# Hard Lefschetz Theorem for Vaisman manifolds

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# l.c.s. manifolds

An l.c.s. structure of the first kind on a manifold $M^{2n+2}$ is a couple $(\omega, \eta)$ of 1-forms such that:

(*i*) $\omega$ is closed;

(*ii*) the rank of $d\eta$ is $2n$ and $\omega \wedge \eta \wedge (d\eta)^n$ is a volume form.

The form $\omega$ is called the Lee 1-form while $\eta$ is said to be the anti-Lee 1-form.

If $(\omega, \eta)$ is an l.c.s. structure of the first kind on $M$ then there exists a unique pair of vector fields $(U, V)$, characterized by

$$\omega(U) = 1, \;\; \eta(U) = 0, \;\; i_U d\eta = 0,$$

$$\omega(V) = 0, \;\; \eta(V) = 1, \;\; i_V d\eta = 0.$$

# l.c.s. manifolds

Let $(\omega, \eta)$ be an l.c.s. structure of the first kind and consider the 2-form

$$\Omega := d\eta + \eta \wedge \omega.$$

Then $\Omega$ is non-degenerate and

$$\begin{aligned} d\Omega &= d(d\eta + \eta \wedge \omega) \\ &= d\eta \wedge \omega - 0 \qquad \text{since } \omega \text{ is closed} \\ &= \omega \wedge \Omega. \end{aligned}$$

Moreover, one has

$$\mathcal{L}_U \Omega = 0.$$

In other words, $\Omega$ is an l.c.s. structure of the first kind in the sense of Vaisman. The converse is also true.

# Vaisman manifolds

A Vaisman manifold is an l.c.s. manifold of the first kind $(M, \omega, \eta)$ which carries a Riemannian metric $g$ such that:

1. The tensor field $J$ of type $(1,1)$ given by

   $$g(X, JY) = \Omega(X, Y), \text{ for } X, Y \in \mathfrak{X}(M),$$

   is a complex structure compatible with $g$, that is,

   $$g(JX, JY) = g(X, Y).$$

   Then, one also says that $(M, J, g)$ is locally conformal Kähler.
2. The Lee 1-form $\omega$ is parallel, that is

   $$\nabla\omega = 0.$$

# A simple example of Vaisman manifold: $SU(2) \times S^1$

Let $X_1, X_2, X_3$ be a basis of left invariant vector fields on $SU(2)$, so that

$$[X_1, X_2] = 2X_3, \quad [X_2, X_3] = 2X_1, \quad [X_3, X_1] = 2X_2.$$

Given $c \in \mathbb{R} \setminus \{0\}$ one defines a Riemannian metric on $SU(2)$ by

$$g(X_1, X_1) = 1, \quad g(X_2, X_2) = g(X_3, X_3) = c^2.$$

With this structure $SU(2) \cong S^3$ is called a Berger sphere.
Now consider a non zero vector field $B$ tangent to $S^1$ and define an almost complex structure on $SU(2) \times S^1$ by

$$JB = X_1, \quad JX_1 = -B, \quad JX_2 = X_3, \quad JX_3 = -X_2.$$

Then, one can check that $J$ is integrable and the Lee vector field is

$$U = -\frac{2}{c^2} B$$

and it is parallel, that is $\nabla U = 0$.

# Properties of Vaisman manifolds

In a Vaisman manifold $M$,

- the couple $(U, V)$ defines a flat foliation of rank 2 which is transversely Kähler;
- the foliation generated by $V$ is transversely co-Kähler;
- the orthogonal bundle to the foliation generated by $U$ is integrable and the leaves are $c$-Sasakian manifolds.

## $c$-Sasakian manifolds

Let $(N^{2n+1}, g)$ be a Riemannian manifold, $\eta$ a 1-form, such that

$$\eta \wedge (\mathrm{d}\eta)^n \quad \text{is a volume form.}$$

Fix $c > 0$, define $\varphi : TN \to TN$ by

$$\mathrm{d}\eta(X, Y) = 2cg(X, \varphi Y), \qquad \text{for any } X, Y \in \Gamma(TN).$$

Let $\xi \in \Gamma(TN)$ be the metric dual of $\eta$ and assume that $\eta(\xi) = 1$. Moreover, suppose that

$$\varphi^2 = -Id + \eta \otimes \xi$$

and the Nijenhuis torsion of $\varphi$ satisfies

$$N_\varphi + 2d\eta \otimes \xi = 0.$$

Then, $(N^{2n+1}, \eta, g)$ is called a $c$-Sasakian manifold.

# Mapping torus

Consider a compact manifold $N$, a diffeomorphism $f : N \longrightarrow N$ and $\alpha > 0$. Define a transformation of $N \times \mathbb{R}$ by

$$(f, T_\alpha)(x, t) = (f(x), t + \alpha).$$

The map $(f, T_\alpha)$ induces an action of $\mathbb{Z}$ on $N \times \mathbb{R}$ defined by

$$(f, T_\alpha)^k(x, t) = (f^k(x), t + k\alpha), \qquad \text{for } k \in \mathbb{Z}.$$

*The mapping torus* of $N$ by $(f, \alpha)$ is the space of orbits

$$N_{f,\alpha} = \frac{N \times \mathbb{R}}{\mathbb{Z}}$$

and we have a canonical projection

$$\pi : N_{f,\alpha} \longrightarrow S^1 = \frac{\mathbb{R}}{\alpha\mathbb{Z}}.$$

# Mapping torus by an isometry

We will denote by $\theta$ the closed 1-form on $N_{f,\alpha}$ given by

$$\theta = \pi^*(\theta_{S^1}),$$

where $\theta_{S^1}$ is the length element of the circle $S^1$.
Then, the vector field $U$ on $N_{f,\alpha}$ induced by $\frac{\partial}{\partial t}$ on $N \times \mathbb{R}$ satisfies

$$\theta(U) = 1.$$

Now, suppose that $h$ is a Riemannian metric on $N$ and that $f$ is an isometry. Then, the metric $h + dt^2$ on $N \times \mathbb{R}$ is $\mathbb{Z}$-invariant and hence induces a metric $g$ on $N_{f,\alpha}$.

## Proposition

*The* 1*-form* $\theta$ *on* $N_{f,\alpha}$ *is unitary and parallel with respect to* $g$ *and*

$$\theta(X) = g(X, U), \quad \text{for } X \in \mathfrak{X}(N_{f,\alpha}).$$

# Properties of Vaisman manifolds

Before we have seen that there is a the close relation between Vaisman manifolds and Sasakian manifolds. In fact,

**Theorem (Ornea-Verbitsky, 2003)**

*Let $M$ be a compact Vaisman manifold of dimension $2n+2$. Then, there exists a compact Sasakian manifold $N$ of dimension $2n+1$, a contact isometry $f : N \longrightarrow N$ and a positive real number $\alpha$ such that $M$ is holomorphically isometric to $N_{f,\alpha} = \dfrac{N \times \mathbb{R}}{\mathbb{Z}}$.*

# Kähler manifolds

Let $(M^{2n}, g)$ be a Riemannian manifold, $\Omega$ a 2-form such that

$$\Omega^n \quad \text{is a volume form,} \qquad d\Omega = 0.$$

Define $J : TM \to TM$ by

$$\Omega(X, Y) = g(X, JY), \qquad \text{for any } X, Y \in \Gamma(TM).$$

Now assume that $J$ is a complex structure on M.
Then, $(M^{2n}, \Omega, g)$ is called a Kähler manifold.
In other words a Kähler manifold is a Vaisman manifold with $\omega = 0$.

# Hard Lefschetz Theorem for Kähler manifolds

**Theorem**

*Let $(M^{2n}, \Omega, g)$ be a compact Kähler manifold and $p \leq n$. Then, the maps*

$$\begin{aligned} H^p(M) &\to H^{2n-p}(M) \\ [\,\alpha\,] &\mapsto [\,\Omega^{n-p} \wedge \alpha\,], \end{aligned}$$

*are isomorphisms.*

# Hard Lefschetz Theorem for Sasakian manifolds

**Theorem (B. Cappelletti-Montano, A.D.N., I. Yudin, 2015)**

*Let $(M^{2n+1}, \eta, g)$ be a compact Sasakian manifold and $p \leq n$. Let $\mathcal{H}\colon \Omega^p(M) \to \Omega^p_\Delta(M)$ be the projection on the harmonic part. Then the map*

$$\begin{aligned} Lef_p\colon H^p(M) &\longrightarrow H^{2n+1-p}(M) \\ [\,\alpha\,] &\longmapsto [\,\eta \wedge (d\eta)^{n-p} \wedge \mathcal{H}\alpha\,], \end{aligned}$$

*is an isomorphism. Furthermore, it does not depend on the choice of the Sasakian metric $g$ on $(M^{2n+1}, \eta)$.*

So, a natural question arise: is there a Hard Lefschetz theorem for a compact Vaisman manifold?
We give a positive answer to this question.

# Hard Lefschetz Theorem for Vaisman manifolds

## Theorem

*Let $M^{2n+2}$ be a compact Vaisman manifold. Then for each $k$, $0 \leq k \leq n$, there exists an isomorphism*

$$Lef_k : H^k(M) \longrightarrow H^{2n+2-k}(M)$$

*which may be computed by using the following properties:*

(1) *For every $[\gamma] \in H^k(M)$, there is $\bar{\gamma} \in [\gamma]$ such that*

$$\mathcal{L}_U \bar{\gamma} = 0, \quad i_V \bar{\gamma} = 0, \quad L^{n-k+2}\bar{\gamma} = 0, \quad L^{n-k+1}\epsilon_\omega \bar{\gamma} = 0.$$

(2) *If $\bar{\gamma} \in [\gamma]$ satisfies the conditions in (1) then*

$$Lef_k[\gamma] = [\epsilon_\eta L^{n-k}(L i_U \bar{\gamma} - \epsilon_\omega \bar{\gamma})].$$

In this theorem, we use the notation $\epsilon_\beta = \beta \wedge -$ and $L = \frac{1}{2} d\eta \wedge -$.

# Auxiliary Theorem

In order to prove the theorem, we used as a first step a result which relates the de Rham cohomology with the basic cohomology.

## Theorem

*Let $W$ be a unitary and parallel vector field on an oriented compact Riemannian manifold $(P, g)$ and let the 1-form $w$ be the metric dual of $W$. Denote by $H_B^*(P)$ the basic cohomology of $P$ with respect to $W$. Then for $0 \leq k \leq \dim P$, the map*

$$H_B^k(P) \oplus H_B^{k-1}(P) \longrightarrow H^k(P)$$

*defined by*

$$([\beta]_B, [\beta']_B) \mapsto [\beta + w \wedge \beta']$$

*is an isomorphism.*

# Basic Hard Lefschetz Theorem

## Theorem

*Let $M$ be a compact Vaisman manifold of dimension $2n+2$. Denote by $H^*_B(M)$ the basic cohomology of $M$ with respect to $U$. Then for each $k$, $0 \leq k \leq n$, there exists an isomorphism*

$$Lef^B_k : H^k_B(M) \longrightarrow H^{2n+1-k}_B(M)$$

*which may be computed by using the following properties:*

(1) *For every $[\beta]_B \in H^k_B(M)$, there is $\beta' \in [\beta]_B$ such that*

$$i_V \beta' = 0, \quad L^{n-k+1}\beta' = 0. \tag{1}$$

(2) *If $\beta' \in [\beta]_B$ satisfies the conditions in (1) then*

$$Lef^B_k[\beta]_B = [\epsilon_\eta L^{n-k}\beta]_B.$$

# A topological obstruction

For a Vaisman manifold $M^{2n+2}$ the couple $(\omega, \eta)$ of the Lee and anti-Lee 1-forms defines a locally conformal symplectic (l.c.s.) structure of the first kind.
Now, assume that we have a compact manifold $M^{2n+2}$ with an l.c.s. structure of the first kind $(\omega, \eta)$.
Then, we introduce the following *Lefschetz relation* between the cohomology groups $H^k(M)$ and $H^{2n+2-k}(M)$ , for $0 \leq k \leq n$,

$$R_{Lef_k} = \Big\{([\gamma], [\epsilon_\eta L^{n-k}(Li_U\gamma - \epsilon_\omega\gamma)])\Big|\, \gamma \in \Omega^k(M),\ d\gamma = 0,$$
$$\mathcal{L}_U\gamma = 0,\ i_V\gamma = 0,\ L^{n-k+2}\gamma = 0,\ L^{n-k+1}\epsilon_\omega\gamma = 0\Big\}.$$

# A topological obstruction

Similarly, we can define the *basic Lefschetz relation* between the basic cohomology groups $H_B^k(M)$ and $H_B^{2n+1-k}(M)$, for $0 \leq k \leq n$, by

$$R^B_{Lef_k} = \Big\{ ([\beta]_B, [\epsilon_\eta L^{n-k}\beta]_B) \,\Big|\, \beta \in \Omega^k_B(M),\ d\beta = 0, \\ i_V\beta = 0,\ L^{n-k+1}\beta = 0 \Big\}.$$

# A topological obstruction

## Definition

An l.c.s. structure of the first kind on a manifold $M^{2n+1-k}$ is said to be:

- *Lefschetz* if, for every $0 \leq k \leq n$, the relation $R_{Lef_k}$ is the graph of an isomorphism $Lef_k : H^k(M) \longrightarrow H^{2n+2-k}(M)$;
- *Basic Lefschetz* if, for every $0 \leq k \leq n$, the relation $R^B_{Lef_k}$ is the graph of an isomorphism $Lef^B_k : H^k_B(M) \longrightarrow H^{2n+1-k}_B(M)$.

# Hard Lefschetz vs basic Hard Lefschetz

**Theorem**

*Let $(M^{2n+2}, \omega, \eta)$ be a compact l.c.s. manifold of the first kind such that the Lee vector field $U$ is parallel with respect to a Riemannian metric $g$ on $M$ and $\omega$ is the metric dual of $U$. Then:*

(1) *The structure $(\omega, \eta)$ is Lefschetz if and only if it is basic Lefschetz.*

(2) *If the structure $(\omega, \eta)$ is Lefschetz (or, equivalently, basic Lefschetz), then for each $1 \leq k \leq n$ there exists a non-degenerate bilinear form*

$$\psi : H_B^k(M) \times H_B^k(M) \longrightarrow \mathbb{R}$$

$$\psi([\beta]_B, [\beta']_B) = \int_M Lef_k[\beta] \cup [\beta']$$

*which is skew-symmetric for odd $k$ and symmetric for even $k$.*

# Betti numbers of Lefschetz l.c.s. manifolds

From the above theorem we get that when $k$ is odd, $H_B^k(M)$ must be of even dimension, that is

$$b_k^B(M) \quad \text{is even, if } k \text{ is odd and } 1 \leq k \leq n,$$

where $b_k^B(M)$ is the $k$th basic Betti number of $M$.
But from our auxiliary theorem we also have that

$$H^k(M) \cong H_B^k(M) \oplus H_B^{k-1}(M).$$

Hence

$$\begin{aligned} b_k - b_{k-1} &= b_k^B + b_{k-1}^B - (b_{k-1}^B + b_{k-2}^B) \\ &= b_k^B - b_{k-2}^B. \end{aligned}$$

Thus

$$b_k(M) - b_{k-1}(M) \quad \text{is even.}$$

# Betti numbers of Lefschetz l.c.s. manifolds

In conclusion we get

## Corollary

*A compact Lefschetz l.c.s. manifold of the first kind $M^{2n+2}$ with parallel Lee vector field with respect to some metric $g$ has*

$$b_k(M) - b_{k-1}(M) \text{ even, if } k \text{ is odd and } 1 \leq k \leq n,$$

*where $b_k(M)$ is the $k\mathrm{th}$ Betti number of $M$.*

In particular

$$b_1(M) \text{ is odd.}$$

We remark that the above properties of the Betti numbers are well-known when the manifold is Vaisman.

# Hard Lefschetz vs basic Hard Lefschetz

**Corollary**

*Let $M^{2n+2}$ be a compact l.c.s. manifold of the first kind such that the space of orbits of the Lee vector field is a contact manifold $N^{2n+1}$. Then, the following conditions are equivalent:*

1. *The l.c.s. structure on $M$ satisfies the Lefschetz property.*
2. *The l.c.s. structure on $M$ satisfies the basic Lefschetz property.*
3. *The contact structure on $N$ satisfies the Lefschetz property.*

Now, let $N$ be a compact contact manifold and consider in the product manifold $M = N \times S^1$ the standard l.c.s. structure of the first kind. Conversely, one has that the space of orbits of the Lee vector field of $M$ is $N$.

# Application: a non-Lefschetz l.c.s. manifold

In 2014, we found examples of non-Lefschetz compact contact manifolds with even Betti numbers $b_{2k+1}$, for $1 \leq 2k+1 \leq n$. Using the above Corollary and taking as $N$ one of these examples, we obtain examples of compact l.c.s. manifolds of the first kind such that

1. Their Betti numbers satisfy the relations
$$b_k(M) - b_{k-1}(M) \quad \text{is even, if } k \text{ is odd and } 1 \leq k \leq n,$$
just as in Vaisman manifolds.
2. They do not satisfy Lefschetz property neither basic Lefschetz property (and, therefore, they do not admit compatible Vaisman metrics).

# A non-Vaisman Lefschetz l.c.s. manifold

On the other hand, in a recent preprint (arXiv:1507.04661), we presented an example of a compact Lefschetz contact manifold $N$ which does not admit any Sasakian structure.

Now, consider $M = N \times S^1$ with the standard l.c.s. structure of the first kind. We get that $M$ is Lefschetz and basic Lefschetz. However, it does not admit any compatible Vaisman metric.

Indeed, recall that for a Vaisman manifold $M$, the distribution orthogonal to the Lee vector field $U$ is integrable and the leaves admit a Sasakian metric, as we recalled at the beginning.

# A Lefschetz non-Sasakian contact manifold

For each $p \neq 0$, a Lefschetz non-Sasakian contact manifold $N_p$ is obtained as follows: consider the Lie group of dimension 5 given as the semi-direct product

$$G(p) = (H(1,1) \rtimes_\psi \mathbb{R}) \rtimes_\phi \mathbb{R},$$

where $\psi : \mathbb{R} \to Aut(H(1,1))$ and $\phi : \mathbb{R} \to Aut(H(1,1) \rtimes_\psi \mathbb{R}u)$ are the representations defined by

$$\psi_u(x,y,z) = (e^{pu}x, e^{-pu}y, z), \quad \phi_t(x,y,z,u) = (x,y,z+tu,u).$$

Then, one proves that there is a discrete subgroup $\Gamma(p)$ such that $N_p := G(p)/\Gamma(p)$ is a compact $K$-contact solvmanifold with no Sasakian structure. Moreover, $N_p$ is formal and of Tievsky type.

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Thank you!