Robust Optimization in Finance

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Coimbra, July 5, 2005

Outline

1. Robust Portfolio Optimization

2. Robust Risk Management

3. Robust Pricing/Hedging Problems

Asset Allocation and Portfolio Selection

Consider an investor who wants to allocate his/her funds among a set of asset classes $S_1, S_2, \ldots, S_n$. The returns from these asset classes at the end of an investment period are random. For asset class j, let μ_j represent its expected return, and let σ_j represent the standard deviation on the return. Also, for asset classes i and j, let σ_{ij} represent the covariance of their returns.

Let x_j denote the proportion of the investor's money to be allocated to asset class j. These are the variables to be determined to form an optimal portfolio of the available asset classes. $x = (x_1, x_2, \ldots, x_n)$ denotes a portfolio whose expected return and variance of return are given by

$$E(x) = \sum_j \mu_j x_j = \mu^T x,$$

and

$$Var(x) = \sum_j \sigma_j^2 x_j^2 + 2 \sum_{i<j} \sigma_{ij} x_i x_j = x^T \Sigma x,$$

where $\Sigma_{ij} = \sigma_{ij}$ and $\Sigma_{jj} = \sigma_j^2$.

Asset Allocation

- Competing objectives: maximize expected return, minimize its variance
- Representing portfolio constraints in the generic form $x \in \mathcal{X}$, we have three "equivalent" formulations:

$$\begin{array}{rr} \max & \mu^T x \\ & x \in \mathcal{X} \\ & x^T \Sigma x \leq \sigma^2 \end{array} \qquad \begin{array}{rr} \min & x^T \Sigma x \\ & x \in \mathcal{X} \\ & \mu^T x \geq R \end{array} \qquad \begin{array}{rr} \max & \mu^T x - \lambda x^T \Sigma x \\ & x \in \mathcal{X} \end{array}$$

- The problem is, μ and Σ are not observable–we must use estimates. Furthermore, solutions are quite sensitive to changes in μ and Σ. Therefore, we explore different robust optimization models.

Scenario-based Models

Consider a finite set of possible scenarios for μ and Σ given as $\mu^1, \ldots, \mu^I$ and $\Sigma^1, \ldots, \Sigma^J$.

In the return-maximization model, we formulate the robust problem as follows:

$$\begin{array}{lr} \max & \min_i (\mu^i)^T x \\ & x \in \mathcal{X} \\ & (\max_j x^T \Sigma^j x) \leq \sigma^2 \end{array}$$

which is equivalent to the following "deterministic" problem:

$$\begin{array}{ll} \max & t \\ & t \leq (\mu^i)^T x, i = 1, \ldots, I \\ & x \in \mathcal{X} \\ & x^T \Sigma^j x \leq \sigma^2, j = 1, \ldots, J \end{array}$$

Simple, intuitive notion of robustness. Leads to a convex optimization problem. Studied by the Imperial College team (Rustem, Settergren, Gülpınar, etc.)

Interval Uncertainty Sets

- T. and Koenig (AOR, 2004) use "box" type uncertainty sets that may be obtained from "confidence intervals" on the elements of the vector μ and the matrix Σ:

$$\mathcal{U} = \{(\mu, \Sigma) : \mu^L \leq \mu \leq \mu^U, \Sigma^L \leq \Sigma \leq \Sigma^U, \Sigma \text{ sym, pos. semidef.}\}.$$

  Their method can also handle ellipsoidal uncertainty sets as in Goldfarb/Iyengar, Ceria/Stubbs models.
- We considered several methods for generating this set. For example we used moving averages and bootstrapped averages from historical data. An alternative method would use a statistical procedure (perhaps confidence intervals) built on top of the particular alpha and risk model one might be using.
- Given a choice x for the decision variables, we are concerned about the worst-case realization of the data from the uncertainty set:

$$\min_{(\mu,\Sigma)\in\mathcal{U}} \mu^T x - \lambda x^T \Sigma x.$$

The Robust Formulation

Worst-case oriented robust optimization formulations seek to find the solution with the best worst-case guarantees:

$$\max_{x \in \mathcal{X}} \{ \min_{(\mu,\Sigma) \in \mathcal{U}} \mu^T x - \lambda x^T \Sigma x \}.$$

Using a saddle-point characterization of optimal solutions to this problem, an interior-point algorithm can be utilized to solve it.

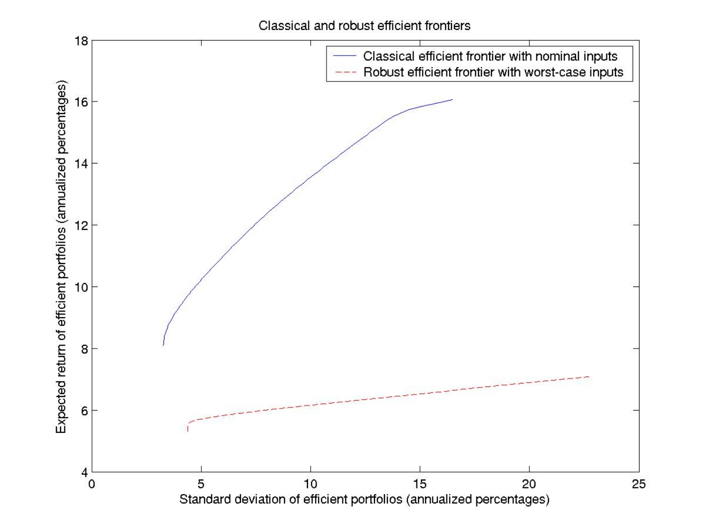
Numerical experiments

Asset classes:

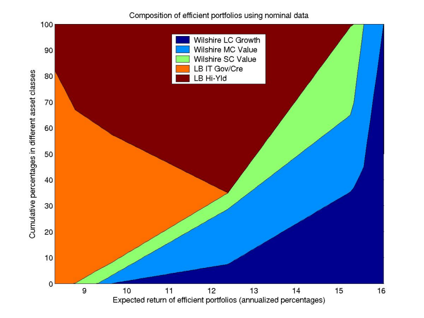
Large-cap growth	Large-cap value
Mid-cap growth	Mid-cap value
Small-cap growth	Small-cap value
International stocks	Real estate securities
Fixed income (govt)	Fixed income (hi yield)

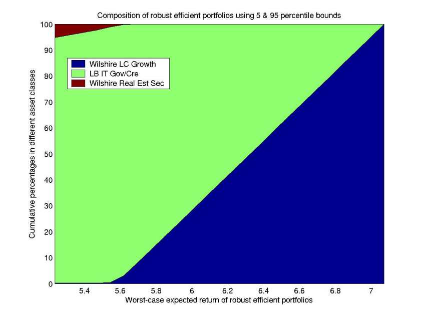
- Wilshire indices for equity and RE classes, LB for fixed income, MSCI EAFE for intl.
- Data: Monthly returns between July 1983 and July 2002
- Uncertainty set: 5 & 95 percentiles of 48-month moving averages

Nominal and robust efficient frontiers

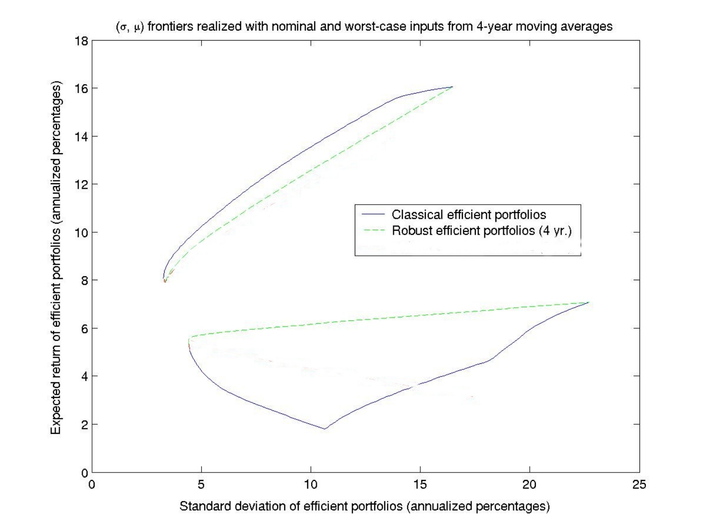


Compositions of nominal and robust efficient portfolios

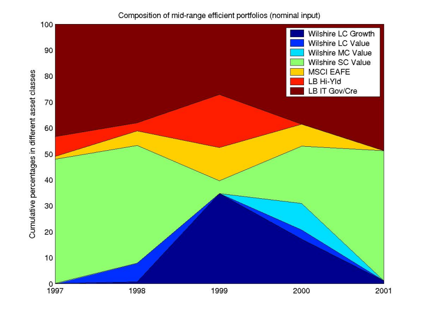


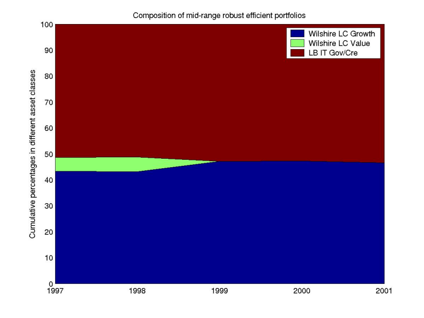


Performance comparison



Stability of robust optimal portfolios over time





Observations

- Robust efficient portfolios have significantly better worst-case behavior and are only slightly inefficient with "average" inputs.
- Robust efficient portfolios remain relevant for long periods: Good for buy-and-hold investors. Also low turnover $\longrightarrow$ low transaction costs
- Worst-case oriented, conservative models $\longrightarrow$ not for everyone
- Semidefiniteness constraints together with $\mathcal{O}(n^2)$ component-wise constraints limit the use of this model to smaller problems (asset allocation rather than portfolio selection).

Alternative Uncertainty Sets

Goldfarb and Iyengar (MOR, 2003) consider a factor model of returns:

$$r = \mu + V^T f + \varepsilon$$

where μ is the mean return, f is the (random) return vector for market factors, V is factor loading matrix, ε is the residual returns.
If these parameters are obtained from time series data via linear regression, confidence regions around the least squares estimates have the following structures:

$$S_v = \{V : V = V_0 + W, \|W_i\|_g \leq \rho_i, i = 1, \ldots, n\},$$

etc. where $\|u\|_g = \sqrt{u^T G u}$. These are ellipsoidal sets.
Such sets are attractive for uncertainty modeling because of their compact representation, nice fit with S-procedure, and intuitive construction.
Goldfarb and Iyengar obtain second-order cone problems from robust portfolio selection formulations with these uncertainty sets.

A Multi-Period Model

- Initial portfolio: $x^0 = (x_1^0, \ldots, x_n^0)$, x_i^0 number of shares of asset i, x_0^0 cash holdings. Time horizon: $l = 1, \ldots, L$.
- Decision variables: b_i^l (s_i^l): additional shares of asset i purchased (sold) at the beginning of period l.
- Parameters: P_i^l: the price of a share of asset i in period l. (wolog assume that $P_i^0 = 1, \forall i$.) Transaction costs: α_i^l and β_i^l for sales and purchases.
- Balance equations:

$$
\begin{gathered}
x_i^l = x_i^{l-1} - s_i^l + b_i^l, \quad i = 1, \ldots, n, \; l = 1, \ldots, L, \\
x_0^l \leq x_0^{l-1} + \textstyle\sum_{i=1}^n (1-\alpha_i) P_i^l s_i^l - \sum_{i=1}^n (1+\beta_i) P_i^l b_i^l, l = 1, \ldots, L.
\end{gathered}
$$

A Multi-Period Model

If we assume that all the future prices P_i^l are known at the time this investment problem is to be solved, we obtain the following deterministic optimization problem:

$$
\begin{array}{lrcl}
\max_{x,s,b,t} & t & & \\
& t & \leq & \sum_{i=0}^n P_i^L x_i^L \\
& x_0^l & \leq & x_0^{l-1} + \sum_{i=1}^n (1-\alpha_i) P_i^l s_i^l - \sum_{i=1}^n (1+\beta_i) P_i^l b_i^l, l = 1, \ldots \\
& x_i^l & = & x_i^{l-1} - s_i^l + b_i^l, \quad i = 1, \ldots, n, \ l = 1, \ldots, L \\
& s_i^l & \geq & 0, \quad i = 1, \ldots, n, \ l = 1, \ldots, L \\
& b_i^l & \geq & 0, \quad i = 1, \ldots, n, \ l = 1, \ldots, L \\
& x_i^l & \geq & 0, i = 0, \ldots, n, \ l = 1, \ldots, L.
\end{array}
$$

A simple linear programming problem.

Parameter Uncertainty

In a realistic setting, we do not know P_i^l's in advance and therefore can not solve the optimal portfolio allocation problem as the linear program we developed above.

Let us denote the expected value of the vector $P^l = \begin{bmatrix} P_1^l \\ \vdots \\ P_n^l \end{bmatrix}$ with

$\mu^l = \begin{bmatrix} \mu_1^l \\ \vdots \\ \mu_n^l \end{bmatrix}$ and its variance with V^l.

Consider the constraint:

$$t \le \sum_{i=0}^{n} P_i^L x_i^L.$$

Letting $x^L = (x_1^L, \ldots, x_n^L)$, the expected value and the standard deviation of the right-hand-side expression are given by $(\mu^L)^T x^L = \sum_{i=1}^n \mu_i^L x_i^L$ and $\sqrt{(x^L)^T V^L x^L}$.

A Constraint Robust Model

If P_i^L quantities are normally distributed, by requiring

$$t \leq E(RHS) - 3STD(RHS) = (\mu^L)^T x^L - 3\sqrt{(x^L)^T V^L x^L}$$

we would guarantee that the (random) inequality $t \leq \sum_{i=0}^n P_i^L x_i^L$ would be satisfied more than 99% of the time.

We regard this last inequality as the "robust" version of $t \leq \sum_{i=0}^n P_i^L x_i^L$.

The idea works also for the cash balance constraints.

This constraint robustness model corresponds to choosing the uncertainty sets for P^l as:

$$\mathcal{U}^l := \{P^l : \sqrt{(P^l - \mu^l)^T (V^l)^{-1} (P^l - \mu^l)} \leq 3\}, l = 1, \ldots, L$$

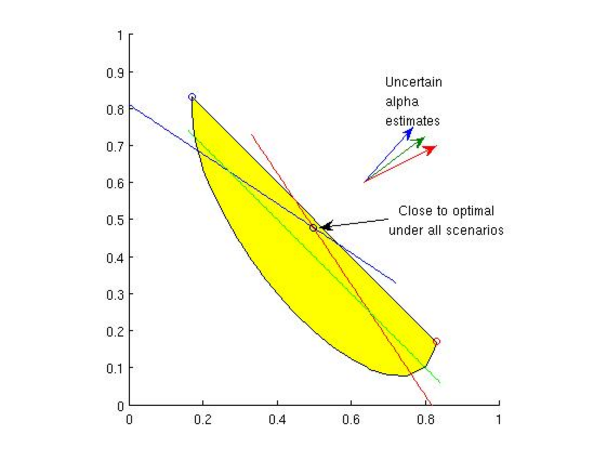
The resulting problem can be written as a second-order cone problem.

This model is based on the work of Ben-Tal, Margalit, and Nemirovski.

Relative Robustness in Portfolio Optimization

- The worst-case orientation in robust optimization and the conservative decisions that come with it are undesirable for most modelers. A more attractive model might measure robustness in a relative sense. This is especially useful when performance is measured relative to ones peers.
- For each scenario p for the uncertain parameters, one can consider a *regret* function that measures the difference between the performance of the solution with and without the benefit of hindsight.
- Then, we choose portfolios that minimize the maximum regret among all scenarios. These formulations are more difficult than usual robust formulations.
- *Relative robust* formulations for mean-variance optimization and Sharpe-ratio maximization problems with scenario based uncertainty structures lead to convex problems.

Relative Robustness: An illustration



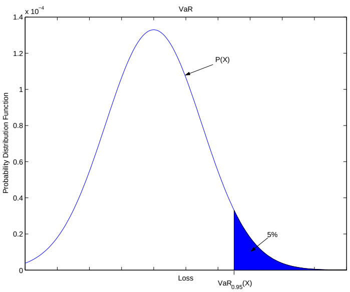
Outline

Value at Risk

- Financial activities involve risk. Financial institutions can and very often must manage risk using sophisticated mathematical techniques.
- Managing risk requires a good understanding of risk which comes from quantitative risk measures that adequately reflect the vulnerabilities of a company.
- Perhaps the best-known risk measure is Value-at-Risk (VaR) developed by financial engineers at J.P. Morgan.
- VaR is a measure related to percentiles of loss distributions and represents the predicted maximum loss with a specified probability level (e.g., 95%) over a certain period of time (e.g., one day).
- VaR suffers from the lack of subadditivity but is still widely used for risk management.

- Consider, for example, a random variable X that represents the loss from an investment portfolio over a fixed period of time. A negative value for X indicates gains.
- Given a probability level α, α-VaR of the random variable X is given by the following relation:

$$\mathrm{VaR}_\alpha(X) := \min\{\gamma : P(X \le \gamma) \ge \alpha\}.$$



Robust VaR

- Given a portfolio selection problem, if x represents the weight vector and r represents the random return vector, the total return of the portfolio $r(x) = r^T x$ is a random variable.
- To compute the VaR for the random variable $r(x)$, we need the joint density of the returns of all assets. This is often hard to obtain. Often, all we have are moment estimates, i.e., means and covariances.
- To determine the robust VaR, we ask the following question: Among all distributions for r with a fixed mean and covariance, which one gives the worst VaR for $r(x)$?
- The worst-case Value at Risk at level α:

$$\min\{\gamma : \inf_q P_q(X \leq \gamma) \geq \alpha\}$$

where the inf is taken over all distributions q with the fixed mean and covariance.

Robust VaR

- El Ghaoui et al. (OR, 2003) formulate this problem and then provide a solution.
- First, using Lagrangian duality on the space of probability distributions, they show that the condition "the worst-case VaR exceeds γ" can be written as two quadratic implications.
- Then, using the S-procedure, they rewrite these implications using semidefiniteness constraints. As a result the worst-case VaR can be computed via semidefinite optimization.
- Their approach is based on the moment bound study by Bertsimas and Popescu which we will see shortly.

Robust VaR

- El Ghaoui et al. also extend their results to the cases where the means and covariances are not fixed.
- They consider polytopic uncertainty, component-wise bounds, ellipsoidal uncertainty, factor models, etc. and obtain min-max formulations involving second-order cone or semidefiniteness constraints and/or plain semidefinite optimization problems.

Outline

Pricing/Hedging Problems

- When pricing securities with complicated payoff structures, one of the strategies analysts use is to develop a portfolio of "related" securities in order to form a super (or sub) hedge and then use no-arbitrage arguments to bound the price of the complicated security.
- Finding the super or sub hedge that gives the sharpest no-arbitrage bounds is formulated as an optimization problem.
- Or, as in the VaR problem above, we have some incomplete information about the underlying distribution and want to determine price bounds based on this information.
- Some of these problems can be addressed using semidefinite optimization thanks to a recent result characterizing polynomial inequalities using semidefiniteness restrictions.

A Useful Characterization

Theorem (Bertsimas and Popescu, OR 2002)

The polynomial $h(x) = \sum_{r=0}^{k} y_r x^r$ satisfies $h(x) \geq 0$ for all $x \in [a, b]$ if and only if there exists a positive semidefinite matrix $X = [x_{ij}]_{i,j=0,\ldots,k}$ such that

$$
\begin{aligned}
\sum_{i,j:i+j=2\ell-1} x_{ij} &= 0, \quad \ell = 1, \ldots, k, \\
\sum_{i,j:i+j=2\ell} x_{ij} &= \sum_{m=0}^{\ell} \sum_{r=m}^{k+m-\ell} y_r \binom{r}{m} \binom{k-r}{\ell-m} a^{r-m} b^m, \\
& \quad \ell = 0, \ldots, k.
\end{aligned}
$$

In other words: A univariate polynomial function remains nonnegative on an interval if and only if a specific square matrix constructed from its coefficients is positive semidefinite. $\rightarrow$ Perfect setting for semidefinite optimization.

Semi-parametric Bounds

- Consider a payoff function for a European option of an underlying security: $f(s)$. Let $\sigma \in \Re^m$ be the given moments of function $f^i(s), i = 1, \ldots, m$.
- Then the semi-parametric upper bound on the fair price of this security is determined by solving

$$\begin{array}{rl} \sup_P & E_P(f(s)) = \int_{\Re} f(s) dP(s) \\ \text{s.t.} & E_P(1) = 1, \\ & E_P(f^i(s)) = \sigma_i, i = 1, \ldots, m, \\ & P \text{ a probability distribution in } \Re_+. \end{array}$$

  And similarly for a lower bound.
- In essence, we are optimizing over all pricing (risk-neutral) measures that are consistent with the observed prices.

Bertsimas-Popescu Results

- Bertsimas and Popescu observed that when the derivative security is a European call option, the dual of the semi-parametric bound problem above has a linear objective and has polynomial constraints that must hold for all nonnegative arguments of the polynomial function.
- Using their structural result about non-negativity of polynomials, they formulate this dual as a semidefinite optimization problem.
- This approach remains valid for options with piecewise polynomial payoff functions.
- In some of the simpler cases, analytical solutions to these optimization problems can be derived.

Zuluaga-Peña Results

- Bertsimas and Popescu also tackle the case of multiple underlying assets but produce mostly negative results.
- In a related study Zuluaga and Peña cast the semi-parametric bound problem as a conic optimization problem where the relevant cones are the cone of moments and its dual, the cone of positive semidefinite polynomials.
- They also discuss the relaxations/approximations of these cones using the cone of semidefinite matrices which lead to computable bounds for the options under consideration.
- Their results include options with multiple underlying assets such as exchange options, max-cap options, rainbow options.

Recap

- Robust optimization models offer intuitive and useful approaches to interpret and manage uncertainty in parameters of optimization problems.
- Different interpretations of uncertainty lead to different optimization problems of varying difficulty. Many open problems waiting to be solved.
- With the inherent uncertainty in their model parameters, financial optimization problems are ideal settings for the use of robust optimization approaches.
- These models are useful and there is real interest in them. They are fun, interesting, and mathematically challenging.