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EMBEDDING TENSORS ON LIE ∞-ALGEBRAS WITH RESPECT TO LIE ∞-ACTIONS

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ABSTRACT: Given two Lie ∞-algebras E and V, any Lie ∞-action of E on V defines a Lie ∞-algebra structure on $E \oplus V$. Some compatibility between the action and the Lie ∞-structure on V is needed to obtain a particular Loday ∞-algebra, the non-abelian hemisemidirect product. These are the coherent actions. For coherent actions it is possible to define non-abelian homotopy embedding tensors as Maurer-Cartan elements of a convenient Lie ∞-algebra. Generalizing the classical case, we see that a non-abelian homotopy embedding tensor defines a Loday ∞-structure on V and is a morphism between this new Loday ∞-algebra and E.

KEYWORDS: Lie ∞-algebra, embedding tensor, hemisemidirect product.

Introduction

Embedding tensors have been widely used and studied in the context of supergravity theory and (higher) gauge theories. The link between embedding tensors, Leibniz algebras and Lie ∞-algebras is exploited in [10] and provides an interesting mathematical perspective that has brought this subject back to the forefront of mathematical/physical interests. Several works have appeared that provide new and different enlightenment on the mathematical framework behind embedding tensors and their tensor hierarchies (see for instance [4, 11, 8, 16, 12, 13, 14] and all the references therein). From a purely algebraic point of view, embedding tensors are also known as averaging operators on Lie/Leibniz algebras (see for instance [1, 18]). Recently, the homotopy version of embedding tensors, their cohomology and deformations was studied in [19]. More recently, in [20] we can find the case where the representation space is also Lie algebra and, in this case, the embedding tensors are called non-abelian.

Leibniz algebras are behind the classical (mathematical) notion of embedding tensors. A (left) Leibniz algebra is a vector space V together with a

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bilinear product $\circ$ satisfying the Jacobi identity:

$$(x \circ y) \circ z - x \circ (y \circ z) + y \circ (x \circ z) = 0, \quad x, y, z \in V.$$

Each Leibniz algebra has an associated Lie algebra called its gauge algebra $V^{Lie} = V / \left\langle x^2,\ x \in V \right\rangle$ such that the projection $p : V \to V^{Lie}$ preserves the products:

$$p(x \circ y) = [p(x), p(y)] .$$

Left multiplication turns V into a left V^{Lie}-module:

$$\Phi_{p(x)} y := p(x) \cdot y = x \circ y, \quad x, y \in V,$$

and $p : V \to V^{Lie}$, called the *embedding tensor* of V, is an equivariant linear map

$$p \circ \Phi_{p(x)} = \mathrm{ad}_{p(x)} \circ p, \quad x \in V.$$

This setting can be generalized to Lie-Leibniz triples [12, 13, 14]. A Lie-Leibniz triple is a triple $(\mathfrak{g}, V, T)$ where $\mathfrak{g}$ is a Lie algebra, V is a $\mathfrak{g}$-module (given by a representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$) and $T : V \to \mathfrak{g}$ is a linear map such that

$$[T(x), T(y)]_{\mathfrak{g}} = T(\rho_{T(x)} y), \quad x, y \in V.$$

It turns out that V acquires a Leibniz product given by $x \circ y := \rho_{T(x)} y$. Therefore, in a Lie-Leibniz triple $(\mathfrak{g}, V, T)$, the embedding tensor T is equivariant (in the image of T):

$$T \circ \rho_{T(x)} = \mathrm{ad}_{T(x)} \circ T, \quad x \in V.$$

Also, it defines a Leibniz product on $V \oplus \mathfrak{g}$ called the hemisemidirect product. Lie ∞-algebras appeared when authors tried to explain the tensor hierarchies associated with embedding tensors. Kotov and Strobl showed in [10] that any Leibniz algebra V gives rise to a universal Lie ∞-algebra structure in the minimal Lie subalgebra of $T(V)_{Lie} = (T(V), [\cdot, \cdot]_c)$. In another direction, Lavau and his co-authors [12], [13], [14] gave different approaches to the tensor hierarchies associated with embedding tensors. The construction given in [10] and the constructions given by Lavau and co-authors are different and natural relations between them are still unclear.

More recently, using the V-data approach by Voronov [22], Sheng et al. [19] introduced the homotopy version of embedding tensors with respect to a Lie ∞-representation. Any Lie ∞-representation $\Phi : E \to \mathrm{End}(V)$ defines a Lie ∞-algebra $E \ltimes V$ on $E \oplus V$ but also a Leibniz ∞-algebra $E \ltimes^{\Phi} V$ in the same space: the hemisemidirect product of E by V. In [19] homotopy embedding tensors are seen as Maurer-Cartan elements of a particular Lie ∞-algebra and the authors define a Leibniz ∞-algebra structure on V. Moreover, they also establish the existence of a functor from Leibniz ∞-algebras category to the Lie ∞-algebras category that, in particular, induces a functor from the category of homotopy embedding tensors to that of Lie ∞-algebras, thus generalizing of the result of Kotov and Strobl [10].

Recently, Tang and Sheng [20] considered the non-abelian case. Given an action $\rho : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$ of a Lie algebra $\mathfrak{g}$ on another Lie algebra $\mathfrak{h}$ the product $(x+v)\circ(y+w) = \rho_x w + [v,w]_{\mathfrak{h}}$ may not be a Leibniz product. For this to happen there has to be a compatibility between the action ρ and the Lie algebra $\mathfrak{h}$. These are the coherent actions. A non-abelian embedding tensor (with respect to a coherent action ρ) is a linear map $T : \mathfrak{h} \to \mathfrak{g}$ such that

$$[Tu, Tv]_{\mathfrak{g}} = T\left(\rho_{Tu} v + [u,v]_{\mathfrak{h}}\right), \quad u, v \in \mathfrak{h}.$$

When $\mathfrak{h}$ is abelian, usual embedding tensors are recovered. Any non-abelian embedding tensor defines a (new) Leibniz algebra structure on $\mathfrak{h}$ and is a Maurer-Cartan element of a differential graded Lie algebra.

In this paper we intend to give a homotopy version of non-abelian embedding tensors, thus generalizing both definitions in [19] and in [20].

Given two Lie ∞-algebras $(E, \{l_k\}_{k\in\mathbb{N}})$ and $(V, \{m_k\}_{k\in\mathbb{N}})$, a Lie ∞-action is a Lie ∞-morphism

$$\Phi : E \to \mathrm{Coder}(\bar{S}(V))[1]$$

and it defines a Lie ∞-algebra structure on $E \oplus V$, the direct product $E \ltimes V$ [15, 5]. We define coherent Lie ∞-actions and prove that being coherent is a necessary and sufficient condition for $E \oplus V$ to have a particular Loday ∞-structure: the non-abelian hemisemidirect product.

Theorem. *Let $\Phi : (E, \{l_k\}_{k\in\mathbb{N}}) \to (\mathrm{Coder}(\bar{S}(V))[1], \partial_{M_V}, [\cdot,\cdot])$ be a Lie ∞-action. Consider $E \oplus V$ equipped with brackets $\{\mathfrak{l}_n\}_{n\in\mathbb{N}}$ defined by:*

$$\begin{aligned}\mathfrak{l}_n(x_1+v_1, \ldots, x_{n-1}+v_{n-1}, x_n+v_n) &= l_n(x_1,\ldots,x_n)\\ &+ \sum_{i=1}^{n-1} \Phi^{i,n-i}(x_1,\ldots,x_i; v_{i+1},\ldots,v_n) + m_n(v_1,\ldots,v_n).\end{aligned}$$

Then, $E \oplus V$ is a Loday ∞-algebra if and only if Φ is a coherent action.

For coherent actions we define non-abelian homotopy embedding tensors. These are (Zinbiel) comorphisms $T : T(V) \to T(E)$ that are related to

Maurer-Cartan elements of a particular Lie ∞-algebra. We also give the explicit equations that define non-abelian homotopy embedding tensors. Then we prove that each non-abelian embedding tensor induces a (descendent) Leibniz ∞-algebra structure on V and, finally, we analyse some examples as well as the Lie ∞-algebra that controls the deformations of these tensors.

The paper is divided into three sections. In the first section we review Lie ∞-algebras and Loday ∞-algebras. First, we introduce notations and conventions on graded vector spaces. Then, we define Lie and Lie[1] ∞-algebras and its properties as well as Loday and Loday[1] ∞-algebras.

The reader should be warned that after Section 1 we will only work with Lie[1] ∞-algebras and Loday[1] ∞-algebra and, for the sake of simplification, we discard the symbol [1].

In Section 2, we define coherent Lie ∞-actions and the non-abelian hemisemidirect product $E \ltimes^{\Phi} V$. We show that Lie ∞-representations on chain complexes are always coherent Lie ∞-actions and we analyse some examples.

In Section 3, we introduce non-abelian homotopy embedding tensors. They are related to Maurer-Cartan elements of a particular Lie ∞-algebra and we establish explicit conditions that reveals what this Maurer-Cartan condition really means. We see how a non-abelian embedding tensor induces a new (descendent) Loday ∞-algebra and we prove that non-abelian embedding tensors behave well under the functor (Loday ∞) $\to$ (Lie ∞) given in [10, 19]. Since the adjoint representation of a Lie ∞-algebra defines a coherent Lie ∞-action, this special case is treated more closely. We finish by looking at deformations of embedding tensors and the Lie ∞-algebra that controls them.

1. Lie and Loday ∞-algebras

1.1. Notations and conventions on graded vector spaces. We will work with finite dimensional $\mathbb{Z}$-graded vector spaces and over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $V = \oplus_{i\in\mathbb{Z}} V_i$ be a graded vector space. Each V_i is the homogeneous component of V of degree i. An element x of V_i is called homogeneous with degree $|x|$.

The *suspension* of V is the graded vector space sV defined by $(sV)_i = V_{i-1}$, $i \in \mathbb{Z}$. The *desuspension* of V is the graded vector space $s^{-1}V$ defined by $(s^{-1}V)_i = V_{i+1}$, $i \in \mathbb{Z}$

The suspension operator $s : V \to sV$ is the linear map that increases the degree of the elements by 1:

$$s : x \in V_i \mapsto sx := x \in (sV)_{i+1} = V_i.$$

The desuspension operator $s^{-1} : V \to s^{-1}V$ lowers the degree of the elements by 1.

For each $k \in \mathbb{Z}$ we will denote by $V[k]$ the graded vector space $(s^{-1})^k V$, that is, $(V[k])_i = V_{i+k}$, $i \in \mathbb{Z}$.

A *morphism* $\Phi : V \to W$ between two graded vector spaces is a degree-preserving linear map, i.e., a collection of linear maps $\Phi_i : V_i \to W_i$, $i \in \mathbb{Z}$. We call $\Phi : V \to W$ a *morphism of degree* k, for some $k \in \mathbb{Z}$, if it is a morphism between V and $W[k]$. The space of morphisms of degree k is denoted by $\mathrm{Hom}^k(V, W)$ and $\mathrm{Hom}(V, W) = \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}^k(V, W)$.

Given two graded vector spaces V and W, their *direct sum* $V \oplus W$ (resp. *tensor product* $V \otimes W$) is the graded vector space with grading

$$(V \oplus W)_i = V_i \oplus W_i \quad (\text{resp. } (V \otimes W)_i = \oplus_{k+j=i} V_j \otimes W_k).$$

For each $k \in \mathbb{N}_0$, let $T^k(V) = \otimes^k V$. The tensor algebra over V is $T(V) = \oplus_{k\geq 0} T^k(V)$ and the reduced tensor algebra over V is $\bar{T}(V) = \oplus_{k\geq 1} T^k(V)$.

We follow the standard Koszul sign convention: for homogeneous morphisms $f : V \to W$ and $g : E \to L$, of degrees $|f|$ and $|g|$, respectively, the tensor product $f \otimes g : V \otimes E \to W \otimes L$ is the morphism of degree $|f| + |g|$ given by

$$(f \otimes g)(x \otimes y) = (-1)^{|g|\,|x|} f(x) \otimes g(y), \quad x \in V,\, y \in E.$$

For each $k \geq 1$, the permutation group of order k, S_k, acts on $T^k(V)$ by

$$\sigma(v_1 \otimes \ldots \otimes v_k) = \epsilon(\sigma; v_1 \otimes \ldots \otimes v_k) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)},$$

for $\sigma \in S_k$ and $v_1, \ldots, v_k \in V$, where $\epsilon(\sigma; v_1 \otimes \ldots \otimes v_k)$ stands for *Koszul sign*. Although the Koszul sign depends both on σ and $v_1 \otimes \ldots \otimes v_k$, for sake of simplification, we will simply denote it by $\epsilon(\sigma)$. A permutation σ of order $n = i_1 + \ldots + i_k$ is called a $(i_1, \ldots, i_k)$-*unshuffle* if

$$\begin{aligned} \sigma(1) < {} & \ldots < \sigma(i_1), \\ \sigma(i_1 + 1) < {} & \ldots < \sigma(i_1 + i_2), \\ & \ldots \\ \sigma(i_1 + \ldots + i_{k-1} + 1) < {} & \ldots < \sigma(n). \end{aligned}$$

The set of $(i_1, \ldots, i_k)$-unshuffles is denoted by $Sh(i_1, \ldots, i_k)$.

An important subset of unshuffles will be considered. An *increasing* $(i_1, \dots, i_k)$*-unshuffle* σ is an element of $Sh(i_1, \dots, i_k)$ such that

$$\sigma(i_1) < \sigma(i_1 + i_2) < \dots < \sigma(n).$$

The set of increasing $(i_1, \dots, i_k)$-unshuffles is denoted by $\widetilde{Sh}(i_1, \dots, i_k)$.

An element of $T^k(V)$ is called a *symmetric tensor* (of order k) if it is invariant by the action of S_k. The set $T(V)^S$ of all symmetric tensors is a graded subalgebra of $T(V)$, $i : T(V)^S \hookrightarrow T(V)$. The symmetrization map $\pi : T(V) \to T(V)^S$,

$$\pi(v_1 \otimes \dots, \otimes v_n) = \sum_{\sigma \in S_k} \frac{\epsilon(\sigma)}{k!} v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(k)}, \qquad v_1, \dots, v_k \in V,$$

allows the identification of the set of all symmetric tensors with the quotient

$$S(V) = T(V) / \left\langle x \otimes y - (-1)^{|x||y|} y \otimes x \right\rangle.$$

Therefore, the quotient $S(V)$ is a graded commutative algebra whose product we denote by $\cdot$:

$$v_1 \cdot \dots \cdot v_n \simeq \pi(v_1 \otimes \dots \otimes v_n), \quad v_1, \dots, v_n \in V.$$

A linear map $F : T(V) \to W$ is *symmetric* if $F \circ \pi = F$. Symmetric linear maps are in one-to-one correspondence with linear maps from $S(V)$ to W and we use this correspondence to write

$$F(v_1, \dots, v_k) = F(v_1 \cdot \dots \cdot v_k), \quad v_1, \dots, v_k \in V.$$

Given a linear map $F : T(V) \to W$, its *symmetrization* is the morphism $F^S : S(V) \to W$, defined by

$$F^S(v_1, \dots, v_k) = \sum_{\sigma \in S_k} \frac{\epsilon(\sigma)}{k!} F(v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(k)}), \quad v_1, \dots, v_k \in V.$$

The reduced tensor algebra $\bar{T}(V)$ can be equipped with the coshuffle coproduct $\Delta^c : \bar{T}(V) \to \bar{T}(V) \otimes \bar{T}(V)$ defined by

$$\Delta^c(v) = 0, \quad v \in V,$$

$$\Delta^c(v_1 \otimes \dots \otimes v_k) = \sum_{\substack{p=1 \\ \sigma \in Sh(p,k-p)}}^{k-1} \epsilon(\sigma) \left(v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(p)}\right) \otimes \left(v_{\sigma(p+1)} \otimes \dots \otimes v_{\sigma(k)}\right),$$

for all $v_1, \ldots, v_k \in V$. The coalgebra $(\bar{T}(V), \Delta^c)$ is non-unital, cocommutative and coassociative. It is called the (reduced)*coshuffle coalgebra* and is denoted by $\bar{T}^c(V)$.

The coshuffle coproduct Δ^c induces a cocommutative coassociative coproduct in $\bar{S}(V)$, which will also be denoted by Δ^c:

$$\Delta^c(v) = 0, \quad v \in V,$$

$$\Delta^c(v_1 \cdot \ldots \cdot v_k) = \sum_{\substack{p=1 \\ \sigma \in Sh(p,k-p)}}^{k-1} \epsilon(\sigma) \left(v_{\sigma(1)} \cdot \ldots \cdot v_{\sigma(p)}\right) \otimes \left(v_{\sigma(p+1)} \cdot \ldots \cdot v_{\sigma(k)}\right),$$

for all $v_1, \ldots, v_k \in V$. Its cogenerator is the projection map $p : \bar{S}(V) \to V$.

Let $\mathrm{Prim}(V) = \{x \in \bar{T}(V) : \Delta^c(x) = 0\}$ be the set of primitive elements of $T^c(V)$. It is a graded Lie subalgebra of $\mathrm{Lie}(V) = (T(V), [\cdot,\cdot]_c)$, where $[\cdot,\cdot]_c$ denotes the commutator of the graded algebra $(T(V), \otimes)$. In fact, $\mathrm{Prim}(V)$ is the free Lie algebra generated by V [17]. Moreover, $(\bar{T}(V), \Delta^c)$ and $(\bar{S}(\mathrm{Prim}(V)), \Delta^c)$ are isomorphic coalgebras [19, 10].

1.2. Lie[1] ∞-algebra. A linear map $Q : \bar{S}(V) \to \bar{S}(V)$ of degree $|Q|$ is called a *coderivation* of $(\bar{S}(V), \Delta^c)$ (of degree $|Q|$), if

$$\Delta^c Q = (Q \otimes \mathrm{Id} + \mathrm{Id} \otimes Q)\Delta^c.$$

Let $\mathrm{Coder}(\bar{S}(V))$ be the set of coderivations of the reduced symmetric coalgebra $(\bar{S}(V), \Delta^c)$. It is well known that $\mathrm{Coder}(\bar{S}(V))$ together with the graded commutator becomes a graded Lie algebra:

$$[Q, P]_c = Q \circ P - (-1)^{|Q||P|} P \circ Q, \quad Q, P \in \mathrm{Coder}(\bar{S}(V)).$$

Definition 1.1. A **Lie**[1] **∞-algebra** (or a symmetric Lie ∞-algebra) is a graded vector space V together with a degree $+1$ coderivation M_V of the reduced symmetric coalgebra $(\bar{S}(V), \Delta^c)$ squaring zero.

By a standard argument, any coderivation $Q \in \mathrm{Coder}(\bar{S}(V))$ defines (and is defined by) its restriction maps $Q_k : S^k(V) \to V$, $k \geq 1$. This way we have an alternative definition of Lie[1] ∞-algebra.

Definition 1.2. A **Lie**[1] **∞-algebra** is a graded vector space V together with a family a degree $+1$ linear maps $\left\{l_k : S^k(V) \to V\right\}_{k\geq 1}$ such that

$$\sum_{\substack{i=1\\ \sigma\in Sh(i,n-i)}}^{n} \epsilon(\sigma) l_{n-i+1}\left(l_i(v_{\sigma(1)},\ldots,v_{\sigma(i)}), v_{\sigma(i+1)},\ldots,v_{\sigma(n)}\right) = 0, \tag{1}$$

for all $n \in \mathbb{N}$, and all homogeneous elements $v_1,\ldots,v_n \in V$.

Remark 1.3. The isomorphisms $\otimes^k s^{-1} : T^k(V) \to T^k(s^{-1}V)$, $k \in \mathbb{N}$, yield an isomorphism between $\mathrm{Hom}(\bar{T}(s^{-1}V), s^{-1}V)$ and $\mathrm{Hom}(\bar{T}(V), V)$, given by:

$$Q_k \in \mathrm{Hom}^{|Q_k|}(T^k(s^{-1}V), s^{-1}V) \mapsto q_k = s\circ Q_k \circ \otimes^k s^{-1} \in \mathrm{Hom}^{|Q_k|+1-k}(T^k(V), V). \tag{2}$$

The inverse morphism is given by $Q_k = (-1)^{\frac{k(k-1)}{2}} s^{-1}\circ q_k \circ \otimes^k s$.

This isomorphism also provides an isomorphism between the symmetric algebra $S(s^{-1}V)$ and the exterior algebra $\wedge V$. Using this isomorphism one has the definition of *Lie ∞-algebra*: a vector space V equipped with a family of skew-symmetric brackets $l'_k : \wedge^k V \to V$, $k \geq 1$, of degree $2-k$, satisfying a collection of equations equivalent to (1).

Example 1.4. A graded Lie[1]-algebra is a Lie[1] ∞-algebra $V = \oplus_{i\in\mathbb{Z}} V_i$ such that $l_n = 0$, for $n \neq 2$. Then the degree 0 bilinear map on $sV = V[-1]$ is defined by:

$$[sx, sy] := l'_2(sx, sy) = (-1)^i s(l_2(x,y)), \quad x \in V_i, y \in V_j, \tag{3}$$

is a graded Lie bracket. In particular, if $V = V_{-1}$ is concentrated in degree -1, we get a Lie algebra structure.

Example 1.5. A differential graded Lie[1]-algebra (DGLA[1]) is a Lie[1] ∞-algebra $V = \oplus_{i\in\mathbb{Z}} V_i$ such that $l_n = 0$, for $n \neq 1$ and $n \neq 2$. Then $\mathrm{d} := l_1$ is a degree $+1$ linear map $\mathrm{d} : V \to V$ squaring zero and satisfies the following compatibility condition with the bracket $\{\cdot,\cdot\} = l_2(\cdot,\cdot)$:

$$\begin{cases} \mathrm{d}\{x,y\} + \{\mathrm{d}(x), y\} + (-1)^{|x|}\{x, \mathrm{d}(y)\} = 0, \\ \{\{x,y\}, z\} + (-1)^{|y||z|}\{\{x,z\}, y\} + (-1)^{|x|}\{x, \{y,z\}\} = 0. \end{cases}$$

Example 1.6. Let $(V = \oplus_{i\in\mathbb{Z}} V_i, \mathrm{d})$ be a cochain complex. Then $\mathrm{End}(V)[1] = (\oplus_{i\in\mathbb{Z}} \mathrm{End}_i V)[1]$ has a natural DGLA[1] structure with $l_1 = \partial_{\mathrm{d}}$, $\;l_2 = [\cdot,\cdot]$ given by:

$$\begin{cases} \partial_{\mathrm{d}}\phi = -\mathrm{d}\circ\phi + (-1)^{|\phi|+1}\phi\circ\mathrm{d}, \\ [\phi,\psi] = (-1)^{|\phi|+1}\left(\phi\circ\psi - (-1)^{(|\phi|+1)(|\psi|+1)}\psi\circ\phi\right), \end{cases}$$

for ϕ, ψ homogeneous elements of $\mathrm{End}(V)[1]$.

Let $(E, M_E \equiv \{l_k\}_{k\in\mathbb{N}})$ and $(V, M_V \equiv \{m_k\}_{k\in\mathbb{N}})$ be Lie ∞-algebras. A *Lie ∞-morphism* $\Phi : E \to V$ is given by a collection of degree zero linear maps:

$$\Phi_k : S^k(E) \to V, \quad k \geq 1,$$

such that, for each $n \geq 1$,

$$\sum_{\substack{k+l=n \\ \sigma\in Sh(k,l) \\ l\geq 0,\, k\geq 1}} \varepsilon(\sigma)\Phi_{1+l}\Big(l_k(x_{\sigma(1)},\dots,x_{\sigma(k)}), x_{\sigma(k+1)},\dots,x_{\sigma(n)}\Big) =$$

$$= \sum_{\substack{k_1+\ldots+k_j=n \\ \sigma\in\widetilde{Sh}(k_1,\ldots,k_j)}} \varepsilon(\sigma)\, m_j\Big(\Phi_{k_1}(x_{\sigma(1)},\dots x_{\sigma(k_1)}), \Phi_{k_2}(x_{\sigma(k_1+1)},\dots x_{\sigma(k_1+k_2)}),\dots, \tag{4}$$
$$\Phi_{k_j}(x_{\sigma(k_1+\ldots+k_{j-1}+1)},\dots,x_{\sigma(n)})\Big).$$

If $\Phi_k = 0$ for $k \neq 1$, then Φ is called a *strict Lie ∞-morphism.*

Considering the coalgebra morphism $\Phi : \bar{S}(E) \to \bar{S}(V)$ defined by the collection of degree zero linear maps

$$\Phi_k : S^k(E) \to V, \quad k \geq 1,$$

we see that Equation (4) is equivalent to Φ preserving the Lie ∞-algebra structures:

$$\Phi \circ M_E = M_V \circ \Phi.$$

Let us now give the definition of a (curved) representation of a Lie ∞-algebra.

A complex (V, d) induces a natural symmetric DGLA[1] structure on $\mathrm{End}(V)[1]$ (see Example 1.6). A *representation* of a Lie ∞-algebra $(E, M_E \equiv \{l_k\}_{k\in\mathbb{N}})$ on a complex (V, d) is a Lie ∞-morphism

$$\Phi : (E, \{l_k\}_{k\in\mathbb{N}}) \to (\mathrm{End}(V)[1], \partial_{\mathrm{d}}, [\cdot,\cdot]),$$

i.e., $\Phi \circ M_E = M_{\mathrm{End}(V)[1]} \circ \Phi$, where $M_{\mathrm{End}(V)[1]}$ is the coderivation determined by $\partial_{\mathrm{d}} + [\cdot,\cdot]$.

Equivalently, a representation of E on (V, d) is defined by a collection of degree $+1$ maps

$$\Phi_k : S^k(E) \to \mathrm{End}(V), \quad k \geq 1,$$

such that, for each $n \geq 1$,

$$\sum_{\substack{i=1 \\ \sigma\in Sh(i,n-i)}}^{n} \varepsilon(\sigma)\Phi_{n-i+1}\left(l_i\left(x_{\sigma(1)},\dots,x_{\sigma(i)}\right), x_{\sigma(i+1)},\dots,x_{\sigma(n)}\right)$$

$$=\partial\Phi_n(x_1,\dots,x_n)+\sum_{\substack{j=1 \\ \sigma\in \widetilde{Sh}(j,n-j)}}^{n-1} \varepsilon(\sigma)\big[\Phi_j(x_{\sigma(1)},\dots,x_{\sigma(j)}), \Phi_{n-j}(x_{\sigma(j+1)},\dots,x_{\sigma(n)})\big].$$

A *Maurer-Cartan element* of a Lie[1] ∞-algebra $(E,\{l_k\}_{k\in\mathbb{N}})$ is a degree zero element e of E such that

$$\sum_{k\geq 1}\frac{1}{k!}\, l_k(e,\dots,e)=0. \tag{5}$$

The set of Maurer-Cartan elements of E is denoted by $\mathrm{MC}(E)$. Let e be a Maurer-Cartan element of $(E,\{l_k\}_{k\in\mathbb{N}})$ and set, for $k\geq 1$,

$$l_k^e(x_1,\dots,x_k) := \sum_{i\geq 0}\frac{1}{i!}\, l_{k+i}(e,\dots,e,x_1,\dots,x_k). \tag{6}$$

Then, $(E,\{l_k^e\}_{k\in\mathbb{N}})$ is a Lie[1] ∞-algebra, called *twisting of E* by e [7]. For filtered, or even weakly filtered Lie ∞-algebras, the convergence of the infinite sums defining Maurer-Cartan elements and twisted Lie ∞-algebras (Equations (5) and (6)) is guaranteed (see [7, 6]).

1.3. Loday ∞-algebra and Loday[1] ∞-algebra. Let us recall the definition and some properties of Loday ∞-algebras and Loday[1] ∞-algebras. More details can be found in [2, 21].

Let V be a graded vector space. The coproduct

$$\Delta^Z:\bar{T}(V)\to\bar{T}(V)\otimes\bar{T}(V)$$

given by

$$\Delta^Z(v)=0,\quad v\in V,$$

$$\Delta^Z(v_1\otimes\ldots\otimes v_k)=\sum_{\substack{p=1 \\ \sigma\in Sh(p,k-p-1)}}^{k-1}\epsilon(\sigma)\left(v_{\sigma(1)}\otimes\ldots\otimes v_{\sigma(p)}\right)\otimes\left(v_{\sigma(p+1)}\otimes\ldots\otimes v_{\sigma(k-1)}\otimes v_k\right),$$

for all $v_1, \ldots, v_k \in V$, provides a (cofree, conilpotent) Zinbiel coalgebra structure to $\bar{T}(V)$, i.e, this coproduct satisfies

$$(\mathrm{Id} \otimes \Delta^Z)\Delta^Z = (\Delta^Z \otimes \mathrm{Id})\Delta^Z + (\tau \circ \Delta^Z \otimes \mathrm{Id})\Delta^Z,$$

where τ is the twisting map: $\tau(v \otimes w) = (-1)^{|v||w|} w \otimes v$.

Remark 1.7. Notice that $\Delta^c = \Delta^Z + \tau \circ \Delta^Z$, this means that the coshuffle coproduct is the symmetrization of the Zinbiel coproduct.

The pair $(\bar{T}(V), \Delta^Z)$ is the *Zinbiel reduced tensor coalgebra* and we will denote it by $T^Z(V)$. The cogenerator of $T^Z(V)$ is the projection map $p : \bar{T}(V) \to V$.

Let $\mathrm{Coder}(T^Z(V))$ be the set of coderivations of $T^Z(V) = (\bar{T}(V), \Delta^Z)$. There is a (standard) isomorphism between $\mathrm{Coder}(T^Z(V))$ and $\mathrm{Hom}(\bar{T}(V), V)$ that we describe now: each linear map $Q_k : T^k(V) \to V$ $(k \geq 1)$ of degree $|Q|$ defines the coderivation Q given by

$$\begin{aligned} Q(v_1 \otimes \ldots \otimes v_n) \;=\; & \sum_{k=1}^{n} \sum_{\substack{i=0,\\ \sigma \in Sh(i,k-1)}}^{k-1} (-1)^{|Q|(|v_{\sigma(1)}|+\ldots+|v_{\sigma(i)}|)} \epsilon(\sigma) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(i)} \otimes \\ & \otimes Q_k(v_{\sigma(i+1)}, \ldots, v_{\sigma(k-1)}, v_k) \otimes v_{k+1} \otimes \ldots \otimes v_n, \end{aligned}$$

for $v_1, \ldots, v_n \in V$, $n \in \mathbb{N}$.

The inverse mapping associates to each coderivation $Q : \bar{T}^Z(V) \to \bar{T}^Z(V)$ its restriction:

$$p \circ Q : \bar{T}(V) \to V,$$

where $p : \bar{T}(V) \to V$ stands for the projection onto V.

The (graded-)commutator gives to $\mathrm{Coder}(T^Z(V))$ a structure of a graded Lie algebra:

$$[Q, P]_c = Q \circ P - (-1)^{|Q||P|} P \circ Q.$$

By the isomorphism defined above, the graded vector space $\mathrm{Hom}(\bar{T}(V), V)$ acquires a graded Lie algebra bracket $[\cdot,\cdot]_B$, known as the *Balavoine bracket* [3].

Remark 1.8. Any coderivation $Q \equiv \{Q_k\}_{k\geq 1}$ of $T^Z(V)$ is also a coderivation of the coshuffle coalgebra $T^c(V)$. Since $\bar{T}^c(V)$ and $\bar{S}(\mathrm{Prim(V)})$ are isomorphic coalgebras, Q defines a coderivation Q^S of $\bar{S}(\mathrm{Prim(V)})$.

Remark 1.9. Each coderivation $Q \equiv \{Q_k\}_{k\geq 1}$ of $(\bar{S}(V), \Delta^c)$ defines a coderivation of $T^Z(V)$, which we will denote by Q^Z (or simply by Q when there is no danger of confusion). It is defined by the same family of (symmetric) linear maps $\left\{Q_k : \bar{S}(V) \to V\right\}_{k\geq 1}$ and these coderivations are related by

$$\pi \circ Q^Z = Q \circ \pi.$$

Definition 1.10. A **Loday ∞-algebra** (V, Q) is graded vector space V together with a coderivation Q of the Zinbiel coalgebra $\bar{T}^Z(s^{-1}V)$, of degree $+1$, that squares zero.

A Loday ∞-algebra (V, Q) may be equivalently defined by a family of degree $+1$ linear maps $Q_k : T^k(s^{-1}V) \to s^{-1}V$ satisfying: for each $n \in \mathbb{N}$,

$$0 = \sum_{k=1}^{n} \sum_{\substack{i=0 \\ \sigma\in Sh(i,k-1)}}^{n-k} (-1)^{|v_{\sigma(1)}|+\ldots+|v_{\sigma(i)}|}\epsilon(\sigma) Q_{n-k+1}\big(v_{\sigma(1)}, \ldots, v_{\sigma(i)}, \qquad (7)$$
$$Q_k(v_{\sigma(i+1)}, \ldots, v_{\sigma(k+i-1)}, v_{i+k}), v_{i+k+1}, \ldots, v_n\big),$$

for all $v_1, \ldots, v_n \in s^{-1}V$.

Remark 1.11. The isomorphism $\mathrm{Hom}(\bar{T}(s^{-1}V), s^{-1}V) \simeq \mathrm{Hom}(\bar{T}(V), V)$ given by (2) yields an equivalent definition of a Loday ∞-algebra in terms of maps $q_k \in \mathrm{Hom}^{2-k}(T^k(V), V)$, $k \in \mathbb{N}$. So, a Loday ∞-algebra may be defined by a family of brackets $q_k \in \mathrm{Hom}^{2-k}(T^k(V), V)$ satisfying a collection of equations equivalent to (7).

A codifferential (i.e. a coderivation of degree $+1$ squaring zero) of $T^Z(V) = (\bar{T}(V), \Delta^Z)$ induces a Loday ∞-algebra structure on sV. This justifies the next definition:

Definition 1.12. A **Loday**[1] **∞-algebra** is a graded vector space V together with a codifferential of the Zinbiel coalgebra $T^Z(V) = (\bar{T}(V), \Delta^Z)$.

Example 1.13. A *graded Loday algebra* is a Loday ∞-algebra V such that $q_n = 0$, for $n \neq 2$. Then the bracket $[\cdot, \cdot] := q_2 : V \otimes V \to V$ satisfies the graded Jacobi identity

$$[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|} [y, [x, z]], \quad x, y, z \in V.$$

In this case, $s^{-1}V$ is a Loday[1] ∞-algebra with codifferential defined by the linear map $Q_2(v, w) := (-1)^{|v|-1} s^{-1} [sv, sw]$, $v, w \in s^{-1}V$, that satisfies

$$Q_2(Q_2(v, u), w) + (-1)^{|v|} Q_2(v, Q_2(u, w)) + (-1)^{(|v|+1)|u|} Q_2(u, Q_2(v, w)) = 0,$$

for all $v, u, w \in s^{-1}V$.

Example 1.14. A *graded differential Loday algebra* is Loday ∞-algebra V such that $q_n = 0$ for $n \neq 1$ and $n \neq 2$. Then $\mathrm{d} := q_1$ is a degree $+1$ linear map squaring zero and $[\,,\,] := q_2$ a Loday bracket satisfying the compatibility condition:

$$\mathrm{d}\,[x, y] = [\mathrm{d}x, y] + (-1)^{|x|}\,[x, \mathrm{d}y]\,, \quad x, y \in V.$$

The graded vector space $s^{-1}V$ is a Loday[1] ∞-algebra with the codifferential Q of $\bar{T}(s^{-1}V)$ given by the maps

$$Q_1(v) = s^{-1}\mathrm{d}sv, \quad v \in s^{-1}V,$$

and

$$Q_2(v, w) = (-1)^{|v|-1}s^{-1}\,[sv, sw]\,, \quad v, w \in s^{-1}V.$$

They satisfy the compatibility conditions

$$\begin{aligned}
&Q_1^2 = 0\\
&Q_1Q_2(v, u) + Q_2(Q_1v, u) + (-1)^{|v|}Q_2(v, Q_1u) = 0\\
&Q_2(Q_2(v, u), w) + (-1)^{|v|}Q_2(v, Q_2(u, w)) + (-1)^{(|v|+1)|u|}Q_2(u, Q_2(v, w)) = 0,
\end{aligned}$$

for all $v, u, w \in s^{-1}V$.

Example 1.15. Let (V, Q_V) and (W, Q_W) be two Loday ∞-algebras. The direct sum $V \oplus W = \bigoplus_{i\in\mathbb{Z}}(V_i \oplus W_i)$ is a Loday ∞-algebra with $Q = Q_V + Q_W$ given by, for each $k \in \mathbb{N}$,

$$Q(v_1 + w_1, \ldots, v_k + w_k) = Q_V(v_1, \ldots, v_k) + Q_W(w_1, \ldots, w_k),$$

for all $v_1 + w_1, \ldots, v_k + w_k \in s^{-1}(V \oplus W) = s^{-1}V \oplus s^{-1}W$.

Example 1.16. Let $(E, M_E \equiv \{l_k\}_{k\in\mathbb{N}})$ be a Lie[1] ∞-algebra. The symmetric brackets $\{l_k\}_{k\in\mathbb{N}}$ define M_E^Z, a coderivation of $T^Z(E)$. Notice that for symmetric brackets Equations (1) and (7) are equivalent and we conclude that M_E^Z is also a codifferential and defines a Loday[1] ∞-algebra structure on E. Obviously this also means that any Lie ∞-algebra structure on a graded vector space defines a Loday ∞-structure on that space.

Definition 1.17. Let (E, Q_E) and (V, Q_V) be Loday ∞-algebras. A **Loday ∞-morphism** $F : (E, Q_E) \to (V, Q_V)$ is a comorphism $F : T^Z(s^{-1}E) \to T^Z(s^{-1}V)$ such that

$$F \circ Q_E = Q_V \circ F.$$

Remark 1.18. A Loday ∞-morphism $F:(E,Q_E\equiv\{l_k\}_{k\geq1})\to(V,Q_V\equiv\{m_k\}_{k\geq1})$ defines and is defined by a family of degree 0 linear maps $F_k: T^k(s^{-1}E)\to s^{-1}V$, $k\geq 1$ satisfying:

$$\begin{aligned}\sum_{k=1}^{n}\sum_{\substack{i=0\\ \sigma\in Sh(i,k-1)}}^{n-k}&(-1)^{|v_{\sigma(1)}|+\ldots+|v_{\sigma(i)}|}\epsilon(\sigma)F_{n-k+1}\big(v_{\sigma(1)},\ldots,v_{\sigma(i)},\\
&\qquad l_k(v_{\sigma(i+1)},\ldots,v_{\sigma(k+i-1)},v_{i+k}),v_{i+k+1},\ldots,v_n\big)\\
=\sum_{k=1}^{n}\sum_{\substack{\sigma\in\widetilde{Sh}(i_1,\ldots,i_k)\\ i_1+\ldots+i_k=n}}&\epsilon(\sigma)m_k\big(F_{i_1}(v_{\sigma(1)},\ldots,v_{\sigma(i_1)}),F_{i_2}(v_{\sigma(i_1+1)},\ldots,v_{\sigma(i_1+i_2)}),\\
&\qquad\ldots,F_{i_k}(v_{\sigma(i_1+\ldots i_{k-1}+1)},\ldots,v_{\sigma(n)})\big)\,,\end{aligned}\tag{8}$$

for all $v_1,\ldots,v_n\in s^{-1}E$.

By isomorphism (2), we can equivalently define a Loday ∞-morphism by a family of linear maps $f_k: T^k(E)\to V$ (of degree $1-k$) that satisfy a collection of equations equivalent to (8).

Remark 1.19. Any (Zinbiel) comorphism $F: T^Z(E)\to T^Z(V)$ is also a comorphism between coshuffle coalgebras. When $F: T^Z(E)\to T^Z(V)$ is a comorphism defined by a family of symmetric maps $\big\{F_k: S^k(E)\to V\big\}_{k\geq1}$ we say it is a *symmetric comorphism*. Hence there is a natural one-to-one correspondence between symmetric (Zinbiel) comorphisms $F: T^Z(E)\to T^Z(V)$ and comorphisms between symmetric coalgebras $F^S:\bar S(E)\to\bar S(V)$. This correspondence is given by:

$$F^S\circ\pi=\pi\circ F.$$

Looking at the formula in Remark 1.18 we conclude that $F:(E,M_E)\to(V,M_V)$ is a Lie ∞-algebra morphism if and only if it defines a symmetric Loday ∞-algebra morphism between (E,M_E^Z) and (V,M_V^Z).

Hence the category of Lie ∞-algebras is a subcategory of the category of Loday ∞-algebras.

Remark 1.20. Let (V,Q_V) be a Loday[1] ∞-algebra. Then Q_V is also a codifferential of the coshuffle coalgebra $T^c(V)$. By the coalgebra isomorphism $T^c(V)\simeq\bar S(\mathrm{PrimV})$, Q_V defines a codifferential of $\bar S(\mathrm{Prim(V)})$. This means that $\mathrm{Prim}(V)$ is a Lie[1] ∞-algebra. Each Loday ∞-morphism $F:(E,Q_E)\to$

(V, Q_V) is also a coshuffle comorphism that preserves codifferentials, consequently, it induces a Lie ∞-morphism between Prim(E) and Prim(V). This means there is a functor between the category of Loday[1]-algebras and Lie[1]-algebras. [19]

Notation: In what follows, we will only use the Lie[1] ∞-algebras and the Loday[1] ∞-algebra approach. For this reason, in the next sections, Lie/Loday ∞-algebras are meant to be Lie/Loday[1] ∞-algebras, this means we will drop the shifting symbol [1].

2. Coherent actions of Lie ∞-algebras

Let $(E, M_E \equiv \{l_k\}_{k\in\mathbb{N}})$ be a Lie ∞-algebra and (V, d) a cochain complex. Example 1.6 describes a DGLA[1] structure in $\mathrm{End}(V)[1]$. Recall that (V, d) is a *(curved) representation* of E if there is a Lie ∞-algebra morphism

$$\rho : (E, M_E \equiv \{l_k\}_{k\in\mathbb{N}}) \to (\mathrm{End}(V)[1], \partial_{\mathrm{d}}, [\cdot,\cdot]).$$

Now suppose (V, M_V) is a Lie ∞-algebra. Since $(\bar{S}(V), M_V)$ is a cochain complex, we have a DGLA[1] structure on $s^{-1}(\mathrm{Coder}\,\bar{S}(V)) = (\mathrm{Coder}\,\bar{S}(V))[1]$ induced by $\mathrm{End}\,\bar{S}(V)[1]$ given by:

$$\begin{cases} \partial_{M_V} Q = -M_V \circ Q + (-1)^{|Q|+1} Q \circ M_V, \\ [Q, P] = (-1)^{|Q|+1}\left(Q \circ P - (-1)^{(|Q|+1)(|P|+1)} P \circ Q\right), \end{cases}$$

with Q and P homogeneous elements of $(\mathrm{Coder}\,\bar{S}(V))[1]$ of degrees $|Q|$ and $|P|$, respectively.

Definition 2.1. Let $(E, M_E \equiv \{l_k\}_{k\in\mathbb{N}})$ and $(V, M_V \equiv \{m_k\}_{k\in\mathbb{N}})$ be Lie ∞-algebras. A **Lie ∞-action** of E on V is a Lie ∞-morphism

$$\Phi : \ (E, \{l_k\}_{k\in\mathbb{N}}) \ \to \ (\mathrm{Coder}(\bar{S}(V))[1], \partial_{M_V}, [\cdot,\cdot]).$$

Any Lie ∞-action is determined by its degree $+1$ restriction maps:

$$\begin{array}{rccl} \Phi^k : & S^k(E) & \to & \mathrm{Coder}(\bar{S}(V)) \\ & x & \mapsto & \Phi^k(x) := p \circ \Phi(x) = \Phi_x, \quad k \in \mathbb{N}. \end{array}$$

Furthermore, any Lie ∞-action Φ is completely determined by the family degree $+1$ of linear maps:

$$\Phi^{k,n} : S^k(E) \times S^n(V) \to V,$$

given by

$$\Phi^{k,n}(x; v) = \Phi^n_x(v) = p \circ \Phi_x(v), \quad x \in S^k(V), v \in S^n(V), \quad k, n \in \mathbb{N}.$$

For each $x \in \bar{S}(E)$ and $v \in \bar{S}(V)$, let $\Phi_{x;v}$ be the coderivation of $\bar{S}(V)$ defined by the family of maps

$$(\Phi_{x;v})_i : \begin{array}{ccc} S^i(V) & \to & V \\ w & \mapsto & \Phi^{\bullet,\bullet}(x; v.w), \end{array}, \quad i \in \mathbb{N}.$$

Remark 2.2. A Lie ∞-action is a (curved) representation of E in $\bar{S}(V)$ but the image of all the restriction maps is contained in $\operatorname{Coder}(\bar{S}(V))[1] \subset \operatorname{End} \bar{S}(V)[1]$.

Example 2.3 (Adjoint representation and adjoint action)**.** Let $\big(E, M_E \equiv \{l_k\}_{k\in\mathbb{N}}\big)$ be a Lie ∞-algebra. The *adjoint representation* of E, (E, l_1), is defined by the collection of degree $+1$ maps

$$\mathrm{ad}_k : \begin{array}{ccc} S^k(E) & \to & \operatorname{End}(E) \\ x_1 \cdot \ldots \cdot x_k & \mapsto & \mathrm{ad}_{x_1\cdot\ldots\cdot x_k} := l_{k+1}\,(x_1, \ldots, x_k, \ - \ ) \end{array}, \quad k \geq 1.$$

The *adjoint action* of E is the Lie ∞-action of E on itself, given by the family of linear maps (see [5], [15]):

$$\begin{array}{cccc} \mathrm{ad}^{k,i} : & S^k(E) \times S^i(E) & \to & E \\ & (x; e) & \mapsto & l_{i+k}(x, e), \quad i, k \geq 1. \end{array}$$

Remark 2.4. If we define $\Phi^0 := M_V$, then an action is equivalent to a curved Lie ∞-morphism between E and the graded Lie algebra $\operatorname{Coder}(\bar{S}(V))$ (compatible with the Lie ∞-structure in V) [15]. In this case, $\Phi = \sum_{k\geq 0} \Phi^k$ is called a *curved Lie ∞-action*.

In this context, we can say that, for each $v \in \bar{S}(V)$, the coderivation Φ^0_v is the image of v by the adjoint action of V:

$$\Phi^0_v = \mathrm{ad}_v\,.$$

Definition 2.5. A Lie ∞-action of E on V

$$\Phi : (E, \{l_k\}_{k\in\mathbb{N}}) \to (\operatorname{Coder}(\bar{S}(V))[1], \partial_{M_V}, [\cdot, \cdot])$$

is said to be **coherent** if

$$[\mathrm{ad}_v, \Phi_x] = 0 \tag{9}$$

and

$$[\Phi_{y;v}, \Phi_x] = 0, \tag{10}$$

for all $x, y \in \bar{S}(E)$ and $v \in \bar{S}(V)$.

Remark 2.6. In the context of curved Lie ∞-actions, Equation (9) can be rewritten as

$$\left[\Phi^0_v, \Phi_x\right] = 0, \quad x \in \bar{S}(E), v \in \bar{S}(V).$$

Remark 2.7. If we write the bracket $[\cdot,\cdot]$ in terms of the usual commutator of coderivations, we have that Equations (9) and (10) are equivalent to

$$[\mathrm{ad}_v, \Phi_x]_c = 0 \quad \text{and} \quad [\Phi_{y;v}, \Phi_x]_c = 0.$$

Example 2.8. Let $(E, M_E \equiv \{l_k\}_{k\in\mathbb{N}})$ be a Lie ∞-algebra and (V, d) a representation of E given by the Lie ∞-morphism $\Phi : E \to \mathrm{End}(V)$. Notice that $m_{\bullet\geq 2} = 0$ so $\mathrm{ad}_v = 0$, $v \in V$. Since Φ is a representation, we have $\Phi^{\bullet,\bullet\geq 2} = 0$ which means that $\Phi_{y;v} \circ \Phi_x(w) = 0$, for all $v, w \in V$, $x, y \in \bar{S}(E)$. This way we see that Φ is a coherent action.

Example 2.9. Let $(E, [\cdot,\cdot]_E)$ and $(V, [\cdot,\cdot]_V)$ be Lie algebras and $\rho : E \to \mathrm{Der}(V)$ a representation of E in the classic sense. The representation ρ is equivalent to the action $\Phi : E \to \mathrm{Coder}(\bar{S}(V))$ defined by the family of maps $\Phi^{k,n} : S^k(E) \times S^n(V) \to V$, $k, n \geq 1$, given by:

$$\Phi^{1,1}(x; v) = \rho(x)(v), \quad x \in E, v \in V$$

$$\Phi^{\bullet,\bullet} = 0, \quad \text{otherwise.}$$

In this case, the action (representation) Φ is coherent if and only if

$$[\mathrm{ad}_v, \Phi_x]_c\,(w) = 0, \quad w \in V,$$

which means that

$$[v, \Phi_x w]_V - \Phi_x\,[v, w]_V = 0, \quad x \in E,\ v, w \in V.$$

Since $\Phi_x = \rho(x)$ is a derivation of $[\ ,\ ]_V$ we end up with

$$[w, \Phi_x v]_V = 0, \quad x \in E,\ v, w \in V.$$

This is equivalent to Φ_x being a central derivation of the Lie algebra V, for all $x \in E$. In case V is abelian, the representation is obviously coeherent. The definition of coherent representation of Lie algebras first appeared in [20].

Remark 2.10. If E is a Lie ∞-algebra, even when V is abelian, that is, $m_k = 0$, for $k > 1$, the coherence of a Lie ∞-action Φ is not guaranteed due to the condition (10):

$$[\Phi_{x;v}, \Phi_y]_c = 0 \quad x, y \in \bar{S}(E),\ v \in \bar{S}(V).$$

Clearly, if $\Phi^{\bullet,\bullet\geq 2} = 0$ then coherence is guaranteed, but this means that Φ is, in fact, a Lie ∞-representation.

The next theorem is one of our main results. We show that the coherence of a Lie ∞-action is a necessary and sufficient condition for the construction of a particular Loday ∞-algebra in the direct sum of the Lie ∞-algebras: the non-abelian hemisemidirect product. This theorem generalizes two results: Proposition 6.3 in [19] on representations of Lie ∞-algebras and Proposition 2.7 in [20] on coherent representations of Lie algebras.

Theorem 2.11. *Let $\Phi : (E, \{l_k\}_{k\in\mathbb{N}}) \to (\mathrm{Coder}(\bar{S}(V))[1], \partial_{M_V}, [\cdot,\cdot])$ be a Lie ∞-action. Consider $E \oplus V$ equipped with brackets $\{\mathfrak{l}_n\}_{n\in\mathbb{N}}$ defined by:*

$$\begin{aligned}\mathfrak{l}_n\left(x_1+v_1,\ldots,x_{n-1}+v_{n-1},x_n+v_n\right) &= l_n(x_1,\ldots,x_n)\\ &+\sum_{i=1}^{n-1}\Phi^{i,n-i}(x_1,\ldots,x_i;v_{i+1},\ldots,v_n)+m_n(v_1,\ldots,v_n)\end{aligned} \tag{11}$$

Then $E \oplus V$ is a Loday ∞-algebra if and only if Φ is a coherent action.

Proof: Let Q be the coderivation of the Zinbiel coalgebra $T^Z(E\oplus V)$ defined by the brackets $\{\mathfrak{l}_n\}_{n\in\mathbb{N}}$. Proving that $Q^2 = 0$ is equivalent to showing that the Jacobiator vanishes:

$$J(x_1+v_1,\ldots,x_n+v_n)=\sum_{k=1}^{n}\mathfrak{l}_k\circ Q\left(x_1+v_1,\ldots,x_n+v_n\right)=0$$

for all $x_1+v_1,\ldots,x_n+v_n \in E\oplus V$. Therefore we want to prove that coherence of the action is equivalent to the vanishing of the Jacobiator.

We immediately see that

$$J(x_1,\ldots,x_n)=J(v_1,\ldots,v_n)=0,\quad x_1,\ldots,x_n\in E, v_1,\ldots,v_n\in V,$$

because $\{l_k\}_{k\in\mathbb{N}}$ and $\{m_k\}_{k\in\mathbb{N}}$ define Lie ∞-structures on E and V, respectively.

Also, since

$$\mathfrak{l}_k(x_1+v_1,\ldots,x_{k-1}+v_{k-1},x_k)=l_k(x_1,\ldots,x_k),\quad k\in\mathbb{N},$$

we have that

$$J(x_1+v_1,\ldots,x_{n-1}+v_{n-1},x_n)=J(x_1,\ldots,x_n)=0,$$

for all $x_1,\ldots,x_n\in E, v_1,\ldots,v_{n-1}\in V$.

It remains to prove $J(x_1+v_1,\ldots,x_{n-1}+v_{n-1},v_n)=0$ if and only if Φ is a coherent action.

In what follows, we will use the Sweedler notation for the coshuffle coproduct:

$$\Delta^c(v)=v_{(1)}\otimes v_{(2)},\quad v\in\bar{T}(V).$$

If $v\in V$, then $\Delta^c(v)=0$ and we consider $v_{(1)}=v_{(2)}=0$.

First notice that, for $y\in\bar{T}(E)$ and $w\in\bar{T}(V)$, we have

$$Q(y\otimes w)=M_E^Z(y)\otimes w+\Phi^Z_{\pi(y)}(w)+(-1)^{|y_{(1)}|}y_{(1)}\otimes\Phi^Z_{\pi(y_{(2)})}(w)+(-1)^{|y|}y\otimes M_V^Z(w).$$

Therefore

$$\begin{aligned} J(y,w) &= \Phi^\bullet_{\pi(M_E^Z(y))}\pi(w)+m_\bullet\circ\pi(\Phi^Z_{\pi(y)}(w))+(-1)^{|y|}\Phi^\bullet_{\pi(y)}\pi(M_V^Z(w)) \\ &\quad +(-1)^{|y_{(1)}|}\Phi^\bullet_{\pi(y_{(1)})}\pi(\Phi^Z_{\pi(y_{(2)})}(w)). \end{aligned} \tag{12}$$

All coderivations in this equation are defined by symmetric linear maps. Also $\pi: T^c(V)\to\bar{S}(V)$ is a comorphism, so we have:

$$\begin{aligned} J(y,w) &= \Phi^\bullet_{M_E(\pi(y))}\pi(w)+m_\bullet(\Phi_{\pi(y)}\pi(w))+(-1)^{|y|}\Phi^\bullet_{\pi(y)}(M_V(\pi(w))) \\ &\quad +(-1)^{|y_{(1)}|}\Phi^\bullet_{\pi(y_{(1)})}\Phi_{\pi(y_{(2)})}(\pi(w)). \end{aligned}$$

Taking into account that Φ is a Lie ∞-action we conclude that $J(y,w)=0$ (for this the coherence of the action is not needed).

Now notice that for $v,w\in\bar{T}(V)$, $y\in\bar{T}(E)$, we have

$$\begin{aligned} Q(v\otimes y\otimes w) &= M_V^Z(v)\otimes y\otimes w+(-1)^{|v|}v\otimes M_E^Z(y)\otimes w \\ &\quad +(-1)^{|v|+|y|}v\otimes y\otimes M_V^Z(w)+(-1)^{|v|+|y_{(1)}|}v\otimes y_{(1)}\otimes\Phi^Z_{\pi(y_{(2)})}(w) \\ &\quad +(-1)^{|v_{(1)}|+|y|(1+|v_{(2)}|)}v_{(1)}\otimes y\otimes \mathrm{ad}^Z_{\pi(v_{(2)})}(w) \\ &\quad +(-1)^{|v|}v\otimes\Phi^Z_{\pi(y)}(w)+(-1)^{|y|(|v|+1)}y\otimes\mathrm{ad}^Z_{\pi(v)}(w) \end{aligned} \tag{13}$$

hence

$$\begin{aligned} J(v,y,w) &= (-1)^{|v|}m_\bullet(v,\Phi^Z_{\pi(y)}(w))+(-1)^{|y|(|v|+1)}\Phi^\bullet_{\pi(y)}(\mathrm{ad}_{\pi(v)}(\pi(w))) \\ &= (-1)^{|v|}m_\bullet(\pi(v),\Phi_{\pi(y)}\pi(w))+(-1)^{|y|(|v|+1)}\Phi^\bullet_{\pi(y)}(\mathrm{ad}_{\pi(v)}(\pi(w))) \\ &= (-1)^{|v|}p\circ\left[\mathrm{ad}_{\pi(v)},\Phi_{\pi(y)}\right]_c(\pi(w)) \\ &= -p\circ\left[\mathrm{ad}_{\pi(v)},\Phi_{\pi(y)}\right](\pi(w)). \end{aligned}$$

Similar calculations lead to the general formula, for $y_1, \ldots, y_n \in \bar{T}(E)$ and $v_1, \ldots, v_n, w \in \bar{T}(V)$:

$$\begin{aligned} J(v_1, y_1, \ldots, v_n, y_n, w) =& (-1)^{\sum_{i=2}^{n}\sum_{j=1}^{i-1}|v_i||y_j|}(-1)^{|v|}p \circ \left[\mathrm{ad}_{\pi(v)}, \Phi_{\pi(y)}\right]_c (\pi(w)), \\ =& -(-1)^{\sum_{i=2}^{n}\sum_{j=1}^{i-1}|v_i||y_j|}p \circ \left[\mathrm{ad}_{\pi(v)}, \Phi_{\pi(y)}\right] (\pi(w)), \end{aligned}$$

where $v = v_1 \otimes \ldots \otimes v_n$, $y = y_1 \otimes \ldots \otimes y_n$ and $n \geq 2$.

Therefore, $J(v_1, y_1, \ldots, v_n, y_n, w) = 0$, for all $v_1, \ldots, v_n, w \in \bar{T}(V)$ and $y_1, \ldots, y_n \in \bar{T}(E)$ if and only if

$$\left[\mathrm{ad}_{\bar{v}}, \Phi_{\bar{y}}\right](\bar{w}) = 0, \quad \text{for all } \bar{v}, \bar{w} \in \bar{S}(V),\ \bar{y} \in \bar{S}(E). \tag{14}$$

Now, for $x \in E$, $v, w \in \bar{T}(V)$, $y \in \bar{T}(E)$, we have

$$\begin{aligned} Q(x \otimes v \otimes y \otimes w) \;=\;& l_1(x) \otimes v \otimes y \otimes w + (-1)^{|x|} x \otimes Q(v \otimes y \otimes w) \\ &+ \Phi_x^Z(v) \otimes y \otimes w + (-1)^{|v|(|x|+1)} v \otimes \mathrm{ad}_x^Z(y) \otimes w \\ &+ (-1)^{|y|(|x|+|v|+1)} y \otimes \Phi^Z_{x;\pi(v)}(w) \\ &+ (-1)^{|v_{(1)}|(|x|+1)+|y|(|x|+|v_{(2)}|+1)} v_{(1)} \otimes y \otimes \Phi^Z_{x;\pi(v_{(2)})}(w) \\ &+ (-1)^{(|v|+|y_{(1)}|)(|x|+1)} v \otimes y_{(1)} \otimes \Phi^Z_{x\cdot\pi(y_{(2)})}(w) \\ &+ (-1)^{|v|(|x|+1)} v \otimes \Phi^Z_{\pi(x\otimes y)}(w) \\ &+ (-1)^{(|v|+|y|)(1+|x|)} v \otimes y \otimes \Phi^Z_x(w). \end{aligned}$$

and

$$\begin{aligned} J(x, v, y, w) =& (-1)^{|y||v|+|x|+|y|}\Phi^{\bullet}_{\pi(x\otimes y)}(\mathrm{ad}_{\pi(v)}(\pi(w))) \\ &+ (-1)^{|v|(|x|+1)} m_{\bullet}(\pi(v), \Phi_{\pi(x\otimes y)}\pi(w)) \\ &+ (-1)^{|v|+|x|}\Phi^{\bullet}_x(\pi(v), \Phi_{\pi(y)}\pi(w)) \\ &+ (-1)^{(|x|+|v|+1)|y|}\Phi^{\bullet}_{\pi(y)}(\Phi_{x;\pi(v)}(\pi(w))) \\ =\;& (-1)^{|x||v|+|v|}p \circ \left[\mathrm{ad}_{\pi(v)}, \Phi_{\pi(x\otimes y)}\right]_c(\pi(w)) \\ &+ (-1)^{|v|+|x|}p \circ \left[\Phi_{x;\pi(v)}, \Phi_{\pi(y)}\right]_c(\pi(w)). \\ =\;& -(-1)^{|x||v|}p \circ \left[\mathrm{ad}_{\pi(v)}, \Phi_{\pi(x\otimes y)}\right](\pi(w)) - p \circ \left[\Phi_{x;\pi(v)}, \Phi_{\pi(y)}\right](\pi(w)). \end{aligned}$$

Taking into account Equation (14) we conclude that $J(x, v, y, w) = 0$, for all $x \in E$, $y \in \bar{T}(E)$, $v, w \in \bar{T}(V)$ if and only if

$$\left[\Phi_{x;\bar{v}}, \Phi_{\bar{y}}\right](\bar{w}), \quad \text{for all } x \in E,\ \bar{y} \in \bar{S}(E),\ \bar{v}, \bar{w} \in \bar{S}(V). \tag{15}$$

Consider $x \in T^n(E)$, $n \geq 2$, and again $v, w \in \bar{T}(V)$, $y \in \bar{T}(E)$. Then,

$$
\begin{aligned}
Q(x \otimes v \otimes y \otimes w) &= M_E^Z(x) \otimes v \otimes y \otimes w + (-1)^{|x|} x \otimes Q(v \otimes y \otimes w) \\
&+(-1)^{|x_{(1)}|} x_{(1)} \otimes \Phi^Z_{\pi(x_{(2)})}(v) \otimes y \otimes w + \Phi^Z_{\pi(x)}(v) \otimes y \otimes w \\
&+(-1)^{|x_{(1)}|+|v|(|x_{(2)}+1|)} x_{(1)} \otimes v \otimes \mathrm{ad}^Z_{\pi(x_{(2)})}(y) \otimes w \\
&+(-1)^{|v|(|x|+1)} v \otimes \mathrm{ad}^Z_{\pi(x)}(y) \otimes w \\
&+(-1)^{|x_{(1)}|+|y|(|x_{(2)}|+|v|+1)} x_{(1)} \otimes y \otimes \Phi^Z_{\pi(x_{(2)};v)}(w) \\
&+(-1)^{|y|(|x|+|v|+1)} y \otimes \Phi^Z_{\pi(x;v)}(w) \\
&+(-1)^{|x_{(1)}|+|v_{(1)}|(|x_{(2)}|+1)+|y|(|x_{(2)}|+|v_{(2)}|+1)} x_{(1)} \otimes v_{(1)} \otimes y \otimes \Phi^Z_{\pi(x_{(2)};v_{(2)})}(w) \\
&+(-1)^{|v_{(1)}|(|x|+1)+|y|(|x|+|v_{(2)}|+1)} v_{(1)} \otimes y \otimes \Phi^Z_{\pi(x;v_{(2)})}(w) \\
&+(-1)^{|x_{(1)}|+|v|(|x_{(2)}|+1)} x_{(1)} \otimes v \otimes \Phi^Z_{\pi(x_{(2)}\otimes y)}(w) \\
&+(-1)^{|v|(|x|+1)} v \otimes \Phi^Z_{\pi(x\otimes y)}(w) \\
&+(-1)^{|x_{(1)}|+(1+|x_{(2)}|)(|v|+|y_{(1)}|)} x_{(1)} \otimes v \otimes y_{(1)} \otimes \Phi^Z_{\pi(x_{(2)}\otimes y_{(2)})}(w) \\
&+(-1)^{|v|(|x|+1)+|y_{(1)}|(|x|+1)} v \otimes y_{(1)} \otimes \Phi^Z_{\pi(x\otimes y_{(2)})}(w) \\
&(-1)^{(|v|+|y|)(1+|x|)} v \otimes y \otimes \Phi^Z_x(w).
\end{aligned}
$$

Equation (13) and the brackets (11) yield

$$
\begin{aligned}
J(x,v,y,w) &= (-1)^{|v|+|x|} \Phi^\bullet_{\pi(x);\pi(v)}(\Phi_{\pi(y)}(\pi(w))) \\
&+(-1)^{|x|+|y|(|v|+1)} \Phi^\bullet_{\pi(x\otimes y)}(\mathrm{ad}_{\pi(v)}(\pi(w))) \\
&+(-1)^{|x_{(1)}|+|y|(|x_{(2)}|+|v|+1)} \Phi^\bullet_{\pi(x_{(1)}\otimes y)}(\Phi_{\pi(x_{(2)});\pi(v)}(\pi(w))) \\
&+(-1)^{|y|(|x|+|v|+1)} \Phi^\bullet_{\pi(y)}(\Phi_{\pi(x);\pi(v)}(\pi(w))) \\
&+(-1)^{|v|(|x|+1)} m_\bullet(v, \Phi^Z_{\pi(x\otimes y)}(w)) \\
&+(-1)^{|x_{(1)}|+|v|(|x_{(2)}|+1)} \Phi^\bullet_{\pi(x_{(1)});\pi(v)}(\Phi_{\pi(x_{(2)}\otimes y)}(\pi(w))) \\
&= \ (-1)^{(|x|+1)|v|} p \circ \left[\mathrm{ad}_{\pi(v)}, \Phi_{\pi(x\otimes y)}\right]_c (\pi(w)) \\
&+(-1)^{|x|+|v|} p \circ \left[\Phi_{\pi(x);\pi(v)}, \Phi_{\pi(y)}\right]_c (\pi(w)) \\
&+(-1)^{(|x_{(2)}|+1)|v|+|x_{(1)}|} p \circ \left[\Phi_{\pi(x_{(1)});\pi(v)}, \Phi_{\pi(x_{(2)}\otimes y)}\right]_c (\pi(w))
\end{aligned}
$$

$$\begin{aligned} = & -(-1)^{|x||v|} p \circ \left[\mathrm{ad}_{\pi(v)}, \Phi_{\pi(x\otimes y)}\right](\pi(w)) \\ & -(-1)^{|x|} p \circ \left[\Phi_{\pi(x);\pi(v)}, \Phi_{\pi(y)}\right](\pi(w)) \\ & -(-1)^{|x_{(2)}||v|+|x_{(1)}|} p \circ \left[\Phi_{\pi(x_{(1)});\pi(v)}, \Phi_{\pi(x_{(2)}\otimes y)}\right](\pi(w)). \end{aligned}$$

Similar computations lead to the general formula:

$$\begin{aligned} J(x, v_1, y_1, v_2, \dots, v_n, y_n, w) &= a(-1)^{|v|+|x||v|} p \circ \left[\mathrm{ad}_{\pi(v)}, \Phi_{\pi(x\otimes y)}\right]_c (\pi(w)) \\ &+ a(-1)^{|x|+|v|} p \circ \left[\Phi_{\pi(x);\pi(v)}, \Phi_{\pi(y)}\right]_c (\pi(w)) \\ &+ a(-1)^{|x_{(1)}|+|v|+|x_{(2)}||v|} p \circ \left[\Phi_{\pi(x_{(1)});\pi(v)}, \Phi_{\pi(x_{(2)}\otimes y)}\right]_c (\pi(w)), \end{aligned}$$

for all $x, y_1, \dots, y_n \in \bar{T}(V)$, $v_1, \dots, v_n \in \bar{T}(V)$, where $v = v_1 \otimes \dots \otimes v_n$, $y = y_1 \otimes \dots \otimes y_n$ and $a = (-1)^{\sum_{i=2}^{n} \sum_{j=1}^{i-1} |v_i||y_j|}$, $n \geq 2$.

An induction argument together with Equations (14) and (15) allows us to conclude that $J(x, v, y, w) = 0$, for all $x, y \in \bar{T}(E)$ and $v, w \in \bar{T}(V)$, if and only if

$$[\Phi_{\bar{x};\bar{v}}, \Phi_{\bar{y}}](\bar{w}), \quad \text{for all } \bar{x}, \bar{y} \in \bar{S}(E), \bar{v}, \bar{w} \in \bar{S}(V). \tag{16}$$

Equations (14), (15) and (16) express the coherence of the action Φ and we conclude that it is equivalent to the Jacobiator being zero. ■

Following [20], we define

Definition 2.12. Let $(E, \{l_k\}_{k\in\mathbb{N}})$ and $(V, M_V \equiv \{m_k\}_{k\in\mathbb{N}})$ be Lie ∞-algebras and $\Phi : (E, \{l_k\}_{k\in\mathbb{N}}) \to (\mathrm{Coder}(\bar{S}(V))[1], \partial_{M_V}, [\cdot,\cdot])$ a coherent Lie ∞-action. The Loday ∞-algebra $(E \oplus V, \{\mathfrak{l}_k\}_{k\in\mathbb{N}})$ given by

$$\begin{aligned} \mathfrak{l}_n(x_1 + v_1, \dots, x_{n-1} + v_{n-1}, x_n + v_n) &= l_n(x_1, \dots, x_n) \\ &+ \sum_{i=1}^{n-1} \Phi^{i,n-i}(x_1, \dots, x_i; v_{i+1}, \dots, v_n) + m_n(v_1, \dots, v_n), \end{aligned}$$

for all $x_1 + v_1, \dots, x_n + v_n \in E \oplus V$, is called the **non-abelian hemisemidirect product** of E and V and is denoted by $E \ltimes^{\Phi} V$.

Example 2.13. Let $(E, [\cdot,\cdot]_E)$ and $(V, [\cdot,\cdot]_V)$ be Lie algebras and $\rho : E \to \mathrm{Der}(V)$ a coherent Lie representation (see Example 2.9):

$$[w, \rho_x(v)]_V = 0, \quad x \in E,\ v, w \in V.$$

This particular case was considered in [20] and the non-abelian hemisemidirect product $E \ltimes^{\Phi} V$ is the Loday algebra (or the left Leibniz algebra) with product given by

$$(x+v)\circ(y+w) = [x,y]_E + \Phi_x w + [v,w]_V\,.$$

Notice that when the representation V is an abelian Lie algebra, then it is always coherent and we obtain the usual hemisemidirect product introduced in [9].

Example 2.14. Let $(E, M_E \equiv \{l_k\}_{k\in\mathbb{N}})$ be a Lie ∞-algebra and $\Phi : \bar{S}(E) \to \mathrm{End}(V)$ a representation on the complex (V, d). The action Φ is coherent (see Example 2.8).

The non-abelian hemisemidirect product $E \ltimes^{\Phi} V$ is defined by the brackets:

$$\begin{aligned} \mathfrak{l}_1(x_1+v_1) &= l_1 x_1 + \mathrm{d}\, v_1, \\ \mathfrak{l}_n(x_1+v_1,\ldots,x_n+v_n) &= l_n(x_1,\ldots,x_n) + \Phi^{n-1,1}(x_1,\ldots,x_{n-1};v_n), \end{aligned}$$

for all $x_1+v_1,\ldots,x_n+v_n \in E\oplus V$, $n\geq 2$.

Remark 2.15. A Lie ∞-action

$$\Phi : (E, M_E \equiv \{l_k\}_{k\in\mathbb{N}}) \to (\mathrm{Coder}(\bar{S}(V))[1], \partial_{M_V}, [\cdot,\cdot]),$$

is a special representation of E on the complex $(\bar{S}(V), M_V)$. Since any representation is a coherent action, we have the non-abelian hemisemidirect product $E \ltimes^{\Phi} \bar{S}(V)$ with brackets:

$$\begin{aligned} \bar{\mathfrak{l}}_1(x_1+v_1) &= l_1(x_1) + M_V(v_1), \\ \bar{\mathfrak{l}}_n(x_1+v_1,\ldots,x_n+v_n) &= l_n(x_1,\ldots,x_n) + \Phi_{x_1\cdot\ldots\cdot x_{n-1}} v_n, \end{aligned}$$

for all $x_1,\ldots,x_n \in E$, $v_1,\ldots,v_n \in \bar{S}(V)$, $n\geq 2$.

If Φ is a coherent Lie ∞-action, then $E \ltimes^{\Phi} V$ is also a Loday ∞-algebra. The brackets $\{\mathfrak{l}_k\}_{k\in\mathbb{N}}$ in $E \ltimes^{\Phi} V$ and the brackets $\{\bar{\mathfrak{l}}_k\}_{k\in\mathbb{N}}$ in $E \ltimes^{\Phi} \bar{S}(V)$ are related by

$$\begin{aligned} \mathfrak{l}_1(x_1+v_1) &= \bar{\mathfrak{l}}_1(x_1+v_1) \\ \mathfrak{l}_n(x_1+v_1,\ldots,x_n+v_n) &= \bar{\mathfrak{l}}_n(x_1,\ldots,x_n) \\ &\quad + p_V \circ \left(\sum_{i=1}^{n-1} \bar{\mathfrak{l}}_{i+1}(x_1,\ldots,x_i, v_{i+1}\cdot\ldots\cdot v_n) + \bar{\mathfrak{l}}_1(v_1\cdot\ldots\cdot v_n) \right), \end{aligned}$$

for all $x_1,\ldots,x_n \in E$, $v_1,\ldots,v_n \in V$, $n\geq 2$, where $p_V : \bar{S}(V) \to V$ is the projection map.

3. Non-abelian embedding tensors

Let $(E, M_E \equiv \{l_k\}_{k\in\mathbb{N}})$ and $(V, M_V \equiv \{m_k\}_{k\in\mathbb{N}})$ be Lie ∞-algebras,

$$\Phi : (E, \{l_k\}_{k\in\mathbb{N}}) \to (\mathrm{Coder}(\bar{S}(V))[1], \partial_{M_V}, [\cdot,\cdot])$$

a coherent Lie ∞-action and $E \ltimes^{\Phi} V$ the non-abelian hemisemidirect product defined by Φ.

The graded vector space of linear maps between $\bar{T}(V)$ and E will be denoted by $\mathfrak{h} := \mathrm{Hom}(\bar{T}(V), E)$. It can be identified with the space of coalgebra morphisms between $T^Z(V)$ and $T^Z(E)$ and also with a subspace of $\mathrm{Coder}(T^Z(E \oplus V))$.

The space $\mathfrak{h}$ together with the coherent action defines a V-data [22] and, consequently, $\mathfrak{h}$ acquires a Lie ∞-algebra structure:

Proposition 3.1. *Let us consider:*

(1) the graded Lie algebra $\mathcal{L} := (\mathrm{Coder}(T^Z(E \oplus V)), [\cdot,\cdot]_c)$;
(2) the abelian graded Lie subalgebra $\mathfrak{h}$;
(3) the projection $\mathcal{P} : \mathcal{L} \to \mathfrak{h}$ onto $\mathfrak{h}$;
(4) the coderivation Q associated with the Loday ∞-algebra $E \ltimes^{\Phi} V$.

Then, $(\mathcal{L}, \mathfrak{h}, \mathcal{P}, Q)$ is a V-data and $\mathfrak{h}$ is equipped with a Lie ∞-algebra structure given by:

$$\partial_k\big(\mathfrak{t}_1, \ldots, \mathfrak{t}_k) = \mathcal{P}([[\ldots[Q, \mathfrak{t}_1]_c \ldots]_c, \mathfrak{t}_k]_c\big), \quad \mathfrak{t}_1, \ldots, \mathfrak{t}_k \in \mathfrak{h},\ k \geq 1.$$

Proof: Although we are dealing with Lie ∞-actions rather than representations, the argument used in the proof of Proposition 6.4 in [19], to show that we have a V-data, can be used in an identical way in this case. ∎

Now we are in position to define a non-abelian embedding tensor with respect to a coherent Lie ∞-algebra action $\Phi : E \to \mathrm{Coder}(\bar{S}(V))$.

Definition 3.2. A **non-abelian embedding tensor** on E with respect to the coherent action Φ is a (Zinbiel) coalgebra morphism $T \equiv \big\{T_k : T^k(V) \to E\big\}_{k\in\mathbb{N}}$, such that the induced coderivation $\mathfrak{t} \in \mathfrak{h}$ is a Maurer-Cartan element of $\mathfrak{h}$:

$$\mathcal{P}\left([Q, \mathfrak{t}]_c + \frac{1}{2}[[Q, \mathfrak{t}]_c, \mathfrak{t}]_c + \ldots\right) = 0. \tag{17}$$

Remark 3.3. By setting

$$e^{[\,,\mathfrak{t}]}(Q) = Q + [Q, \mathfrak{t}]_c + \frac{1}{2}[[Q, \mathfrak{t}]_c, \mathfrak{t}]_c + \ldots + \frac{1}{n!}[\ldots[[Q, \mathfrak{t}]_c, \mathfrak{t}]_c \ldots \mathfrak{t}]_c + \ldots$$

we can rewrite the Equation (17) as

$$\mathcal{P}\left(e^{[\,,\mathfrak{t}]}(Q)\right)=0.$$

Let $T : T^Z(V) \to T^Z(E)$ be a non-abelian embedding tensor with respect to the coherent action Φ. Suppose T is defined by the family of maps $\{T_k : T^k(V) \to E\}_{k\geq 1}$ and consider $\mathfrak{t} \in \mathrm{Coder}(T^Z(E \oplus V))$ the associated coderivation.

The projection $p_V : E \oplus V \to V$ defines the strict comorphism $p_{T(V)} : T^Z(E \oplus V) \to T^Z(V)$. The composition $T \circ p_{T(V)}$ is the comorphism defined by the family of linear maps $\{T_k \circ p_{T(V)} : T^k(E \oplus V) \to E\}_{k\geq 1}$. This composition can be see as an extension of T to $\bar{T}(E \oplus V)$ so we also denote it by T.

Lemma 3.4. *The exponential of the coderivation* $\mathfrak{t}$ *is the comorphism* $\mathrm{Id}+T : T^Z(E \oplus V) \to T^Z(E \oplus V)$ *:*

$$e^{\mathfrak{t}}=\sum_{k=0}^{\infty}\frac{\mathfrak{t}^k}{k!}=\mathrm{Id}+T.$$

Proof: The exponential of a (conilpotent) coderivation of degree 0 is a comorphism. The restriction maps of $e^{\mathfrak{t}}$ are given by

$$\begin{aligned}(e^{\mathfrak{t}})_1(x+v) &= x+v+T_1(v), \quad x\in E, v\in V\\ (e^{\mathfrak{t}})_n(x_1+v_1,\ldots,x_n+v_v) &= T_n(v_1,\ldots,v_n), \quad n\geq 2,\end{aligned}$$

for all $x_1+v_1,\ldots,x_n+v_n \in E\oplus V$, and this yields the result. ■

Lemma 3.5. *The map* $p_{T(V)}Qe^{\mathfrak{t}}$ *is a coderivation with respect to the comorphism* $p_{T(V)}$*. Its restriction to* $T(V)$*,* $p_{T(V)}Qe^{\mathfrak{t}}_{|TV}$*, is defined by the restriction maps:*

$$p_VQe^{\mathfrak{t}}(v_1)=m_1(v_1),$$

$$p_VQe^{\mathfrak{t}}(v_1\otimes\ldots\otimes v_n)=m_n(v_1,\ldots,v_n)+\sum_{k=1}^{n-1}\Phi^{\bullet}_{\pi T(v_1\otimes\ldots\otimes v_k)}(v_{k+1},\ldots,v_n) \quad (18)$$

for all $v_1,\ldots,v_n\in V$*,* $n\geq 2$*.*

Proof: The projection map $p_{T(V)} : T^Z(E\oplus V)\to T^Z(V)$ is a comorphism. Since $e^{\mathfrak{t}}$ is also a comorphism, the composition $p_{T(V)}Qe^{\mathfrak{t}}$ is a coderivation with respect to the comorphism $p_{T(V)} \circ e^{\mathrm{t}} = p_{T(V)} \circ (\mathrm{Id} + T) = p_{T(V)}$. In fact,

$$\begin{aligned}
\Delta^Z p_{T(V)} Q e^{\mathrm{t}} &= p_{T(V)} \otimes p_{T(V)} \left(\mathrm{Id} \otimes Q + Q \otimes \mathrm{Id}\right)(e^{\mathrm{t}} \otimes e^{\mathrm{t}})\Delta^Z \\
&= \left(p_{T(V)} e^{\mathrm{t}} \otimes (p_{T(V)} Q e^{\mathrm{t}}) + (p_{T(V)} Q e^{\mathrm{t}}) \otimes p_{T(V)} e^{\mathrm{t}}\right) \Delta^Z \\
&= \left(p_{T(V)} \otimes (p_{T(V)} Q e^{\mathrm{t}}) + (p_{T(V)} Q e^{\mathrm{t}}) \otimes p_{T(V)}\right) \Delta^Z,
\end{aligned}$$

which means that $p_{T(V)} Q e^{\mathrm{t}}$ is a $p_{T(V)}$-coderivation.

Let $v_1, \ldots, v_n \in V$. For $n = 1$, the restriction map of $p_{T(V)} Q e^{\mathrm{t}}_{|TV}$ is given by:

$$p_V Q e^{\mathrm{t}}(v_1) = p_V Q(\mathrm{Id} + T)(v_1) = m_1(v_1).$$

For $n \geq 2$, the restriction map of $p_{T(V)} Q e^{\mathrm{t}}_{|TV}$ is given by:

$$\begin{aligned}
p_V Q e^{\mathrm{t}}(v_1 \otimes \ldots \otimes v_n) &= p_V Q(\mathrm{Id} + T)(v_1 \otimes \ldots \otimes v_n) \\
&= p_V Q \sum_{\substack{i_1 + \ldots + i_k = n \\ \sigma \in \widetilde{Sh}(i_1, \ldots, i_k)}} \epsilon(\sigma)(\mathrm{Id} + T)_{i_1}(v_{\sigma(1)}, \ldots, v_{\sigma(i_1)}) \otimes \ldots \\
&\qquad \ldots \otimes (\mathrm{Id} + T)_{i_k}(v_{\sigma(i_1 + \ldots + i_{k-1} + 1)}, \ldots, v_{\sigma(n)}).
\end{aligned}$$

Now Equation (18) follows if we keep in mind three things:

(i) The restriction maps of the identity are given by: $\mathrm{Id}_1 = \mathrm{Id}_{E \oplus V}$ and $\mathrm{Id}_k = 0$, for $k > 1$.

(ii) $\sigma \in \widetilde{Sh}(i_1, \ldots, i_k)$ means that

$$\sigma(i_1) < \sigma(i_1 + i_2) < \ldots < \sigma(i_1 + \ldots + i_{n-1}) < \sigma(i_1 + \ldots + i_n) = n.$$

(iii) By (11) we have

$$\begin{aligned}
&p_V Q(v_1 \otimes \ldots \otimes v_n) = m_n(v_1, \ldots, v_n), \\
&p_V Q(x_1 \otimes \ldots \otimes x_k \otimes v_1 \otimes \ldots \otimes v_p) = \Phi^{\bullet,\bullet}(x_1, \ldots, x_k; v_1, \ldots, v_p), \\
&\qquad \text{for all } x_1, \ldots, x_k \in E, \\
&p_V Q = 0 \text{ in all the other cases.}
\end{aligned}$$

Therefore,

$$p_V Q e^{\mathrm{t}}(v_1 \otimes \ldots \otimes v_n) = m_n(v_1, \ldots, v_n) + \sum_{k=1}^{n-1} \Phi^{\bullet}_{\pi T(v_1 \otimes \ldots \otimes v_k)}(v_{k+1}, \ldots, v_n). \quad \blacksquare$$

Next proposition gives an explicit description of non-abelian embedding tensors.

Proposition 3.6. *Equation (17) is equivalent to*

$$l_1 \circ T_1 = T_1 \circ m_1$$

and

$$l_\bullet(T(v)) = T_\bullet\left(M_V^Z(v)\right) + \sum_{k=2}^{n}\sum_{i=0}^{k-2}\sum_{\substack{j=i+1\\ \sigma\in Sh(i,k-i-1)}}^{k-1} \epsilon(\sigma)(-1)^{|v_{\sigma(1)}|+\ldots+|v_{\sigma(i)}|}$$

$$T_\bullet\Big(v_{\sigma(1)},\ldots,v_{\sigma(i)},\Phi^\bullet_{\pi T(v_{\sigma(i+1)}\otimes\ldots\otimes v_{\sigma(j)})}\left(v_{\sigma(j+1)},\ldots,v_{\sigma(k-1)},v_k\right),v_{k+1},\ldots,v_n\Big),$$

for all $v = v_1\otimes\ldots\otimes v_n \in T^n(V)$, $n\geq 2$.

Proof: First notice that

$$p_E\mathfrak{t}^k = 0, \quad k\geq 2.$$

Thus

$$p_E(e^{[\,,\mathfrak{t}]}(Q)) = p_E(Qe^{\mathfrak{t}} - \mathfrak{t}Qe^{\mathfrak{t}})$$

and Equation (17) is equivalent to

$$p_EQe^{\mathfrak{t}}_{|_{T(V)}} = p_E\mathfrak{t}Qe^{\mathfrak{t}}_{|_{T(V)}}$$

or to

$$l_\bullet p_{T(E)}e^{\mathfrak{t}}_{|_{T(V)}} = T_\bullet p_{T(V)}Qe^{\mathfrak{t}}_{|_{T(V)}}.$$

Let $v_1,\ldots,v_n\in V$. By Lemma 3.4, we have

$$p_EQe^{\mathfrak{t}}(v_1\otimes\ldots\otimes v_n) = l_\bullet p_{T(E)}e^{\mathfrak{t}}(v_1\otimes\ldots\otimes v_n) = l_\bullet T(v_1\otimes\ldots\otimes v_n).$$

On the other hand, by Lemma 3.5 we know that $p_{T(V)}Qe^{\mathfrak{t}}_{|_{T(V)}}$ is the $p_{T(V)}$-coderivation defined by the linear maps (18). This yields

$$p_{T(V)}Qe^{\mathfrak{t}}(v_1\otimes\ldots\otimes v_n) = M_V^Z(v_1\otimes\ldots\otimes v_n)$$

$$+\sum_{k=2}^{n}\sum_{\substack{i=0\\ \sigma\in Sh(i,k-i-1)}}^{k-2}\sum_{j=i+1}^{k-1}\epsilon(\sigma)(-1)^{(|v_{\sigma(1)}|+\ldots+|v_{\sigma(i)}|)}v_{\sigma(1)}\otimes\ldots\otimes v_{\sigma(i)}\otimes$$

$$\otimes\Phi^\bullet_{\pi T(v_{\sigma(i+1)}\otimes\ldots\otimes v_{\sigma(j)})}\left(v_{\sigma(j+1)},\ldots,v_{\sigma(k-1)},v_k\right)\otimes v_{k+1}\otimes\ldots\otimes v_n. \quad \blacksquare$$

Non-abelian embedding tensors define a Loday ∞-structure on V.

Proposition 3.7. *A non-abelian embedding tensor T with respect to a coherent Lie ∞-algebra action $\Phi : (E, M_E) \to (\mathrm{Coder}(\bar{S}(V))[1], \partial_{M_V}, [\cdot,\cdot])$ induces on V a Loday ∞-algebra structure Q_V^T, given by the brackets*

$$q_1(v_1) = m_1(v_1)$$

$$q_n(v_1, \dots, v_n) = \sum_{k=1}^{n-1} \Phi^{\bullet}_{\pi T(v_1\otimes\ldots\otimes v_k)}(v_{k+1}, \dots, v_n) + m_n(v_1, \dots, v_n),$$

for all $v_1, \dots, v_n \in V$, $n \geq 2$.

Moreover, $T : (V, Q_V^T) \to (E, M_E^Z)$ is a Loday ∞-morphism.

Proof: Let Q be the codifferential (of degree $+1$) of the non-abelian hemisemidirect direct product $E \ltimes^{\Phi} V$. Since

$$e^{[\,,\mathrm{t}]}(Q) \circ e^{[\,,\mathrm{t}]}(Q) = e^{[\,,\mathrm{t}]}(Q^2) = 0,$$

also $e^{[\,,\mathrm{t}]}(Q)$ is a coderivation of degree $+1$ of $T^Z(E \oplus V)$ and, consequently, it defines Loday ∞-structure on $E \oplus V$.

By definition T is a non-abelian embedding tensor if $\mathcal{P}\left(e^{[\,,\mathrm{t}]}(Q)\right) = 0$, therefore

$$p_V\left(e^{[\,,\mathrm{t}]}(Q)\right)_{|_{T(V)}}$$

defines a Loday ∞-structure on V. Noting that $p_V \circ \mathrm{t} = 0$, we have $p_V\left(e^{[\,,\mathrm{t}]}(Q)\right) = p_V \circ Q \circ e^{\mathrm{t}}$ and the Lemma 3.5 gives the Loday ∞-brackets

$$\begin{aligned} q_1(v_1) &= m_1(v_1) \\ q_n(v_1, \dots, v_n) &= m_n(v_1, \dots, v_n) + \sum_{k=1}^{n-1} \Phi^{\bullet}_{\pi T(v_1\otimes\ldots\otimes v_k)}(v_{k+1}, \dots, v_n), \end{aligned}$$

for all $v_1, \dots, v_n \in V$, $n \geq 2$.

Finally, let $Q_V^T = p_{T(V)} Q e^{\mathrm{t}}_{|_{T(V)}}$ be the codifferential of $T^Z(V)$ defined by these brackets. Proposition 3.6 guarantees that T is a non-abelian embedding tensor if and only if $M_E^Z \circ T = T \circ Q_V^T$. This means that T is a Loday ∞-morphism. ∎

Following the nomenclature used in [20]:

Definition 3.8. The Loday ∞-structure on V given by a non-abelian embedding tensor with respect to a coherent Lie ∞-action is called the **descendent Loday ∞-algebra**.

Corollary 3.9. *Let (E, M_E) and (V, M_V) be Lie ∞-algebras and T a symmetric non-abelian embedding tensor with respect to a coherent action Φ such that $\pi(\operatorname{Im} T) \subset \ker \Phi$, then T defines a Lie ∞-morphism.*

Proof: Let T be a non-abelian embedding tensor with respect to an coherent action Φ such that $\pi(\operatorname{Im} T) \subset \ker \Phi$. Then, by Proposition 3.7,

$$M_E^Z \circ T = T \circ M_V^Z.$$

The (Zinbiel) comorphism $T \equiv \left\{T_k : T^k(V) \to E\right\}_{k \in \mathbb{N}}$ is symmetric if $T_\bullet = T_\bullet \circ \pi$ (see Remark 1.19). In this case, the same family of linear maps defines a comorphism $T^S : \bar{S}(V) \to \bar{S}(E)$ that is a Lie ∞-morphism because T is a Loday ∞-morphism. ∎

Corollary 3.10. *Let (E, M_E) and (V, M_V) be Lie ∞-algebras and T a non-abelian embedding tensor with respect to a coherent action Φ. Then T induces a Lie ∞-morphism $\bar{T} : \operatorname{Prim}(V) \to E$.*

Proof: The non-abelian embedding tensor is a Loday ∞-morphism

$$T : (V, Q_V^T) \to (E, M_E^Z),$$

therefore it is also a comorphism between coshuffle coalgebras and preserves the codifferentials.

Recall that, for any graded vector space V, there is a (coalgebra) isomorphism $\Psi : \bar{T}^c(V) \to \bar{S}(\operatorname{Prim}(V))$ (see [19]).

In our case, (V, Q_V^T) and (E, M_E^Z) are Loday ∞-algebras so we can use the isomorphism and the codifferentials Q_V^T and M_E^Z to turn $\operatorname{Prim}(V)$ and $\operatorname{Prim}(E)$ into Lie ∞-algebras:

$$Q_V^{T, Lie} := \Psi \circ Q_V^T \circ \Psi^{-1},$$
$$M_E^{Lie} := \Psi \circ M_E^Z \circ \Psi^{-1}.$$

We have the following diagram:

$$\begin{tikzcd}
(V, Q_V^T) \subset T^c(V) \arrow[r, "T"] \arrow[d, shift left, "\Psi"] & (E, M_E^Z) \subset T^c(E) \arrow[r, "\pi"] \arrow[d, shift left, "\Psi"] & (E, M_E) \subset \bar{S}(E) \\
(\operatorname{Prim}(V), Q_V^{T, Lie}) \arrow[r, "T^{Lie}"] \arrow[u, shift left, "\Psi^{-1}"] & (\operatorname{Prim}(E), M_E^{Lie}) \arrow[u, shift left, "\Psi^{-1}"] &
\end{tikzcd}$$

The symmetrization map $\pi : \bar{T}^c(E) \to \bar{S}(E)$ is a comorphism such that

$$\pi \circ M_E^Z = M_E \circ \pi.$$

Therefore, the map $\bar{T} = \pi \circ T \circ \Psi^{-1} : \bar{S}(\mathrm{Prim}(V)) \to \bar{S}(E)$ is a comorphism that preserves the codifferentials, which means it is a Lie ∞-algebra morphism. ■

Example 3.11. In the case where V is simply a vector space (i.e. $M_V = 0$) and $\rho : (E, M_E) \to \mathfrak{gl}(V)$ is a (non-curved) representation of the Lie ∞-algebra (E, M_E), a non-abelian embedding tensor of ρ is a homotopy embedding tensor of the representation ρ, that was defined in [19].

Example 3.12. Let $\Phi : (E, M_E = \{l_k\}_{k\in\mathbb{N}}) \to (\mathrm{End}(V), \mathrm{d})$ be a Lie ∞-representation. A (Zinbiel) comorphism $T \equiv \left\{T_k : T^k(V) \to E\right\}_{k\in\mathbb{N}}$ is a non-abelian embedding tensor if and only if

$$
\begin{aligned}
l_1 \circ T_1 &= T_1 \circ \mathrm{d}\\
l_\bullet T(v_1 \otimes \ldots \otimes v_n) &= \sum_{i=1}^{n} (-1)^{|v_1|+\ldots+|v_{i-1}|} T_n(v_1 \ldots, \mathrm{d}v_i, \ldots, v_n)\\
&\quad + \sum_{k=2}^{n} \sum_{\substack{i=0\\ \sigma\in Sh(i,k-i-1)}}^{k-2} \epsilon(\sigma)(-1)^{(|v_{\sigma(1)}|+\ldots+|v_{\sigma(i)}|)}\\
&\qquad T_\bullet\left(v_{\sigma(1)}, \ldots, v_{\sigma(i)}, \Phi_{\pi T(v_{\sigma(i+1)}\otimes\ldots\otimes v_{\sigma(k-1)})} v_k, v_{k+1}, \ldots, v_n\right),
\end{aligned}
$$

for all $v_1, \ldots, v_n \in V$, $n \geq 2$.

The Loday ∞-structure on V defined by the non-abelian embedding tensor T is given by the brackets:

$$
\begin{aligned}
q_1(v_1) &= \mathrm{d}(v_1)\\
q_n(v_1, \ldots, v_n) &= \Phi_{\pi T(v_1\otimes\ldots\otimes v_{n-1})} v_n,
\end{aligned}
$$

for all $v_1, \ldots, v_n \in V$, $n \geq 2$.

In particular, a strict morphism $T = T_1 : V \to E$ is an non-abelian embedding tensor if and only if

$$
\begin{aligned}
&l_1 \circ T = T \circ \mathrm{d}\\
&l_n\left(T(v_1), \ldots, T(v_n)\right) = T\left(\Phi^{n-1,1}\left(T(v_1), \ldots, T(v_{n-1}); v_n\right)\right),
\end{aligned}
$$

and the descendent Loday ∞-structure on V induced by T is

$$
\begin{aligned}
q_1(v_1) &= \mathrm{d}(v_1)\\
q_n(v_1, \ldots, v_n) &= \Phi^{n-1,1}(T(v_1), \ldots, T(v_{n-1}); v_n),
\end{aligned}
$$

for all $v_1, \ldots, v_n \in V$, $n \geq 2$.

Example 3.13. Let $(E, [\cdot,\cdot]_E)$ and $(V, [\cdot,\cdot]_V)$ be Lie algebras and $\Phi : E \to \operatorname{Coder} \bar{S}(V)$ the coherent action induced by a coherent Lie representation $\rho : E \to \operatorname{Der}(V)$:

$$[\rho_x v, w]_V = 0, \quad v, w \in V,\ x \in E.$$

A non-abelian embedding tensor $T \equiv \left\{T_k : T^k(V) \to E\right\}_{k\in\mathbb{N}}$ must satisfy

$$\begin{aligned}
&\sum_{i=1}^{n-1} \sum_{\sigma\in Sh(i,n-1-i)} \epsilon(\sigma)\left[T_\bullet(v_{\sigma(1)}, \ldots, v_{\sigma(i)}), T_\bullet(v_{\sigma(i+1)}, \ldots, v_{\sigma(n-1)}, v_n)\right]_E = \\
&= \sum_{k=2}^{n}\sum_{i=0}^{k-2} \sum_{\sigma\in Sh(i,k-1-i)} \epsilon(\sigma)(-1)^{|v_{\sigma(1)}|+\ldots+|v_{\sigma(i)}|} \\
&\qquad T_\bullet\left(v_{\sigma(1)}, \ldots, v_{\sigma(i)}, \rho_{T_\bullet(v_{\sigma(i+1)},\ldots,v_{\sigma(k-1)})} v_k, v_{k+1}, \ldots, v_n\right) \\
&\qquad + \sum_{j=2}^{n}\sum_{i=1}^{j-1} (-1)^{|v_i|(\sum_{k=i}^{j-1}|v_k|)+\sum_{k=1}^{j-1}|v_k|} T_{n-1}(v_1, \ldots, \widehat{v_i}, \ldots, v_{j-1}, [v_i, v_j]_V, v_{j+1}, \ldots, v_n).
\end{aligned}$$

If $T : V \to E$ is a strict morphism, we recover the definition of a non-abelian embedding tensor with respect to a coherent action in the sense of [20], because for $n = 2$ the last equation reduces to

$$[T(v_1), T(v_2)]_E = T\left(\rho_{T(v_1)} v_2\right) + T\left[v_1, v_2\right]_V, \quad v_1, v_2 \in V.$$

3.1. Non-abelian embedding tensors of the adjoint representation. Example 3.12 can be adapted to the adjoint representation of a Lie ∞-algebra $(E, M_E \equiv \{l_k\}_{k\in\mathbb{N}})$.

For the adjoint representation, $T \equiv \{T_k : T(E) \to E\}_{k\in\mathbb{N}}$ is an non-abelian embedding tensor if and only if

$$\begin{aligned}
&l_1 \circ T_1 = T_1 \circ l_1, \\
&l_\bullet T(x_1 \otimes \ldots \otimes x_n) = \sum_{i=1}^{n} (-1)^{(|x_1|+\ldots+|x_{i-1}|)} T_n(x_1, \ldots, l_1(x_i), \ldots, x_n) \\
&\quad + \sum_{k=2}^{n} \sum_{\substack{i=0 \\ \sigma\in Sh(i,k-i-1)}}^{k-2} \epsilon(\sigma)(-1)^{(|x_{\sigma(1)}|+\ldots+|x_{\sigma(i)}|)} \\
&\quad T_\bullet\left(x_{\sigma(1)}, \ldots, x_{\sigma(i)}, l_\bullet(T(x_{\sigma(i+1)} \otimes \ldots \otimes x_{\sigma(k-1)}), x_k), x_{k+1}, \ldots, x_n\right).
\end{aligned}$$

The descendent Loday ∞-structure on E defined by the non-abelian embedding tensor T is given by the brackets:

$$\begin{aligned} q_1(x_1) &= l_1(x_1) \\ q_n(x_1, \dots, x_n) &= l_\bullet(T(x_1 \otimes \dots \otimes x_{n-1}), x_n), \end{aligned}$$

for all $x_1, \dots, x_n \in E$, $n \geq 2$.

In particular, if we ask $T : E \to E$ to be strict we must have

$$\begin{aligned} l_1 \circ T &= T \circ l_1 \\ l_n\left(T(x_1), \dots, T(x_n)\right) &= T\left(l_n\left(T(x_1), \dots, T(x_{n-1}), x_n\right)\right), \end{aligned} \tag{19}$$

and the Loday ∞-brackets are:

$$\begin{aligned} q_1(x_1) &= l_1(x_1) \\ q_n(x_1, \dots, x_n) &= l_\bullet(T(x_1), \dots, T(x_{n-1}), x_n), \end{aligned}$$

for all $x_1, \dots, x_n \in E$, $n \geq 2$.

Remark 3.14. The identity map is obviously a non-abelian embedding tensor for the adjoint representation and, as it should be, the "new" Loday brackets induced by the identity map are the original symmetric brackets that define the Lie ∞-structure.

Remark 3.15. Notice that the brackets $l_\bullet$ are symmetric, hence, for every $n > 1$ and each $i = 1, \dots, n$, we have

$$l_n\left(T(x_1), \dots, T(x_n)\right) = T\left(l_n(T(x_1), \dots, x_i, \dots, T(x_n))\right), \quad x_1, \dots, x_n \in E.$$

Remark 3.16. Let $T : E \to E$ be a strict comorphism such that $l_1 \circ T = T \circ l_1$ and $\mathrm{Im}(T) \subset \mathrm{Ker}(\mathrm{ad})$. Then, T is an embedding tensor for the adjoint representation but the new Loday ∞-algebra is abelian, i.e. $q_k = 0$, for all $k > 1$.

Equations (19) yield that, for the adjoint representation, the composition of strict non-abelian embedding tensors is a strict non-abelian embedding tensor, therefore:

Proposition 3.17. *The space of strict non-abelian embedding tensors with respect to the adjoint representation is an associative algebra with unit, called the* ***strict embedding algebra****.*

Let us call *centroid of the Lie ∞-algebra* $(E, \{l_k\}_{k\in\mathbb{N}})$ the set of strict comorphisms $F : E \to E$ such that

$$\begin{aligned} l_1 \circ F &= F \circ l_1 \\ \mathrm{ad}_x \circ F &= F \circ \mathrm{ad}_x, \quad x \in \bar{S}(E). \end{aligned}$$

Corollary 3.18. *Let E be a Lie ∞-algebra. Then the centroid of E is a subalgebra (with unit) of the strict embedding algebra.*

Moreover, any surjective strict non-abelian embedding tensor is an element of the centroid and the group of invertible elements of the centroid coincides with the group of invertible elements of the strict embedding algebra.

3.2. Strict non-abelian embedding tensors of the adjoint action. The adjoint action $\mathrm{ad} : (E, M_E \equiv \{l_k\}_{k\in\mathbb{N}}) \to (\mathrm{Coder}(\bar{S}(E))[1], \partial_{M_E}, [\cdot,\cdot])$ is a special representation of E on the cochain complex $(\bar{S}(E), M_E)$.

A strict non-abelian embedding tensor of this representation is a strict comorphism $T : \bar{S}(E) \to E$ satisfying

$$\begin{aligned} l_1 \circ T(v_1) &= T \circ M_E(v_1) \\ l_n(T(v_1), \ldots, T(v_n)) &= T\left(\mathrm{ad}_{T(v_1)\cdot\ldots\cdot T(v_{n-1})} v_n\right), \end{aligned}$$

for all $v_1, \ldots, v_n \in \bar{S}(E)$.

In particular, for $v_1, \ldots, v_n \in E$, we have

$$\begin{aligned} l_1 \circ T(v_1) &= T \circ l_1(v_1) \\ l_n(T(v_1), \ldots, T(v_n)) &= T\left(l_n(T(v_1), \ldots, T(v_{n-1}), v_n)\right) \end{aligned}$$

and we conclude that $T_{|_E} : E \to E$ is a strict non-abelian embedding tensor for the adjoint representation.

3.3. Deformations of non-abelian embedding tensors. Let T be a non-abelian embedding tensor with respect to the coherent Lie ∞-action $\Phi : (E, M_E) \to (\mathrm{Coder}(\bar{S}(V))[1], \partial_{M_V}, [\cdot,\cdot])$. Since T defines (and is defined) by $\mathfrak{t}$, a Maurer-Cartan element of the Lie ∞-algebra $\mathfrak{h}$, we have two different V-data $(\mathrm{Coder}\, T^Z(E\oplus V), \mathfrak{h}, \mathcal{P}, e^{[\,,\mathfrak{t}]}(Q))$ and $(\mathrm{Coder}\, T^Z(E\oplus V), \mathfrak{h}, \mathcal{P}\circ e^{[\,,\mathfrak{t}]}, Q)$, both of them inducing the same twisting Lie ∞-structure in $\mathfrak{h}$:

$$\partial_k^T(\mathfrak{t}_1, \ldots, \mathfrak{t}_k) = \sum_{i\geq 0} \frac{1}{i!}\partial_{k+i}(\mathfrak{t}, \ldots, \mathfrak{t}, \mathfrak{t}_1, \ldots, \mathfrak{t}_k), \quad \mathfrak{t}_1, \ldots, \mathfrak{t}_k \in \mathfrak{h}, \quad k \geq 1.$$

Let $T' : T^Z(V) \to T^Z(E)$ be another comorphism and consider $\mathfrak{t}'$ the coderivation of $T^Z(E \oplus V)$ defined by T'. By definition, $T + T'$ is an embedding tensor if and only if $\mathfrak{t} + \mathfrak{t}'$ is a Maurer-Cartan element of $\mathfrak{h}$:

$$\sum_{k \geq 1} \frac{1}{k!} \partial_k(\mathfrak{t} + \mathfrak{t}', \ldots, \mathfrak{t} + \mathfrak{t}') = 0.$$

Taking into account that $\mathfrak{t} \in \mathrm{MC}(\mathfrak{h})$, last equation can be rewritten [6] as

$$\partial_1^T(\mathfrak{t}') + \frac{1}{2}\partial_2^T(\mathfrak{t}', \mathfrak{t}') + \ldots + \frac{1}{k!}\partial_k^T(\mathfrak{t}', \ldots, \mathfrak{t}') + \ldots = 0.$$

Therefore $T + T'$ is a non-abelian embedding tensor with respect to the same Lie ∞-action as T if and only if T' is a Maurer-Cartan element of the deformed Lie ∞-algebra $\mathfrak{h}^T = (\mathfrak{h}, \partial^T)$.

Proposition 3.19. *The Lie ∞-algebra $\mathfrak{h}^T = (\mathfrak{h}, \partial^T)$ controls the deformations of T.*

Definition 3.20. The cohomology of the cochain complex $(\bar{S}(\mathfrak{h}), \partial^T)$ is called the **cohomology of the non-abelian embedding tensor** T.

4. Conclusions

In [20] a definition of non-abelian embedding tensors on a Lie algebra with respect to a coherent action was given. In this work we give a possible generalization of this definition and of some results in [19, 20] to the Lie ∞-setting. We propose a definition of coherent Lie ∞-actions and, for these actions, we construct the non-abelian hemisemidirect product and study non-abelian embedding tensors. The question of whether such a generalization could be carried out was posed by Y. Sheng to one of the co-authors during a Colloquium, last year. There are still open questions concerning this generalization.

Firstly, we can ask about the existence of a Lie ∞-representation/action of the Lie ∞-algebra $\mathrm{Prim}(V)$ on V, such that the projection $T(V) \to \mathrm{Prim}(V)$ is an embedding tensor. Secondly, since there is a straight relation between non-abelian embedding tensors for Lie algebras and Leibniz-Lie algebras [20], we wonder what is the algebraic structure that is related to the embedding tensors we consider. What about cohomology of this algebraic structure? We think the answer to these questions may go through understanding what a Loday ∞-action should be and these problems will deserve our attention in a future work.

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