Optimization in pricing and hedging options

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CIM Workshop on Optimization in Finance
Coimbra 5-8 July 2005

The market

- Two dates: $t = 0$, $t = 1$
- Finite sample space

$$\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$$

- n securities $S^1, S^2, \ldots, S^n$
- $\mathbf{S_0} = \left(S_0^1, S_0^2, \ldots, S_0^n\right)$ is the n-vector of prices at time $t = 0$
- $\mathbf{S_1} = \left(\mathbf{S_1^1} | \mathbf{S_1^2} | \ldots | \mathbf{S_1^n}\right)$ is a *mxn* matrix.

Portfolios of securities

- At time $t = 0$ it is possible to take any position ("long" or "short") on the n securities.
- Let $\mathbf{x}$ be a portfolio of securities

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

The cost (at $t = 0$) of $\mathbf{x}$ is $\mathbf{S_0 x}$ (a scalar)
The payoff (at $t = 1$) of $\mathbf{x}$ is $\mathbf{S_1 x}$ (m-vector)

B-Arbitrages

- Let us find the minimum cost portfolio with a positive payoff
- The Primal Problem (P)

$$\min_{\mathbf{x}} \mathbf{S_0 x}$$
$$\mathbf{S_1 x} \geq \mathbf{0}$$

- (P) is feasible, hence it is either bounded or unbounded
- If (P) is unbounded it is possible to realize an initial earning ($\mathbf{S_0 x} < \mathbf{0}$), with no future liabilities ($\mathbf{S_1 x} \geq \mathbf{0}$). This is called "arbitrage" (of type B).
- (P) bounded $\Leftrightarrow$ No B-arbitrages
- B-arbitrages are not consistent with economic equilibrium

The Dual problem

- The Dual of (P) is the following LP (D)

$$\max_{\mathbf{y}} \mathbf{y} \cdot \mathbf{0}$$
$$\mathbf{S_1}'\mathbf{y} = \mathbf{S_0}$$
$$\mathbf{y} \geq \mathbf{0}$$

- (D) feasible $\Leftrightarrow$ (P) bounded $\Leftrightarrow$ No B-arbitrages

A-Arbitrages

- There is a second type of arbitrage: a free lottery ticket.
- Portfolio $\mathbf{x}$ is an A-arbitrage if

$$\begin{aligned} \mathbf{S_0 x} &= \mathbf{0} \\ \mathbf{S_1 x} &\geq \mathbf{0} \\ \mathbf{S_1 x}(\omega_\mathbf{i}) &> \mathbf{0} \quad \text{for some } \omega_i \in \Omega \end{aligned}$$

- A-arbitrages are not consistent with economic equilibrium
- A model is "arbitrage-free" is there are neither A nor B arbitrages.

Arbitrage-free models

- A market model is arbitrage-free $\Leftrightarrow$ There is a strictly positive solution to

$$\mathbf{S_1}'\mathbf{y} = \mathbf{S_0}$$
$$\mathbf{y} \geq \mathbf{0}$$

- Define $B_0 = \sum_{i=1}^{m} y_i$
- $\mathbf{q} = \mathbf{y}/\mathbf{B_0}$ is a probability on Ω
- $\mathbf{q}$ is called "risk-neutral" because

$$\mathbf{S_0} = B_0 E^q \mathbf{S_1}$$

- No arbitrage $\Leftrightarrow \exists \mathbf{q}$ "risk-neutral"

Pricing contingent claims

- Assume there are no arbitrages
- A contingent claim $\mathbf{b}$ is a random variable on Ω.
- A portfolio $\mathbf{x}$ "replicates" $\mathbf{b}$ if

$$\mathbf{S}_1\mathbf{x} = \mathbf{b}$$

- A claim is "attainable" if it admits a replicating portfolio
- The no-arbitrage price of an attainable claim $\mathbf{b}$ is

$$\mathbf{S}_0\mathbf{x} = \mathbf{x} \cdot \mathbf{B}_0\mathbf{E}^{\mathbf{q}}\mathbf{S}_1 = \mathbf{B}_0\mathbf{E}^{\mathbf{q}}\mathbf{S}_1\mathbf{x} = \mathbf{B}_0\mathbf{E}^{\mathbf{q}}\mathbf{b}$$

Complete markets

- A market is "complete" if all claims are attainable
- Market is complete $\Leftrightarrow$ lin $< \mathbf{S_1^1}, \ldots, \mathbf{S_1^n} >= \Re^m$
- That is $n \geq m$ and

$$\text{rank}(\mathbf{S_1}) = m$$

- The Dual (D) has a unique solution
- Completeness (and No-arbitrages) $\Leftrightarrow \exists! \mathbf{q}$

Incomplete markets

- Suppose a claim **b** is not attainable
- We can determine the minimum price V^+ for a "super-replicating" strategy
- This is called the "buyer's" problem: if one buys the claim at any price greater than V^+ there is an arbitrage opportunity for the writer of the option
- Analogously, the "writer's" problem finds the maximum price V^- for a sub-replicating strategy

The buyer's problem

- Consider the LP

$$V^+ = \min_{\mathbf{x}} \mathbf{S_0 x}$$
$$\mathbf{S_1 x} \geq \mathbf{b}$$

- Its dual is

$$\max_{\mathbf{y}} \mathbf{b}'\mathbf{y}$$
$$\mathbf{S_1}'\mathbf{y} = \mathbf{S_0}$$
$$\mathbf{y} \geq \mathbf{0}$$

- Therefore

$$V^+ = \max_{\mathbf{q}} B_0 E^{\mathbf{q}} \mathbf{b}$$

Arbitrage-free prices of non-attainable claims

- The analogous "seller's problem" yields to

$$V^- = \min_{\mathbf{q}} B_0 E^{\mathbf{q}} \mathbf{b}$$

- Any price V, such that $V^- \leq V \leq V^+$ is an arbitrage free price (all inequalities are strict if $V^- < V^+$)
- A claim is attainable iff it has a unique arbitrage-free price

Market frictions I

- Suppose there are different bid-ask prices
- The primal problem (P) becomes

$$\min_{\mathbf{x}^a,\mathbf{x}^b} \mathbf{S_0}^a\mathbf{x}^a - \mathbf{S_0}^b\mathbf{x}^a$$

$$\mathbf{S_1}^a\mathbf{x}^a - \mathbf{S_1}^b\mathbf{x}^b \geq \mathbf{0}$$

$$\mathbf{x}^a \geq \mathbf{0}$$

$$\mathbf{x}^b \geq \mathbf{0}$$

Market frictions I

- The dual problem is

$$\max_{\mathbf{y}} \mathbf{y} \cdot \mathbf{0}$$
$$\mathbf{S_0}^a \leq \mathbf{S_1}'\mathbf{y} \leq \mathbf{S_0}^{\mathbf{a}}$$
$$\mathbf{y} \geq \mathbf{0}$$

- No arbitrage $\Leftrightarrow$ a risk-neutral measure separates bid and ask prices

Single-period: main results

- No arbitrages $\Leftrightarrow$ There is a risk-neutral measure
- No Arb. + Completeness $\Leftrightarrow$ There is a unique risk-neutral measure
- If there are no arbitrages
  - $\mathbf{b}$ attainable $\Leftrightarrow V_0(\mathbf{b}) = \mathbf{B_0 E^q b}, \forall \mathbf{q}$
  - $\mathbf{b}$ not attainable $\Rightarrow V^-(\mathbf{b}) \le \mathbf{V}(\mathbf{b}) \le \mathbf{V}^+(\mathbf{b})$
  - $\exists \mathbf{q}$ separating bid and ask prices

The market

- T dates: $t = 0, 1, \ldots, T$
- Probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t=0}^{T}, P)$
- n securities $S^1, S^2, \ldots, S^n$
- S^i is a (discrete-time) stochastic process
- S^i_t is $\mathcal{F}_t$-measurable
- Assume $S^1_t > 0, \forall t$. S^1 is called "numeraire". Define the "discounted process" $\mathbf{Z} = \mathbf{S}/S^1$.

Dynamic portfolios

- A "dynamic portfolio" (or "market strategy") is a stochastic process θ. θ_t^i is the number of shares of security S^i held between $t-1$ and t. θ_t^i is $\mathcal{F}_{t-1}$-measurable
- The discounted value of the portfolio at time t is $\theta_t \cdot \mathbf{Z_t}$
- A portfolio strategy is "self-financing" if

$$\theta_t \cdot \mathbf{Z_t} = \theta_{t+1} \cdot \mathbf{Z_t}$$

Arbitrages

- Because there is a numeraire, any B-arbitrage can be transformed into an A-arbitrage.
- A dynamic portfolio θ is an arbitrage if

$$
\begin{aligned}
E\theta_T \cdot \mathbf{Z}_T &> 0 \\
\theta_1 \cdot \mathbf{Z}_0 &= 0 \\
\theta_t \cdot \mathbf{Z_t} &= \theta_{t+1} \cdot \mathbf{Z_t}, \quad t = 0, \ldots, T-1 \\
\theta_T \cdot \mathbf{Z}_T &\geq 0
\end{aligned}
$$

The arbitrage problem

- The arbitrage problem can be set in many equivalent ways. Here we follow the non-recombining tree representation (King (2002)).
- Denote $\mathcal{N}_t$ the set of states at time t. For any state $s \in \mathcal{N}_t$, let $a(s) \subset \mathcal{N}_{t-1}$ be the parent of s and let $c(s) \subset \mathcal{N}_{t+1}$ be the set of childs of s.
- To find arbitrages one can solve

$$
\begin{aligned}
\max_{\theta} \sum_{s \in \mathcal{N}_T} p_s \mathbf{Z}_s \cdot \theta_s & & \\
\mathbf{Z}_0 \cdot \theta_0 = 0 & & : y_0 \\
\mathbf{Z}_s \cdot [\theta_s - \theta_{a(s)}] = 0 & \quad (s \in \mathcal{N}_t, t \geq 1) & : y_s \\
\mathbf{Z}_s \cdot \theta_s \geq 0 & \quad (s \in \mathcal{N}_T) & : w_s
\end{aligned}
$$

Lagrangian

- The Lagrangian is

$$
\begin{aligned}
L(\theta; y, w) &= \sum_{s\in\mathcal{N}_T} p_s \mathbf{Z}_s \cdot \theta_s - \sum_{t=0}^{T} \sum_{s\in\mathcal{N}_t} y_s \mathbf{Z}_s \cdot [\theta_s - \theta_{a(s)}] \\
&- \sum_{s\in\mathcal{N}_T} w_s \mathbf{Z}_s \cdot \theta_s, \quad (w_s \le 0) \\
&= \sum_{s\in\mathcal{N}_T} [p_s - w_s - y_s] \mathbf{Z}_s \cdot \theta_s - \\
&\sum_{t=0}^{T-1} \sum_{s\in\mathcal{N}_t} [y_s \mathbf{Z}_s - \sum_{m\in c(s)} y_m \mathbf{Z}_m] \cdot \theta_s \\
&(w_s \le 0)
\end{aligned}
$$

Dual problem

- From the Lagrangian follows the Dual problem

$$\begin{aligned} w_s \leq 0 & \qquad (s \in \mathcal{N}_T) \\ (p_s - w_s - y_s)\,\mathbf{Z}_s = 0 & \qquad (s \in \mathcal{N}_T) \\ y_s \mathbf{Z}_s - \sum_{m \in c(s)} y_m \mathbf{Z}_m & \quad (s \in \mathcal{N}_t, t \leq T-1) \end{aligned}$$

- There are no arbitrages if and only if the Dual is feasible.
- No arbitrages $\Leftrightarrow \exists \mathbf{q} \sim \mathbf{p}$ s.t. $\mathbf{Z_{t-1}} = \mathbf{E^q}[\mathbf{Z_t} | \mathcal{N}_{\mathbf{t-1}}]$
- The risk neutral measure $\mathbf{q}$ does not depend on $\mathbf{p}$.

Discrete-time: main results

- Same results as in the single-period case
- Fundamental Theorem of Arbitrage:
  No arbitrage $\Leftrightarrow$ There is an equivalent martingale measure

Continuous time models

- For infinite times, infinite states models, there are some complications...
- Most famous example: Black-Scholes model

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu dt + \sigma dZ, \\ \frac{dB_t}{B_t} &= r dt \end{aligned}$$

- This is a complete model, i.e. there is a unique martingale measure and all contingent claims are attainable
- Set constraints on trading strategies and determine minimal super-replication cost

In the workshop

- Touzi, "Hedging with controlled sensitivities"
- Bouchard, "Explicit characterization of the super-replication strategy in financial markets with partial transaction costs"
- Judice, "Foundations and applications of Good-Deal pricing in single-period market models"
- Balbas, "Outperforming revealed prices in imperfect markets"
- Favero, "Long and short term arbitrages: A comment on an example by Pham and Touzi"

The problem

- Minimize the risk of hedging a contingent claim H when the market is incomplete
- Stochastic optimization problem with a quadratic objective function

A single-period model I

- The model: an asset $X_t, t = 0, 1$ and a bank account. We indicate with ξ the shares of X bought at time 0 and by $\eta_t, t = 0, 1$ the money in the bank account. (Assume zero interest rate).
- Value of portfolio at time t

$$V_t = \xi X_t + \eta_t$$

- Determine an "optimal" hedge for a claim with payoff H at time $t = 1$.
- Can always get $V_1 = H$ by setting

$$\eta_1 = H - \xi X_1$$

- Non-self-financing strategies

A single-period model II

- Let C_t be the cumulative cost of the strategy. The initial cost is

$$C_0 = V_0$$

the additional cost at time $t = 1$ is

$$\begin{aligned} C_1 - C_0 &= \eta_1 - \eta_0 \\ &= H - V_0 - \xi \Delta X \end{aligned}$$

- Determine V_0 and ξ to minimize the expected quadratic cost

$$\min_{V_0, \xi} R := E\left(H - V_0 - \xi \Delta X\right)^2$$

The solution I

- It is a linear regression problem
- Optimal investment strategy

$$\xi = \frac{\operatorname{Cov}[H, \Delta X]}{\operatorname{Var}[\Delta X]} = \frac{\operatorname{Cov}[H, X_1]}{\operatorname{Var}[X_1]}$$

- "Fair price" of H

$$V_0 = E(H) - \xi E(\Delta X)$$

- "Residual" or "Unhedgeable" risk

$$R_{\text{min}} = \operatorname{Var}[H]\,(1 - \rho(H, X_1))^2$$

The solution II

- Note that
$$C_0 = E(C_1)$$
the strategy is "mean self-financing".
- All depends on P
- When $R_{\min} = 0$ the claim H is attainable, the strategy is self-financing, the solution does not depend on P.

A discrete-time model I

- The model: a probability space $\left(\Omega, \{\mathcal{F}_k\}_0^T, P\right)$,
- One asset $X_k, k = 0, 1, \dots, T$ and one bank account.
- ξ_k are the shares of X bought at time $k-1$ ($\mathcal{F}_{k-1}$-measurable, "predictable")
- η_k the money in the bank account ($\mathcal{F}_k$-measurable, "adapted")

A discrete-time model II

- Value of portfolio at time t

$$V_t = \xi_t X_t + \eta_t$$

- The problem: Determine an "optimal" hedge for a claim with payoff H at time $t = T$
- Can always get $V_T = H$
- Non self-financing strategy

A discrete-time model III

- Let C_t be the cumulative cost of the strategy,

$$C_t = V_t - \sum_{j=1}^{t} \xi_j \Delta X_j$$

- The "local risk" at time t is

$$E[(C_{t+1} - C_t)^2 | \mathcal{F}_t] = E[(V_{t+1} - V_t - \xi_{t+1} \Delta X_{t+1})^2 | \mathcal{F}_t]$$

- The solution is determined by backward recursion

The solution

- The cost process
$$C_t = E_t[C_{t+1}]$$
is *mean self-financing* (i.e., a Martingale)
- The optimal strategy

$$\begin{aligned}
\xi_t &= \frac{\mathrm{Cov}_{t-1}\left[H - \sum_{j=t+1}^{T} \xi_j \Delta X_j, \Delta X_t\right]}{\mathrm{Var}_{t-1}[\Delta X_t]} \\
\eta_t &= E_t[H - \sum_{j=t+1}^{T} \xi_j \Delta X_j] - \xi_t X_t
\end{aligned}$$

When X is a martingale

- Follmer and Sondermann (1986)
- Let us assume

$$X_t = E_t[X_{t+1}]$$

- Remember that

$$C_t = V_t - \sum_{j=1}^{t} \xi_j \Delta X_j$$

- Since C is also a martingale, V is a martingale and

$$V_t = E_t V_T = E_t H$$

When X is a martingale

- (Kunita-Watanabe decomposition)

$$H = V_0 + \sum_{j=1}^{T} \xi_j^H \Delta X_j + L_T^H$$

where L^H is a martingale orthogonal to X, that is

$$E_{t-1}[\Delta L_t^H \Delta X_t] = 0$$

- $V_t = E_t H = V_0 + \sum_{j=1}^{t} \xi_j^H \Delta X_j + L_t^H$

- Local risk

$$\begin{aligned}
R_{t-1} &= E_{t-1}\left(C_t - C_{t-1}\right)^2 \\
&= E_{t-1}\left(V_t - V_{t-1} - \xi_t \Delta X_t\right)^2 \\
&= E_{t-1}\left(\xi_t^H \Delta X_t + \Delta L_t^H - \xi_t \Delta X_t\right)^2 \\
&= E_{t-1}(\Delta L_t^H)^2 + (\xi_t - \xi_t^H)^2 E_{t-1}(\Delta X_t)^2
\end{aligned}$$

- Hedging strategy

$$\xi_t = \xi_t^H$$

- Bank account

$$\begin{aligned}
\eta_t &= V_t - \xi_t^H X_t \\
&= V_0 + \sum_{j=1}^{t} \xi_j^H \Delta X_j + L_t^H - \xi_t^H X_t
\end{aligned}$$

When X is a martingale and $L^H = 0$

- In this case
$$H = V_0 + \sum_{j=1}^{T} \xi_j^H \Delta X_j$$
- $R_t = 0 \Rightarrow C_{t+1} = C_t = C_0$
- Bank account
$$\eta_t = V_{t-1} - \xi_t X_{t-1}$$
the strategy can be fixed at the beginning of each period.
- The quantities do not depend on P.
$$V_0 = E_0^P(H)$$
for any martingale measure P.

When X is a semi-martingale

- This case is more complicate. It was studied by Schweizer (1988).
- There exists a decomposition (Follmer-Schweizer)
- Non linear stochastic optimality equation
- In continuous time one possibility is to compute the "minimal martingale measure" $\hat{P}$ and then find

$$V_0 = E_0^{\hat{P}}(H)$$

- In general it is not solvable by recursion. Some explicit results for specific cases.

Other objective functions

- So far we have considered non-self-financing strategies with final value equal to H.
- Another possibility is to adopt self-financing strategies and minimize the final shortfall.
- This results in a problem of projections in linear space.
- The two problems are "equivalent" when the *mean-variance tradeoff*

$$\frac{(E_{k-1}\Delta S_k)^2}{\mathrm{Var}_{k-1}\Delta S_k}$$

  is deterministic. (Schall 1994)
- Bertsimas, et al. (2002) propose a DP explicitly solved for some specific cases

In this workshop

- Uryasaev: Pricing options in incomplete market.
- Biagini: A unifying framework for utility maximization problems.
- Consiglio: Evaluation of insurance products with guarantee: A stochastic programming approach

Some questions

- Knowing the prices of some options, can we recover the "risk-neutral probabilities"?
- When are observed prices consistent with no-arbitrage assumptions?
- Given the prices of some derivatives, what can we say about the prices of other derivatives on the same asset?
- Knowing the prices of some derivatives, can we get any bound on the moments of the underlying?
- Assuming the knowledge of the first k moments of the underlying, can we get any bound on the prices of the derivatives?

From prices to probabilities

- Given the prices of some derivatives, what can we infer about the distribution of the underlying?
- First approach: If there are Call prices for any strike K,

$$c(K) = B(0, T) \int_K^{+\infty} (x - K) q(x) dx$$

and then

$$c''(K) = B(0, T) * q(K)$$

Implied binomial trees, Rubinstein (1994)

- Determine the risk-neutral probability P of a binomial model which is closer to a "prior" probability P' and consistent with observed prices.
- Solve the following QP

$$\min_{P_j} \sum_j (P_j - P'_j)^2$$

$$\sum_j P_j = 1, P_j \geq 0$$

$$C_i^b \leq v^n \sum_j P_j (S_j - K_i)^+ \leq C_i^a$$

- Rubinstein solved the problem for 200-step tree, with SP500 options observed three times a day from 1986 to 1992
- Construct the inner probabilities of the tree. This is an-over parameterized problem. Some further assumptions are needed.

Arbitrages for call options

- Let $C_i, i = 1, \ldots, n$ be the prices of n call options written on the same underlying S, with same maturity T and strike prices K_i.
- Let $\Pi_i(s) = (s - K_i)^+$ be the payoff of call i when $S_T = s$
- Without making any assumption on the dynamics of S between 0 and T and on its distribution on T, what can we say on the prices of the options? Are they "Arbitrage-free"?
- Is there a portfolio $\mathbf{x}$ of options with a positive payoff and a negative price?

Arbitrages for call options

- Let $\Pi(s) := [\Pi_1(s), \Pi_2(s), \ldots, \Pi_n(s)]$,

$$\min C \cdot \mathbf{x}$$
$$\Pi(s)\mathbf{x} \geq 0$$

- Need to check feasibility only on the nodes
- Obtain a finite LP
- No arbitrage $\Leftrightarrow$ $C(K)$ is positive, decreasing and convex (for details see H. (2003))

Bounds on prices (Bertsimas and Popescu, 2000)

- Determine the maximum price of a call compatible with some observed moments of the underlying

$$
\begin{aligned}
\max_q E^q(X-K)^+ &= \int_0^{+\infty}(x-K)^+ q(x)dx \\
\int_0^{\infty} x^i q(x)dx &= m_i \quad i=1,\ldots,n \\
q(x) &\geq 0
\end{aligned}
$$

Bounds on prices

- The dual is

$$\min \sum_{i=0}^{n} y_i m_i$$

$$\sum_{i=0}^{n} y_i x^i \geq (x-K)^+, \qquad \forall x \in \Re^+$$

- Strong duality holds (Isii 1963).

Bounds on prices

- Using results like *The polinomial* $g(x) = \sum_{r=0}^{2k} y_r x^r$ *satisfies* $g(x) \geq 0$ *if and only if there exists a positive semidefinite matrix* $X = [x_{ij}]_{i,j=0,\ldots,k}$ *such that*

$$y_r = \sum_{i,j:i+j=r} x_{ij}, \quad r = 0, \ldots, 2k, \quad X \succeq 0$$

  can show that the dual problem is equivalent to a semi-definite programming
- In the workshop
  - Zuluaga, "Optimal semi-parametric bounds for European rainbow options"
  - Prieto-Rumeau, "Pricing exotic options with semidefinite programming"

American options

- An option is called "American" when one can exercise it at any time before expiration.
- To hedge an American option must consider that the buyer acts optimally
- It is an optimal stopping time problem

$$\max_{\tau} E^q f(S_\tau)$$

- For discrete time models can use dynamic programming
- Pennanen and King (2004) show that it can be formulated as a stochastic LP problem
- In continuous time there are not explicit solutions

In this workshop

- Byun: Properties of integral equations arising in the valuation of American options
- Ferulano: Enhanced Monte-Carlo methods for American options

Robust hedging of barrier options

- Brown, Hobson, Rogers (2001)
- Barrier option: an option that starts to exist (or vanishes) when the underlying crosses a given level
- Determine a model-independent hedging strategy
- Assumptions: interest rate is zero, calls of any strike expiring at T are available at time 0
- This is equivalent to setting the pricing measure μ at maturity

An example

- Up-and-In Put with barrier at the strike
- Consider the strategy: *Buy a call with strike K, sell forward the underlying at the instant (if ever) it reaches the level K*
- The cost of the strategy is the cost of the call
- The payoff is equal to the barrier put
- The strategy is robust and the price of the barrier put must be equal to the price of the call

Digital barrier options I

- Digital Barrier Option: an option that pays 1 iff the underlying S crosses B before time $T = 1$.
- $H_B = \inf\{t : S_t \geq B\}$
- Payoff of the digital: $\mathbb{I}_{H_B \leq 1}$

Digital barrier options II

- for any $y < B$,

$$\mathbb{I}_{H_B \leq 1} \leq \frac{(S_1 - y)^+}{B - y} + \frac{B - S_1}{B - y} \mathbb{I}_{H_B \leq 1}$$

- Taking expectations and observing that $E \frac{B-S_1}{B-y} \mathbb{I}_{H_B \leq 1} \leq 0$:

$$\mathbb{P}(H_B \leq 1) \leq \inf_{y \leq B} \frac{C(y)}{B - y}$$

- The minimum is attained at a point $a = a(\mu, B)$. The "robust" upper bound is

$$\frac{C(a)}{B - a}$$

- In the workshop Maruhn: Adding robustness to static hedge portfolios for Barrier options

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