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K-SPLINES ON SPD MANIFOLDS

MARGARIDA CAMARINHA, LUÍS MACHADO AND FÁTIMA SILVA LEITE

Abstract: The generalization of Euclidean splines to Riemannian manifolds was initially motivated by trajectory planning problems for rigid body motion. The increased interest in non-Euclidean splines was essentially due to its relevance in many areas of science and technology. Lie groups and symmetric spaces play an important role in this context. The manifold of symmetric positive definite (SPD) matrices is used, in particular, in computer vision, with emphasis in medical imaging. Different Riemannian structures have been considered in the SPD, in part to reduce the computational effort. In this paper, we first review the theory of high-order geometric splines for general Riemannian manifolds and its specialization to Lie groups. We then solve the variational problem that gives rise to spline curves on the manifold of symmetric positive definite matrices, equipped with the Log-Cholesky metric and having a Lie group group structure introduced in [4]. This enables considerable simplifications and, as a consequence, closed form expressions for higher-order polynomial splines are obtained.

Keywords: SPD manifolds, variational approach, Riemannian splines, Cholesky decomposition, Log-Cholesky metric, Lie group structures.

1. Introduction

Symmetric positive definite (SPD) matrices are widely used in data science applications. In computer vision, for instance, image and video information is encoded by SPD matrices. Identified as diffusion tensors, the SPD matrices are fundamental tools in medical imaging for many neuroscientific studies, including schizophrenia, multiple sclerosis, autism, depression, hypoxia-ischemia, trauma, Alzheimer's disease and other dementias. More details about these applications can be found, for instance, in [11] and references therein. An essential task in image processing requires to interpolate known data to obtain new data. In this context, developing interpolation schemes for SPD matrices is clearly very important. But working on the interpolation of SPD matrices can be quite demanding, since the geometry of the SPD space has to be chosen to comply with the specific area of application and properties of the data.

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The most natural geometric framework is given by a Riemannian metric, and in this context Lie groups and symmetric spaces also play a role. Different Riemannian and Lie group structures on the SPD space have been considered in the literature (see for instance the recent work of Lin [4], Lin et al. [5], Pennec [12], Thanwerdas and Pennec [15], and Arsigny et al. [1]). The choice of the metric should be guided by the invariance and computational properties that are important for each specific application. This analysis can be given by observing different effects that can occur (swelling, fattening and shrinking effects). One inconvenient feature of Frobenius and other metrics is the swelling effect in the geodesics connecting two SPD matrices. The swelling can hinder the correct interpretation of the information carried by the SPD matrices, with evident consequences in data interpolation problems. The Log-Cholesky and Log-Euclidean metrics were designed to eliminate this setback. A comparison of the properties of the most commonly used metrics in the SPD space can be found in [12, 15].

The main objective of this paper is to study high-order interpolation on SPD manifolds. A high-order interpolation method in Riemannian manifolds based on the optimization properties of the Euclidean splines was introduced in Camarinha et al. [2]. This method gave rise to the so-called geometric splines. The generalization of Euclidean splines to Riemannian manifolds, initiated with the work of Noakes et al. [7] and Crouch and Silva Leite [3], was motivated by trajectory planning problems for rigid body motion, but quickly became quite relevant in many other areas of science and technology. In SPD manifolds, geometric splines were studied in Zhang and Noakes [16] and in Machado and Silva Leite [6].

In this paper, we first review the theory of high-order geometric splines for general Riemannian manifolds and its specialization to Lie groups, based on the work of Camarinha et al. [2] and Popiel [14], respectively. We then study geometric splines in SPD manifolds, by considering the Log-Cholesky metric and the Lie group structure introduced in [4]. Using that geometric structure, we derive a necessary and sufficient condition for a curve in SPD to be a geometric spline. We also present a closed form expression for cubic polynomials satisfying boundary conditions on position and velocity. The choice of the Log-Cholesky metric enables to obtain easy-to-compute expressions for higher order interpolation curves. The Cholesky factor representation of SPD matrices reduces substantially the computational costs in comparison with other metrics, which makes possible to work with larger dimensional input data.

2. Riemannian Splines

Let $(M, \langle\cdot,\cdot\rangle)$ be a n-dimensional Riemannian manifold. Denote by $\frac{D}{dt}$ the covariant derivative along a curve associated with the Levi-Civita connection on M, and by R the curvature tensor. For a curve x in M, the notation $\dfrac{D^{i+1}x}{dt^{i+1}}$ will be used to represent $\dfrac{D^i}{dt^i}\Big(\dfrac{dx}{dt}\Big)$, $i \geq 0$.

We consider the following natural generalization of the variational problem that gave rise to the Euclidean splines of odd degree.

Problem ($\mathcal{P}$):

$$\min_{x\in\Gamma} \ \frac{1}{2}\int_0^1 \Big\langle \frac{D^m x}{dt^m}, \frac{D^m x}{dt^m}\Big\rangle \, dt, \tag{1}$$

over the class Γ of $\mathcal{C}^{2m-3}$ paths x on M satisfying $x|_{[t_i,t_{i+1}]}$ is smooth,

$$x(t_i) = x_i, \ \ 0 \leq i \leq N, \tag{2}$$

for a distinct set of points $x_i \in M$ and fixed times t_i, $0 \leq i \leq N$, where $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = 1$, and, in addition,

$$\frac{D^j x}{dt^j}(0) = v_0^j, \quad \frac{D^j x}{dt^j}(1) = v_1^j, \quad 1 \leq j \leq m-1, \tag{3}$$

where v_i^j, with $i = 0,1$ and $1 \leq j \leq m-1$, are fixed tangent vectors.

Proposition 1. *([2]) A necessary condition for x to be a minimizer of the functional (1) is that x is $\mathcal{C}^{2m-2}$ and, for $0 \leq i \leq N-1$,*

$$\frac{D^{2m}x(t)}{dt^{2m}} + \sum_{j=2}^{m}(-1)^j R\left(\frac{D^{2m-j}x(t)}{dt^{2m-j}}, \frac{D^{j-1}x(t)}{dt^{j-1}}\right)\frac{dx(t)}{dt} = 0, \ \forall t \in [t_i, t_{i+1}]. \tag{4}$$

Definition 1. *We say that a curve $x \in \Gamma$ is a* geometric spline of degree $2m-1$ on M *if x is $\mathcal{C}^{2m-2}$ and each curve segment $x|_{[t_i,t_{i+1}]}$ satisfies equation (4).*

In the absence of interpolating points, these curves are smooth and we naturally call them *geometric polynomials of degree* $2m-1$. Hence, geometric splines of degree $2m-1$ can be described as $\mathcal{C}^{2m-2}$ curves obtained by concatenation of geometric polynomials at the interpolating points.

Geometric polynomials of degree one are geodesics in a Riemannian manifold and geometric polynomials of degree three are given through the equation

$$\frac{D^4x}{dt^4}+R\Big(\frac{D^2x}{dt^2},\frac{dx}{dt}\Big)\frac{dx}{dt}=0. \tag{5}$$

These curves were studied in Noakes et al. in [7] and Crouch and Silva Leite [3], to develop dynamical interpolation schemes on Lie groups and symmetric spaces.

Suppose now that the Riemannian manifold M is a connected Lie group G endowed with a bi-invariant metric and denote by $\mathfrak{g}$ its Lie algebra. Equation (4) can be reduced to a $2m-1$ order differential equation in $\mathfrak{g}$ using the so-called Lie reduction of a vector field along a curve. Given a curve x in G, we define the curve in $\mathfrak{g}$ by $V=d\ell_x^{-1}\circ\frac{dx}{dt}$, where ℓ_x denotes the left translation in G. We denote by $V^{(s)}$ the usual s-order derivative of V. In order to write the reduced equation in V for all values of m, we define $V_0:=V$ and introduce the following auxiliary variables V_k, $k=1,\dots,2m-2$, and Z_m.

$$\begin{aligned} V_k &= V_{k-1}^{(1)}+\frac{1}{2}[V,V_{k-1}],\ k=1,\dots,2m-2, &(6)\\ Z_m &= V_{2m-2}+Y_m, &(7)\end{aligned}$$

with $Y_m=\frac{1}{2}\sum_{j=2}^{m}(-1)^j[V_{2m-j-1},V_{j-2}]$. This method was proposed in [14] and permits to express the equation (4) in terms of a Lax equation.

Proposition 2. ([14]) *A curve x is a geometric polynomial of degree $2m-1$ iff*

$$\begin{aligned} \frac{dx}{dt} &= d\ell_x\circ V &(8)\\ Z_m^{(1)} &= [Z_m,V], &(9)\end{aligned}$$

with Z_m given by (6–7).

The reduced equation in $\mathfrak{g}$ is highly complex, but the auxiliary variable Z_m enables to rewrite it in Lax form and easily identify conserved quantities. There are very few examples where a closed form expression for geometric polynomials is known, even for the lowest values of m. For the case $m = 2$, it is important to mention the extensive work done by Noakes and its collaborators [8, 9, 10]. In this case, the equation (9) is

$$Z_2^{(1)} = [Z_2, V], \tag{10}$$

with $Z_2 = V_2 + Y_2$ and $V_2 = V_1^{(1)} - Y_2$. Then $Z_2 = V^{(2)}$ and we have

$$V^{(3)} = [V^{(2)}, V]. \tag{11}$$

Equation (11) was first obtained for $G = SO(3)$ in [7] and for general Lie groups in [3].

When the Lie group G is Abelian, the equations (6)-(9) simplify substantially. Geometric polynomials are then obtained through Euclidean polynomials in the Lie algebra. Moreover, they give rise to the solution of Problem $(\mathcal{P})$, since the necessary condition in Proposition 1 is also sufficient.

In the next section, we obtain geometric splines in SPD manifolds endowed with the so-called Log-Cholesky metric introduced by Lin in [4].

3. Splines on SPD manifolds

3.1. Geometry of SPD with respect to the Log-Cholesky metric. In this subsection, we review the results introduced in [4] that are most relevant to establish our main result.

Given a $n \times n$ real matrix $A = [a_{i,j}]$, we use $\mathbb{L}(A)$ to represent the $n \times n$ matrix whose (i, j) element is a_{ij} whenever $i > j$ and 0 otherwise. We also use $\mathbb{D}(A)$ to denote the diagonal matrix whose (i, i) element is a_{ii}.

Denote by $\mathfrak{t}(n)$ the set of $n \times n$ lower triangular matrices. A matrix $X \in \mathfrak{t}(n)$ can be written as $X = \mathbb{L}(X) + \mathbb{D}(X)$, where $\mathbb{L}(X)$ is the strictly lower triangular part and $\mathbb{D}(X)$ is the diagonal part of X.

Now, denote by $\mathfrak{t}^+(n)$ the subset of $\mathfrak{t}(n)$ with positive diagonal elements. A matrix $L \in \mathfrak{t}^+(n)$ can be parametrized by a lower triangular matrix via the diffeomorphism

$$\begin{array}{rccl} \mathscr{D}: & \mathfrak{t}^+(n) & \longrightarrow & \mathfrak{t}(n) \\ & L & \longmapsto & \mathscr{D}(L) = \mathbb{L}(L) + \log(\mathbb{D}(L)). \end{array} \tag{12}$$

We can use the map $\mathscr{D}$ to pull back to $\mathfrak{t}^+(n)$ the additive product in $\mathfrak{t}(n)$, obtaining the following product $\odot$ in $\mathfrak{t}^+(n)$

$$L \odot K = \mathbb{L}(L) + \mathbb{L}(K) + \mathbb{D}(L)\mathbb{D}(K), \tag{13}$$

which gives an Abelian Lie group structure to $\mathfrak{t}^+(n)$. Then the diffeomorphism $\mathscr{D}$ becomes an isomorphism between the Lie groups $(\mathfrak{t}^+(n), \odot)$ and $(\mathfrak{t}(n), +)$.

The tangent map of the left translation ℓ_L at $K \in \mathfrak{t}^+(n)$ is given by

$$d(\ell_L)_K(Y) = \mathbb{L}(Y) + \mathbb{D}(L)\mathbb{D}(Y), \quad \text{for } Y \in T_K\mathfrak{t}^+(n), \tag{14}$$

where the tangent space of $\mathfrak{t}^+(n)$ at a point K is identified with $\mathfrak{t}(n)$.

Let $\langle\ .\ ,\ .\ \rangle_F$ denote the Frobenius inner product on $\mathfrak{t}(n)$, given by $\langle A, B\rangle_F = \mathrm{Tr}(A^\top B)$. The following defines an inner product on each tangent space of $\mathfrak{t}^+(n)$, obtained throughout the diffeomorphism $\mathscr{D}$,

$$\begin{aligned} \ll X, Y \gg_L \;=\; & \langle \mathbb{L}(X), \mathbb{L}(Y)\rangle_F \\ & + \langle \mathbb{D}(L)^{-1}\mathbb{D}(X), \mathbb{D}(L)^{-1}\mathbb{D}(Y)\rangle_F, \qquad L \in \mathfrak{t}^+(n). \end{aligned} \tag{15}$$

Consequently, $\mathfrak{t}^+(n)$ is a Riemannian manifold with bi-invariant metric $\ll \cdot, \cdot \gg$.

Now, let $\mathfrak{s}(n)$ be the set of $n \times n$ real symmetric matrices and $\mathfrak{s}^+(n)$ the open convex half cone of symmetric and positive definite matrices. Given $P \in \mathfrak{s}^+(n)$, there exists a unique $L \in \mathfrak{t}^+(n)$, such that $P = LL^\top$. The matrix L is called the *Cholesky factor* of P. The *Cholesky map* is the following diffeomorphism,

$$\begin{aligned} \mathscr{L} :\ \mathfrak{s}^+(n) &\longrightarrow \mathfrak{t}^+(n) \\ P &\longmapsto \mathscr{L}(P) = L, \end{aligned} \tag{16}$$

where $P = LL^\top$. Its inverse is defined by

$$\begin{aligned} \mathscr{S} :\ \mathfrak{t}^+(n) &\longrightarrow \mathfrak{s}^+(n) \\ L &\longmapsto \mathscr{S}(L) = LL^\top. \end{aligned} \tag{17}$$

The composition $\mathscr{D} \circ \mathscr{L}$ gives the Log-Cholesky parametrization presented in [13]. Using this single chart, a matrix $P = LL^\top \in \mathfrak{s}^+(n)$ can be represented by the lower triangular matrix $\mathbb{L}(L) + \log(\mathbb{D}(L))$.

In order to equip the SPD manifold $\mathfrak{s}^+(n)$ with a Riemannian structure, define

$$S_{\frac{1}{2}} := \mathbb{L}(S) + \frac{1}{2}\mathbb{D}(S), \quad \text{for a square matrix } S. \tag{18}$$

The tangent map of $\mathscr{S}$ at $L \in \mathfrak{t}^+(n)$ is given by

$$\begin{aligned} d\mathscr{S}_L :\ T_L\mathfrak{t}^+(n) &\longrightarrow T_{LL^\top}\mathfrak{s}^+(n) \\ X &\longmapsto XL^\top + LX^\top, \end{aligned} \tag{19}$$

with inverse

$$\begin{array}{rccc} d\mathscr{L}_{LL^\top}: & T_{LL^\top}\mathfrak{s}^+(n) & \longrightarrow & T_L\mathfrak{t}^+(n) \\ & W & \longmapsto & L(L^{-1}WL^{-\top})_{\frac{1}{2}} \end{array} . \tag{20}$$

Now, given $P = LL^\top \in \mathfrak{s}^+(n)$ and $V, W \in T_P\mathfrak{s}^+(n)$, the diffeomorphism $\mathscr{S}$ induces a Riemannian metric in $\mathfrak{s}^+(n)$, called *Log-Cholesky metric*, defined by

$$\big\langle V, W \big\rangle_P = \ll L(L^{-1}VL^{-\top})_{\frac{1}{2}}, L(L^{-1}WL^{-\top})_{\frac{1}{2}} \gg_L . \tag{21}$$

This Riemannian structure is clearly based on the Log-Cholesky parametrization mentioned above.

Introducing the multiplication $\otimes$ on $\mathfrak{s}^+(n)$, such that the Log-Cholesky map $\mathscr{L}$ is a homomorphism, $(\mathfrak{s}^+(n), \otimes)$ becomes an Abelian Lie group and the Log-Cholesky metric is bi-invariant.

These Riemannian and Lie structures are clearly based on the Log-Cholesky parametrization mentioned above.

3.2. k-Splines on SPD manifolds with the Log-Cholesky metric. The main goal of this section is to solve Problem $(\mathcal{P})$ when the SPD manifold $\mathfrak{s}^+(n)$ is equipped with the Lie group structure $\otimes$.

Theorem 1. *A necessary and sufficient condition for x to be a minimizer of the functional (1) over the class Γ of $\mathcal{C}^{2m-3}$ paths x on $\mathfrak{s}^+(n)$, such that $x|_{[t_i,t_{i+1}]}$ is smooth, satisfies $x(t_i) = P_i$, $0 \le i \le N$, and also*

$$\frac{D^j x}{dt^j}(0) = V_0^j, \quad \frac{D^j x}{dt^j}(1) = V_1^j, \quad 1 \le j \le m-1,$$

is that, x is $\mathcal{C}^{2m-2}$, and, $\forall t \in [t_i, t_{i+1}]$ and $0 \le i \le N-1$, the following holds

$$x(t) = \mathscr{S}\Big(\mathbb{L}(L_i) + \sum_{j=1}^{2m-1} \frac{(t-t_i)^j}{j!}\mathbb{L}(Y_j) + \mathbb{D}(L_i)\exp\Big(\sum_{j=1}^{2m-1} \frac{(t-t_i)^j}{j!}\,\mathbb{D}(Y_j)\Big)\Big), \tag{22}$$

where $L_i = \mathscr{L}(P_i)$ and $Y_j \in \mathfrak{t}(n)$, $j = 1, \dots, 2m-1$, are determined by the interpolation and boundary conditions.

Proof: Let us consider a curve $x \in \Gamma$ satisfying the necessary conditions of Proposition 1. Since the Log-Cholesky map $\mathscr{L}$ in (16) defines an isometry between $\mathfrak{s}^+(n)$ and $\mathfrak{t}^+(n)$, the curve $\widetilde{x} = \mathscr{L}(x)$ in $\mathfrak{t}^+(n)$ also satisfies equation (4). But, being $(\mathfrak{t}^+(n), \odot)$ an Abelian Lie group, the Lie bracket vanishes identically and equations (6-9) simply reduce to

$$\widetilde{V}^{(2m-1)}(t) = 0, \; t \in [t_i, t_{i+1}], \; (0 \le i \le N-1), \tag{23}$$

where $\widetilde{V}$ is the curve in $\mathfrak{t}(n)$ defined by $\widetilde{V} = d\ell_{\widetilde{x}}^{-1} \circ \dfrac{d\widetilde{x}}{dt}$. Therefore,

$$\widetilde{V}(t) = \sum_{j=0}^{2m-2} \frac{(t-t_i)^j}{j!} Y_{j+1}, \quad \text{with} \quad Y_j \in \mathfrak{t}(n),\ j = 1, \ldots, 2m-1.$$

Writing $\widetilde{V} = \mathbb{L}(\widetilde{V}) + \mathbb{D}(\widetilde{V})$, and using the expression (14) for the differential of left translation, equation (23) can be decomposed in the following two equations.

$$\frac{d\mathbb{L}(\widetilde{x})}{dt} = \mathbb{L}(\widetilde{V}), \quad \frac{d\mathbb{D}(\widetilde{x})}{dt} = \mathbb{D}(\widetilde{x})\mathbb{D}(\widetilde{V}). \tag{24}$$

On the other hand, the decomposition $Y_j = \mathbb{L}(Y_j) + \mathbb{D}(Y_j)$, $j = 1, \ldots, 2m-1$, allows us to obtain

$$\mathbb{L}(\widetilde{V}(t)) = \sum_{j=0}^{2m-2} \frac{(t-t_i)^j}{j!} \mathbb{L}(Y_{j+1}), \quad \mathbb{D}(\widetilde{V}(t)) = \sum_{j=0}^{2m-2} \frac{(t-t_i)^j}{j!} \mathbb{D}(Y_{j+1}).$$

Now, integrating equations (24) in the interval $[t_i, t_{i+1}]$, it is immediate to conclude that $\widetilde{x}$ is given explicitly by $\widetilde{x}(t) = \mathbb{L}(\widetilde{x}(t)) + \mathbb{D}(\widetilde{x}(t))$, where

$$\begin{aligned} \mathbb{L}(\widetilde{x}(t)) &= \mathbb{L}(L_i) + \sum_{j=1}^{2m-1} \frac{(t-t_i)^j}{j!} \mathbb{L}(Y_j), \\ \mathbb{D}(\widetilde{x}(t)) &= \mathbb{D}(L_i) \exp\Big(\sum_{j=1}^{2m-1} \frac{(t-t_i)^j}{j!} \mathbb{D}(Y_j) \Big). \end{aligned} \tag{25}$$

Moreover, taking into account that diagonal matrices commute with each other, one can also write the analytical expression of $\widetilde{x}$, in the interval $[t_i, t_{i+1}]$, $0 \le i \le N-1$, as

$$\widetilde{x}(t) = \mathbb{L}(L_i) + \sum_{j=1}^{2m-1} \frac{(t-t_i)^j}{j!} \mathbb{L}(Y_j) + \mathbb{D}(L_i) \prod_{j=1}^{2m-1} \exp\Big(\tfrac{(t-t_i)^j}{j!}\, \mathbb{D}(Y_j)) \Big).$$

Since the necessary conditions satisfied by $\widetilde{x}$ are also sufficient and the matrices Y_j, $j = 1, \ldots, 2m-1$, are uniquely determined from the regularity conditions and from the following interpolation and boundary conditions,

$$\widetilde{x}(t_i) = L_i, \quad \frac{D^j \widetilde{x}}{dt^j}(0) = L_0 (L_0^{-1} V_0^j L_0^{-\top})_{\frac{1}{2}}, \quad \frac{D^j \widetilde{x}}{dt^j}(1) = L_1 (L_1^{-1} V_1^j L_1^{-\top})_{\frac{1}{2}},$$

the result follows.

$\square$ $\blacksquare$

The problem of finding the geodesic connecting two SPD matrices P_0 and P_1 is the case $m = 1$ of Problem $(\mathcal{P})$. Using the boundary conditions, we obtain

$$\mathbb{L}(Y_1) = \frac{\mathbb{L}(L_1) - \mathbb{L}(L_0)}{t_1 - t_0}, \quad \mathbb{D}(Y_1) = \frac{\log(\mathbb{D}(L_0)^{-1}\mathbb{D}(L_1))}{t_1 - t_0},$$

and the following holds.

Corollary 1. *The geodesic in $\mathfrak{s}^+(n)$ connecting the point P_0 (at $t = t_0$) to the point P_1 (at $t = t_1$) is given explicitly by*

$$x(t) = \mathscr{S}\Big(\mathbb{L}(L_0)+\tfrac{t - t_0}{t_1 - t_0}(\mathbb{L}(L_1)-\mathbb{L}(L_0))+\mathbb{D}(L_0)\exp\big(\tfrac{t - t_0}{t_1 - t_0}\log(\mathbb{D}(L_0)^{-1}\mathbb{D}(L_1))\big)\Big),$$

where $L_i \in \mathfrak{t}^+(n)$ is the Cholesky factor of P_i, $i = 0, 1$, $t \in [t_0, t_1]$.

Corollary 2. *The cubic polynomial x in $\mathfrak{s}^+(n)$ satisfying the boundary conditions*

$$x(t_0) = P_0, \quad \frac{Dx}{dt}(t_0) = V_0, \qquad x(t_1) = P_1, \quad \frac{Dx}{dt}(t_1) = V_1,$$

is given explicitly by

$$x(t) = \mathscr{S}\Big(\mathbb{L}(L_0) + \sum_{j=1}^{3}\tfrac{(t-t_0)^j}{j!}\mathbb{L}(Y_j) + \mathbb{D}(L_0)\exp\Big(\sum_{j=1}^{3}\tfrac{(t-t_0)^j}{j!}\mathbb{D}(Y_j)\Big)\Big), \; t \in [t_0, t_1],$$

where Y_i, $i = 1, 2, 3$, are given by

$$\begin{aligned}
\mathbb{L}(Y_1) &= \mathbb{L}(X_0),\\
\mathbb{L}(Y_2) &= \tfrac{2}{(t_1 - t_0)^2}\Big(3\big(\mathbb{L}(L_1) - \mathbb{L}(L_0)\big) - (t_1 - t_0)\big(2\mathbb{L}(X_0) + \mathbb{L}(X_1)\big)\Big),\\
\mathbb{L}(Y_3) &= \tfrac{6}{(t_1 - t_0)^3}\Big(2(\mathbb{L}(L_0) - \mathbb{L}(L_1)) + (t_1 - t_0)\big(\mathbb{L}(X_0) + \mathbb{L}(X_1)\big)\Big),
\end{aligned}$$

$$\begin{aligned}
\mathbb{D}(Y_1) &= \mathbb{D}(X_0)\mathbb{D}(L_0)^{-1},\\
\mathbb{D}(Y_2) &= \tfrac{2}{(t_1 - t_0)^2}\Big((t_0 - t_1)(2\mathbb{D}(L_0)^{-1}\mathbb{D}(X_0) + \mathbb{D}(L_1)^{-1}\mathbb{D}(X_1)) + 3\log(\mathbb{D}(L_1)\mathbb{D}(L_0)^{-1})\Big),\\
\mathbb{D}(Y_3) &= \tfrac{6}{(t_0 - t_1)^3}\Big((t_0 - t_1)(\mathbb{D}(L_0)^{-1}\mathbb{D}(X_0) + \mathbb{D}(L_1)^{-1}\mathbb{D}(X_1)) + 2\log(\mathbb{D}(L_1)\mathbb{D}(L_0)^{-1})\Big),
\end{aligned}$$

where L_i is the Cholesky factor of P_i and $X_i = L_i(L_i^{-1}V_iL_i^{-\top})_{\frac{1}{2}}, i = 1, 2$.

Figure 1 bellow illustrates geometric polynomials of degree 1 and 3 using the Log-Cholesky metric and the ones obtained in [6] for the Log-Euclidean metric. In Table 1, we register the corresponding determinants. With respect to swelling effect, we don't observe significant changes in the value of those determinants when the degree of the polynomial increases.

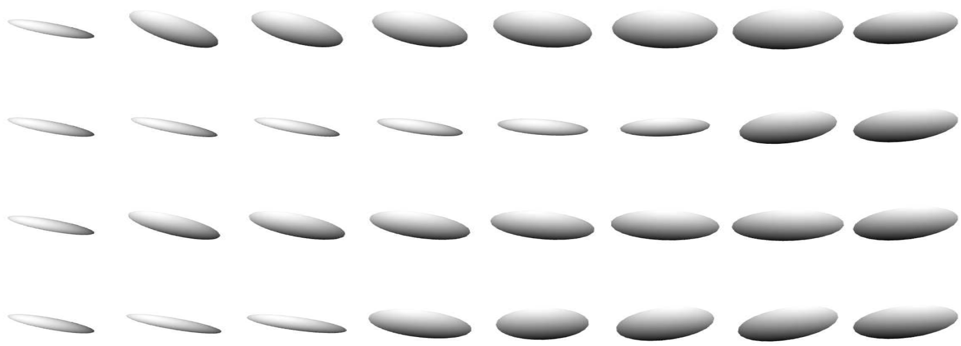

FIGURE 1. Interpolation through geodesics and geometric cubic polynomials joining the same elements. First row: Log-Cholesky geodesic interpolation. Second row: Log-Cholesky cubic interpolation. Third row: Log-Euclidean geodesic interpolation. Fourth row: Log-Euclidean cubic interpolation.

Geodesic	36.3214	34.8781	33.4923	32.1615	30.8836	29.6564	28.4780	27.3465
Cubic	36.3214	32.7492	23.3877	15.4079	10.9067	9.6617	12.4752	27.3465

TABLE 1. Values of the determinant of each iteration of the curve joining P_0 to P_1.

4. Conclusion

In this article, we constructed high-order polynomial spline curves on the SPD Riemannian manifold, equipped with the Log-Cholesky metric. These smooth curves minimize a certain energy functional, interpolate a given set of data points, and satisfy some boundary conditions.

The Abelian Lie group structure on the SPD manifold introduced in [4] enables considerable simplifications. In particular, the variational problem could be solved efficiently and closed form expressions for polynomial splines were obtained.

With the chosen structure, the interpolation problem was reduced to the Lie algebra, as shown in the proof of Theorem 1. This easily follows from the fact that the Riemannian exponential map is a global diffeomorphism that coincides at the identity with the Lie group exponential.

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Margarida Camarinha
University of Coimbra, CMUC, Department of Mathematics, 3000-143 Coimbra, Portugal
E-mail address: mmlsc@mat.uc.pt

Luís Machado
Institute of Systems and Robotics - University of Coimbra, Rua Silvio Lima - Polo II, 3030-290 Coimbra, Portugal and Department of Mathematics - University of Trás-os-Montes e Alto Douro (UTAD), Quinta de Prados, 5000-801 Vila Real, Portugal
E-mail address: lmiguel@utad.pt

Fátima Silva Leite
University of Coimbra, CMUC, Department of Mathematics, 3000-143 Coimbra, Portugal, and Institute of Systems and Robotics, University of Coimbra - Pólo II, Pinhal de Marrocos, 3030-290 Coimbra, Portugal
E-mail address: fleite@mat.uc.pt