Generating Functions for Lattice Points

With an integer vector $m = (\mu_1, \ldots, \mu_d)$, we associate a monomial $\mathbf{x}^m = x_1^{\mu_1} \cdots x_d^{\mu_d}$ in d complex variables. With a set $S \subset \mathbb{Z}^d$ of integer points we associate the *generating function*

$$f(S, \mathbf{x}) = \sum_{m \in S} \mathbf{x}^m,$$

which is a Laurent polynomial in $\mathbf{x}$ when S is finite or a Laurent series in $\mathbf{x}$ if S is infinite.

In many interesting cases, it turns out that the "long" polynomial or series $f(S, \mathbf{x})$ can be expressed as a "short" rational function. For example, if S is the set of integers in the interval $[0, n]$, then the "long" polynomial $f(S, x) = 1 + x + x^2 + \ldots + x^n$ can be written as a "short" rational function $(1 - x^{n+1})/(1 - x)$. This simple example admits some far-reaching extensions. In particular, if $S = P \cap \mathbb{Z}^d$ is the set of integer points in a given rational polyhedron $P \subset \mathbb{R}^d$, then, as long as the dimension d is fixed, the generating function $f(S, \mathbf{x})$ can be computed in time polynomial in the *input size* of P as a short rational function (the input size of P is the number of bits needed to write down the inequalities that define P). Similar phenomenon is observed for more complicated sets S, such as integer semigroups, test sets in integer programming, etc, and conjectured to hold for a yet wider class of sets described by formulas of Presburger arithmetic of a fixed combinatorial complexity.

In the course, I am planning to discuss the emerging "calculus" of short rational generating functions and how this approach is used to solve integer programming problems and count integer points. Some of the algorithms have been implemented by J. De Loera "LattE" project, see http://www.math.ucdavis.edu/$\sim$ latte/