{Prm}(L)$ and hence $S \subseteq \mathfrak{b}(p)^{\mathsf{c}}$ for all $p \in \mathsf{Prm}(L)$. ■

**5.3.2. Proposition.** *Let $L$ be a $T_D$-locale. Then $\mathsf{Sp}L = \Phi L^{\#}$. Hence it is in the Booleanization $\mathsf{S}_{\mathsf{b}}(L)$ of $\mathsf{S}(L)$, and $\mathsf{Sp}L^{\#} = \Phi L^{\#\#}$.*

*Thus we have for the supplement of the spatial part the formula*

$$\mathsf{Sp}L^{\#} = \bigvee\{\mathfrak{c}(a) \cap \mathfrak{o}(b) \,|\, \forall p \in \mathsf{Prm}(L), a \not\leq p \text{ or } b \leq p\}.$$

*If $L$ is subfit, then $\mathsf{Sp}L^{\#} \in \mathsf{S}_{\mathsf{c}}(L)$ and*

$$\mathsf{Sp}L^{\#} = \bigvee\{\mathfrak{c}(a) \,|\, \forall p \in \mathsf{Prm}(L), a \vee p = 1\} = \bigvee\{\mathfrak{c}(a) \,|\, \Sigma_a = L\}.$$

*Proof*: We have

$$\begin{aligned}\Phi L^{\#} &= (\bigcap\{\mathfrak{b}(p)^{\mathsf{c}} \,|\, p \in \mathsf{Prm}(L)\})^{\#} = \\ &= \bigvee\{\mathfrak{b}(p)^{\mathsf{cc}} \,|\, p \in \mathsf{Prm}(L)\}) = \bigvee\{\mathfrak{b}(p) \,|\, p \in \mathsf{Prm}(L)\}) = \mathsf{Sp}L\end{aligned}$$

and hence $\mathsf{Sp}L^{\#} = \Phi L^{\#\#}$.

Now recall (1.4.1). We have

$$\begin{aligned}\mathsf{Sp}L^{\#} &= \bigvee\{\mathfrak{c}(a) \cap \mathfrak{o}(b) \,|\, \mathfrak{c}(a) \cap \mathfrak{o}(b) \subseteq \Phi L\} = \\ &= \bigvee\{\mathfrak{c}(a) \,|\, \forall p \in \mathsf{Prm}(L), p \notin \mathfrak{c}(a) \cap \mathfrak{o}(b)\} = \\ &= \bigvee\{\mathfrak{c}(a) \cap \mathfrak{o}(b) \,|\, \forall p \in \mathsf{Prm}(L), a \not\leq p \text{ or } b \leq p\}.\end{aligned}$$

[3] This intersection has been used by R. Ball in his study of the point-free reflection.

In the subfit case recall 4.1.1: by [16, 14] the double supplement of $S$ in $\mathsf{S}(L)$ is $S^{\#\#} = \bigvee\{\mathfrak{c}(a) \mid \mathfrak{c}(a) \subseteq S\}$. Hence we have by 5.3 and (1.4.1)

$$\begin{aligned}\mathsf{Sp}L^{\#} = \Phi L^{\#\#} &= \bigvee\{\mathfrak{c}(a) \mid \mathfrak{c}(a) \subseteq \bigcap\{\mathfrak{o}(p) \mid p \in \mathsf{Prm}(L)\}\} = \\ &= \bigvee\{\mathfrak{c}(a) \mid \forall p \in \mathsf{Prm}(L), \mathfrak{c}(a) \subseteq \mathfrak{o}(p)\} = \\ &= \bigvee\{\mathfrak{c}(a) \mid \forall p \in \mathsf{Prm}(L), a \vee p = 1\} = \bigvee\{\mathfrak{c}(a) \mid \forall p \in \mathsf{Prm}(L), a \not\leq p\}. \quad \blacksquare\end{aligned}$$

**5.3.3.** Let $L$ be subfit and $T_D$. Since $x \geq a$ and $a \not\leq p$ makes $x \not\leq p$, the last formula above can be rewritten as

$$\mathsf{Sp}L^{\#} = \{\bigwedge_i a_i \mid \forall i\ \forall p \in \mathsf{Prm}(L), a_i \not\leq p\}.$$

**5.4. Note.** $\mathsf{Sp}L^{\#} = \Phi L^{\#\#}$ is typically much smaller then $\Phi L$. One has

**Proposition.** *Let $L$ be $T_1$. Then $\Phi L \cap \mathsf{Sp}L$ is the sublocale*

$$\{\bigwedge A \,|\, A \subseteq \mathsf{Prm}(L) \quad \textit{such that} \quad \forall p \in \mathsf{Prm}(L), \bigwedge(A \smallsetminus {\downarrow}p) = \bigwedge A\}.$$

*Proof*: An element of the intersection is a $\bigwedge A$, $A \subseteq \mathsf{Prm}(L)$ such that for all $p \in \mathsf{Prm}(L)$, $\bigwedge A \in \mathfrak{o}(p)$, that is, $p \to \bigwedge A = \bigwedge A$. We have, by the formula above

$$p \to \bigwedge A = \bigwedge\{p \to q \,|\, q \in A\} = \bigwedge\{p \to q \,|\, q \in A, q \not\leq p\} = \bigwedge(A \smallsetminus {\downarrow}p)$$

and this has to be $\bigwedge A$. ■

Let us illustrate it on the spatial case.

Consider a $T_1$-space $X$. Then the primes in $L = \Omega(X)$ are the $X \smallsetminus \{x\}$ and the $\bigwedge A$ with $A \subseteq \mathsf{Prm}(L)$ are

$$\operatorname{int}(\bigcup\{X \smallsetminus \{x\} \,|\, x \in M\}) = \operatorname{int}(X \smallsetminus M) = X \smallsetminus \overline{M}.$$

Hence the $U \in \Phi L$, because here $\mathsf{Sp}L$ is the whole of $\Omega(X)$, are all the $X \smallsetminus M$ with $M$ closed such that $X \smallsetminus \overline{M \smallsetminus \{x\}} \subseteq X \smallsetminus \{x\}$ for all $x \in M$, that is

$$\forall x \in M,\ x \in \overline{M \smallsetminus \{x\}}.$$

Since, however, $\mathsf{Sp}(\Omega(X))$ is the whole of $\Omega(X)$, $\Phi\Omega(X)^{\#\#}$ is $\mathsf{O}$.

# 6. Complemented, closed and open $\mathsf{Sp}L$

**6.1.** The sublocale $\mathsf{Sp}L$ is complemented if and only if its supplement is a complement, that is, $\mathsf{Sp}L \cap \mathsf{Sp}L^{\#} = \mathsf{O}$. Hence we obtain immediately from 5.3.3

**Proposition.** *If $L$ is subfit and $T_D$ then $\mathsf{Sp}L$ is complemented if and only if*
$$\forall M \subseteq \mathsf{Prm}(L),\ \textit{if } \bigwedge M = \bigwedge_i a_i \textit{ then } a_i \leq q \textit{ for some } i \textit{ and some } q \in \mathsf{Prm}(L).$$

**6.2. Proposition.** *The following are equivalent for a subfit $T_D$-locale $L$:*

(1) $\mathsf{Sp}L$ *is complemented.*
(2) *If $S \neq \mathsf{O}$ is a spatial sublocale, then* $\mathsf{int}_{\overline{S}}(\mathsf{Sp}\overline{S}) \neq \mathsf{O}$ (*i.e.* $\mathsf{int}_{\overline{S}}(\mathsf{Sp}\overline{S})$ *has a point*).
(3) *If $S \neq \mathsf{O}$ is a spatial sublocale, then the smooth part of the pointless part of $\overline{S}$ is not dense in $\overline{S}$.*

*Proof*: First, note that the equivalence between (2) and (3) follows from 1.4.1. Let us check that (1) implies (3). Let $S = \bigvee_i \mathfrak{b}(p_i) \neq \mathsf{O}$ be spatial, and set $x = \bigwedge_i p_i \neq 1$. Assume, for sake of contradiction, that
$$\phi(\overline{S})^{\#_{\overline{S}}\#_{\overline{S}}} = (\mathsf{Sp}\overline{S})^{\#_{\overline{S}}}$$
is dense in $\overline{S}$, that is, $\bigwedge \overline{S} = x \in \phi(\overline{S})^{\#_{\overline{S}}\#_{\overline{S}}}$ (the notation $(-)^{\#_T}$ denotes supplements in $\mathsf{S}(T)$). Since closed sublocales of a subfit locale are subfit and $x \in \phi(\overline{S})^{\#_{\overline{S}}\#_{\overline{S}}}$ we obtain from 5.3.3 (applied to $\overline{S}$) a family $\{a_j\}_{j\in J}$ such that $a_j \not\leq p$ for all $j$ and $p \in \mathsf{Prm}(\overline{S})$ and $x = \bigwedge_j a_j$. Then, by Proposition 6.1, there is a $p \in \mathsf{Prm}(L)$ and a $j$ with $a_j \leq p$. This is a contradiction.

Finally, let us show that (3) implies (1). Let $1 \neq x = \bigwedge_i p_i = \bigwedge_j a_j$ for some $\{p_i\}_{i\in I} \subseteq \mathsf{Prm}(L)$ and $\{a_j\}_{j\in J} \subseteq L$. Assume, by way of contradiction, that for all $i$, $a_i \not\leq p$ for all $p \in \mathsf{Prm}(L)$. If we set $S = \bigvee_i \mathfrak{b}(p_i) \neq \mathsf{O}$, we see that $\mathfrak{c}_{\overline{S}}(a_i) \subseteq \phi(\overline{S})^{\#_{\overline{S}}\#_{\overline{S}}}$ for every $i$. Then
$$\bigvee_i \mathfrak{c}_{\overline{S}}(a_i) \subseteq \phi(\overline{S})^{\#_{\overline{S}}\#_{\overline{S}}},$$
and since the left hand side is (obviously) dense in $\overline{S}$, so is $\phi(\overline{S})^{\#_{\overline{S}}\#_{\overline{S}}}$, which yields a contradiction. ■

**6.3.** Let us say that a $T_D$-locale $L$ is *weakly spatial* if $\Phi(L)^{\#\#}$ is not dense. Clearly, an $L \neq \mathsf{O}$ which is spatial is also weakly spatial.

**Corollary.** *A subfit $T_D$-locale $L$ has complemented spatial part if and only if every nonzero closed sublocale with dense spatial part is weakly spatial.*

**6.4.** Regarding the case whether the spatial part is closed, we have two independent criteria, one for the subfit case and the other for the $T_1$ case.

**6.4.1. Proposition.** *Let $L$ be subfit. Then* $\mathsf{Sp}L$ *is closed if and only if for any $a \in L$ such that $\Sigma_a = \mathsf{Prm}(L)$, one has $a \vee \bigwedge \mathsf{Prm}(L) = 1$.*

*Proof*: The condition $\Sigma_a = \mathsf{Prm}(L)$ is equivalent to $\mathsf{Sp}L \subseteq \mathfrak{o}(a)$, and $a \vee \bigwedge \mathsf{Prm}(L) = 1$ is equivalent to

$$\overline{\mathsf{Sp}L} = \mathfrak{c}(\bigwedge \mathsf{Prm}(L)) \subseteq \mathfrak{o}(a).$$

Hence, the condition in the statement is equivalent to the fittings of $\mathsf{Sp}L$ and $\overline{\mathsf{Sp}L}$ being equal. The "only if" part thus follows trivially. For the "if" part, note that if the fittings of $\mathsf{Sp}L$ and $\overline{\mathsf{Sp}L}$ in $L$ are equal, then the fitting of $\mathsf{Sp}L$ inside $\overline{\mathsf{Sp}L}$ is the whole $\overline{\mathsf{Sp}L}$. But subfitness is hereditary w.r.t. closed sublocales so it follows from [6, Thm. 5.1.2] that $\mathsf{Sp}L = \overline{\mathsf{Sp}L}$. ■

**6.4.2. Proposition.** *The following are equivalent for a $T_1$-locale $L$:*

(1) $\mathsf{Sp}L$ *is closed.*
(2) *The meet $\bigwedge \mathsf{Prm}(L)$ is exact.*
(3) *Each of the meets $\bigwedge\{p \in \mathsf{Prm}(L) \,|\, a \leq p\}$ is exact.*

*Proof*: (1)⇒(2): Let $\uparrow a = \mathsf{Sp}L$. Then, in particular, $a \in \mathsf{Sp}L$ and since it is smallest in $\mathsf{Sp}L$, $a = \bigwedge \mathsf{Prm}(L)$. Let $x \in L$ be arbitrary. Then $x \vee a \in \mathsf{Sp}L$ and hence

$$x \vee \bigwedge \mathsf{Prm}L = x \vee a = \bigwedge\{p \,|\, x \vee a \leq p\} = \bigwedge\{p \,|\, x \leq p\}$$

since $a \leq p$ for all $p$. But we also have $\bigwedge\{x \vee p \,|\, p \in \mathsf{Prm}(L)\} = \bigwedge\{p \,|\, x \leq p\}$ because $x \vee p$ is, by maximality, equal to $p$ if $x \leq p$ and 1 otherwise.
(2)⇒(1): Set $a = \bigwedge \mathsf{Prm}(L)$. If this meet is exact then for every $x \geq a$,

$$x = x \vee \bigwedge \mathsf{Prm}(L) = \bigwedge\{x \vee p \,|\, p \text{ prime and hence maximal}\} = \bigwedge\{p \,|\, x \leq p\}$$

since if $x \not\leq p$, $x \vee p = 1$ by maximality. Thus, $\uparrow a \subseteq \mathsf{Sp}L$, while $\mathsf{Sp}L \subseteq \uparrow a$ trivially.

Trivially (3)⇒(2), and (2)⇒(3) follows immediately by computing

$$(\bigwedge \mathsf{Prm}(L)) \vee a \vee b = \bigwedge\{p \in \mathsf{Prm}(L) \,|\, a \leq p\} \vee b. \quad ■$$

**6.5. Note.** The question naturally arises what about strengthening (3) to:

*Each of the meets $\bigwedge M$ with $M \subseteq \mathsf{Prm}(L)$ is exact.*

By using the fact that a meet $\bigwedge A$ is exact if and only if the sublocale $\bigvee_{a \in A} \mathfrak{c}(a)$ is closed, it follows readily that the condition that each of the meets $\bigwedge M$ with $M \subseteq \mathsf{Prm}(L)$ is exact is equivalent to every spatial sublocale being closed. This is in turn equivalent to the fact that

$\mathsf{Sp}L$ *is closed and* $\Sigma L$ *is discrete.*

We have

**6.5.1. Proposition.** *The following are equivalent for a* $T_1$*-locale* $L$*:*

(1) $\Sigma L$ *is discrete.*
(2) *For every* $M \subseteq \mathsf{Prm}(L)$, $\uparrow\bigwedge M = M$.
(3) *For every* $M \subseteq \mathsf{Prm}(L)$ *and every* $p \in \mathsf{Prm}(L)$, $(\bigwedge M) \vee p = \bigwedge\{m \vee p \,|\, m \in M\}$.

*Proof*: (1)⇒(2): Let $\Sigma L$ be discrete. Then for each $M \subseteq \mathsf{Prm}(L)$ there is an $x$ such that $\mathsf{Prm}(L) \smallsetminus M = \Sigma_x$. Then $x \nleq p$ iff $p \notin M$, hence $x \leq p$ iff $p \in M$, and hence $x \leq \bigwedge M$. Thus,

$$\bigwedge M \leq p \;\Rightarrow\; x \leq p \;\Rightarrow\; p \in M,$$

and $p \in M \;\Rightarrow\; \bigwedge M \leq p$ is trivial.
(2)⇒(3): We have $(\bigwedge M) \vee p = p$ iff $\bigwedge M \leq p$ and $\bigwedge\{m \vee p \,|\, m \in M\}$ iff $p \in M$, and under (2) $\bigwedge M \leq p$ and $b \in M$ are the same.
(3)⇒(1): For $M \subseteq \mathsf{Prm}(L)$ take $N = \mathsf{Prm}(L) \smallsetminus M$. Then $p \in M$ iff $p \notin N$ iff $\bigwedge N \nleq p$ iff $p \in \Sigma_{\bigwedge N}$. ∎

We may now apply Propositions 6.4.1 and 6.4.2 in order to obtain a characterization of $T_D$-locales $L$ such that the spatial part of $\mathsf{S}(L)^{\mathsf{op}}$ is closed.

**6.6. Corollary.** *The following are equivalent for a* $T_D$*-locale* $L$*:*

(1) $\mathsf{SpS}(L)^{\mathsf{op}}$ *is closed.*
(2) $\Phi L$ *is complemented.*
(3) $\mathsf{Sp}L$ *is totally spatial, that is, every sublocale of* $\mathsf{Sp}L$ *is spatial.*

**Note.** The conditions in the Corollary are very strong. This suggests that a sufficient condition on $L$ for $\mathsf{Sp}L$ being closed in terms of a separation-type property only does not seem possible (recall that $\mathsf{S}(L)^{\mathsf{op}}$ is always zero-dimensional).

### 6.7. Open spatial parts.

**6.7.1. Proposition.** *Let* $L$ *be subfit. Then* $\mathsf{Sp}L$ *is open if and only if there is a least* $a \in L$ *such that in the spectrum of* $L$, $\Sigma_a = \mathsf{Prm}(L)$*, more explicitly, if and only if there is an* $a \in L$ *such that*

$$\Sigma_x = \mathsf{Prm}(L) \;\Leftrightarrow\; x \geq a.$$

*Proof*: Since the condition $\Sigma_x = \Sigma L$ is equivalent to $\mathsf{Sp}L \subseteq \mathfrak{o}(x)$, the condition of the proposition means that $\mathsf{Sp}L \subseteq \mathfrak{o}(x)$ iff $\mathfrak{o}(a) \subseteq \mathfrak{o}(x)$, but this means precisely that the fitting of $\mathsf{Sp}L$ is $\mathfrak{o}(a)$. Under subfitness the latter implies that $\mathsf{Sp}L$ is open (see [6]). ■

Applying the result to a locale of the form $\mathsf{S}(L)^{\mathsf{op}}$, we immediately obtain the following

**6.7.2. Corollary.** *A frame $L$ has a largest pointless sublocale if and only if $\mathsf{Sp}(\mathsf{S}(L)^{\mathsf{op}})$ is open. In particular, by 5.3.1, for any $T_D$-locale, $\mathsf{Sp}(\mathsf{S}(L)^{\mathsf{op}})$ is open.*

**Acknowledgements.** The authors acknowledge financial support from the Basque Government (grant IT1483-22), from the Centre for Mathematics of the University of Coimbra (UIDB/00324/2020, funded by the Portuguese Government through FCT/MCTES) and from the Department of Applied Mathematics (KAM) of Charles University. The first named author also acknowledges support from a postdoctoral fellowship of the Basque Government (grant POS-2022-1-0015).

# References

[1] I. Arrieta, *On joins of complemented sublocales*, Algebra Univ. 83 (2022) Art. no. 1, 11 pp.
[2] I. Arrieta, *A study of localic subspaces, separation, and variants of normality and their duals*, PhD Thesis, University of Coimbra and University of the Basque Country UPV/EHU, 2022.
[3] C.E. Aull and W.J. Thron, *Separation axioms between $T_0$ and $T_1$*, Indag. Math. 24 (1963) 26–37.
[4] B. Banaschewski and A. Pultr, *Variants of openness*, Appl. Categ. Structures 2 (1994) 331–350.
[5] B. Banaschewski and A. Pultr, *On covered prime elements and complete homomorphisms of frames*, Quaest. Math. 37 (2014) 451—454.
[6] M.M. Clementino, J. Picado and A. Pultr, *The other closure and complete sublocales*, Appl. Categ. Structures 26 (2018) 891–906; errata ibid 907–908.
[7] M. Erné, J. Picado and A. Pultr, *Adjoint maps between implicative semilattices and continuity of localic maps*, Algebra Universalis 83 (2022) Art. no. 13, 23 pp.
[8] M.J. Ferreira, S. Pinto and J. Picado, *Remainders in point-free topology*, Topology Appl. 245 (2018) 21–45.
[9] J.R. Isbell, *Atomless parts of spaces*, Math. Scand. 31 (1972) 5–32.
[10] J.R. Isbell, *First steps in descriptive theory of locales*, Trans. Amer. Math. Soc. 327 (1991) 353—371; errata ibid 341 (1994) 467—468.
[11] P.T. Johnstone, *Stone Spaces*, Cambridge Univ. Press, Cambridge, 1982.
[12] P.T. Johnstone and S.-H. Sun *Weak products and Hausdorff locales*, in: *Categorical Algebra and its applications*, Lecture Notes in Mathematics, vol. 1348, pp. 173–193, Springer, Berlin (1988).

[13] J. Paseka and B. Šmarda, *$T_2$-frames and almost compact frames*, Czechoslovak Math. J. 42 (1992) 297–313.
[14] J. Picado and A. Pultr, *Frames and Locales: topology without points*, Frontiers in Mathematics, Vol. 28, Birkhäuser-Springer, Basel (2011).
[15] J. Picado and A. Pultr, *Separation in Point-Free Topology*, Birkhäuser-Springer, Cham (2021).
[16] J. Picado, A. Pultr and A. Tozzi, *Joins of closed sublocales*, Houston J. Math. 45 (2019) 21–38.
[17] T. Plewe, *Higher order dissolutions and Boolean coreflections of locales*, J. Pure Appl. Algebra 154 (2000) 273–293.
[18] J. Rosický and B. Šmarda, *$T_1$-locales*, Math. Proc. Cambridge Phil. Soc. 98 (1958) 81–86.
[19] H. Simmons, *A framework for topology*, in: Logic Colloquium '77, Studies in Logic and the Foundations of Mathematics, vol. 96, pp. 239–251, North-Holland, Amsterdam-New York (1978).
[20] H. Simmons, *The lattice theoretic part of topological separation properties*, Proc. Edinburgh Math. Soc. (2) 21 (1978) 41–48.
[21] A.L. Suarez, *On the relation between subspaces and sublocales*, J. Pure Appl. Alg. 226 (2022) Art. no. 106851, 24 pp.
[22] W.J. Thron, *Lattice-equivalence of topological spaces*, Duke Math. J. 29 (1962) 671–679.

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