Geometry and topology of 3-quasi-Sasakian manifolds

Antonio De Nicola

joint work with B. Cappelletti Montano (Univ. Cagliari) and I. Yudin (CMUC)

CMUC, Department of Mathematics,
University of Coimbra

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Almost contact manifolds

- An **almost contact manifold** \((M, \phi, \xi, \eta)\) is an odd-dimensional manifold \(M\) which carries a \((1, 1)\)-tensor field \(\phi\), a vector field \(\xi\), a 1-form \(\eta\), satisfying

\[
\phi^2 = -I + \eta \otimes \xi \quad \text{and} \quad \eta(\xi) = 1.
\]

- It follows that

\[
\phi \xi = 0 \quad \text{and} \quad \eta \circ \phi = 0.
\]

- An almost contact manifold manifold of dimension \(2n + 1\) is said to be a **contact manifold** if

\[
\eta \wedge (d\eta)^n \neq 0.
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An almost contact manifold \((M, \phi, \xi, \eta)\) is said to be \textbf{normal} if
\[
[\phi, \phi] + 2d\eta \otimes \xi = 0.
\]

\(M\) is normal iff the almost complex structure \(J\) on the product \(M \times \mathbb{R}\) defined by setting, for any \(X \in \Gamma(TM)\) and \(f \in C^\infty(M \times \mathbb{R})\),
\[
J \left( X, f \frac{d}{dt} \right) = \left( \phi X - f \xi, \eta(X) \frac{d}{dt} \right)
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is integrable.
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Almost contact metric manifolds

- Every almost contact manifold admits a compatible metric $g$, i.e. such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all $X, Y \in \Gamma(TM)$.

- By putting $\mathcal{H} = \ker(\eta)$ one obtains a $2n$-dim. distribution on $M$ and $TM$ splits as the orthogonal sum

$$TM = \mathcal{H} \oplus \langle \xi \rangle.$$
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  \[
  TM = \mathcal{H} \oplus \langle \xi \rangle.
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A quasi-Sasakian structure on a \((2n + 1)\)-dimensional manifold \(M\) is a normal almost contact metric structure \((\phi, \xi, \eta, g)\) such that \(d\Phi = 0\), where \(\Phi\) is defined by

\[
\Phi(X, Y) = g(X, \phi Y).
\]

They were introduced by Blair in 1967 in the attempt to unify Sasakian geometry \((d\eta = \Phi)\) and cosymplectic geometry \((d\eta = 0, \ d\Phi = 0)\).

A quasi-Sasakian manifold is said to be of rank \(2p + 1\) if

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for some \(p \leq n\).
Quasi-Sasakian manifolds

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3-quasi-Sasakian manifolds

Definition

A 3-quasi-Sasakian manifold is given by a \((4n + 3)\)-dimensional manifold \(M\) endowed with three quasi-Sasakian structures \((\phi_1, \xi_1, \eta_1, g), (\phi_2, \xi_2, \eta_2, g), (\phi_3, \xi_3, \eta_3, g)\) satisfying the following relations, for any even permutation \((\alpha, \beta, \gamma)\) of \(\{1, 2, 3\}\),

\[
\begin{align*}
\phi_\gamma &= \phi_\alpha \phi_\beta - \eta_\beta \otimes \xi_\alpha, \\
\xi_\gamma &= \phi_\alpha \xi_\beta, \\
\eta_\gamma &= \eta_\alpha \circ \phi_\beta.
\end{align*}
\]

(For odd permutations, there is a change of signs).

The class of 3-quasi-Sasakian manifolds \((d\Phi_\alpha = 0)\) includes as special cases the 3-cosymplectic manifolds \((d\eta_\alpha = 0, d\Phi_\alpha = 0)\), and the 3-Sasakian manifolds \((d\eta_\alpha = \Phi_\alpha)\).
A **3-quasi-Sasakian manifold** is given by a $(4n + 3)$-dimensional manifold $M$ endowed with three quasi-Sasakian structures $(\phi_1, \xi_1, \eta_1, g), (\phi_2, \xi_2, \eta_2, g), (\phi_3, \xi_3, \eta_3, g)$ satisfying the following relations, for any even permutation $(\alpha, \beta, \gamma)$ of $\{1, 2, 3\}$,

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\phi_\gamma = \phi_\alpha \phi_\beta - \eta_\beta \otimes \xi_\alpha,
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The canonical foliation of a 3-quasi-Sasakian manifold

Theorem

Let \((M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)\) be a 3-quasi-Sasakian manifold. Then the 3-dimensional distribution \(V := \langle \xi_1, \xi_2, \xi_3 \rangle\) is integrable. Moreover, it defines a totally geodesic and Riemannian foliation.

- The distribution \(\mathcal{H} := \bigcap_{\alpha=1}^{3} \ker (\eta_\alpha)\) has dimension 4\(n\), and \(TM\) splits as the orthogonal sum

\[
TM = \mathcal{H} \oplus V.
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Structure of the leaves of $\mathcal{V}$

**Theorem**

Let $(M, \phi_{\alpha}, \xi_{\alpha}, \eta_{\alpha}, g)$ be a 3-quasi-Sasakian manifold. Then, for any even permutation $(\alpha, \beta, \gamma)$ of $\{1, 2, 3\}$ and for some $c \in \mathbb{R}$

$$[\xi_\alpha, \xi_\beta] = c\xi_\gamma.$$ 

So we can divide 3-quasi-Sasakian manifolds in two main classes according to the behaviour of the leaves of $\mathcal{V}$: those 3-quasi-Sasakian manifolds for which each leaf of $\mathcal{V}$ is locally $SO(3)$ (or $SU(2)$) (which corresponds to take in the above theorem the constant $c \neq 0$), and those for which each leaf of $\mathcal{V}$ is locally an abelian group (the case $c = 0$).
In a 3-quasi-Sasakian manifold one has, in principle, the three odd ranks $r_1, r_2, r_3$ of the 1-forms $\eta_1, \eta_2, \eta_3$, since we have three distinct, although related, quasi-Sasakian structures. We prove that these ranks coincide and their value has great influence on the geometry of the manifold.
The rank of a 3-quasi-Sasakian manifold

Theorem

Let \((M^{4n+3}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)\) be a 3-quasi-Sasakian manifold. Then the 1-forms \(\eta_1, \eta_2\) and \(\eta_3\) have all the same rank \(4l + 3\), for some \(l \leq n\), or rank 1, according to \([\xi_\alpha, \xi_\beta] = c\xi_\gamma\) with \(c \neq 0\), or \([\xi_\alpha, \xi_\beta] = 0\), respectively.

The above theorem allows to define the rank of a 3-quasi-Sasakian manifold \((M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)\) as the rank shared by the 1-forms \(\eta_1, \eta_2\) and \(\eta_3\).

Theorem

Every 3-quasi-Sasakian manifold of rank 1 is 3-cosymplectic.

Theorem

Every 3-quasi-Sasakian manifold of maximal rank is 3-\(\alpha\)-Sasakian.
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Toward a decomposition theorem

Besides the vertical distribution $\mathcal{V}$ we proved that the following two fundamental distributions are Riemannian and totally geodesic.

- $\mathcal{E}^{4m} := \{ X \in \mathcal{H} \mid i_X d\eta_\alpha = 0, \text{ for } \alpha = 1, 2, 3 \}$,
- $\mathcal{E}^{4l+3} := \mathcal{E}^{4l} \oplus \mathcal{V}$,

where $\mathcal{E}^{4l}$ is the orthogonal complement of $\mathcal{E}^{4m}$ in $\mathcal{H}$. 
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3-quasi-Sasakian manifolds of rank $4l + 3$

The following decomposition theorem holds.

**Theorem**

Let $(M^{4n+3}, \phi_{\alpha}, \xi_{\alpha}, \eta_{\alpha}, g)$ be a 3-quasi-Sasakian manifold of rank $4l + 3$ with $[\xi_{\alpha}, \xi_{\beta}] = 2\xi_{\gamma}$. Then $M^{4n+3}$ is locally the Riemannian product of a 3-Sasakian manifold $M^{4l+3}$ and a hyper-Kähler manifold $M^{4m}$, with $m = n - l$. 
Nontrivial examples of 3–quasi-Sasakian manifolds

Example

Let $M$ be a compact Riemannian manifold and $G$ a finite group freely acting on $M$. Then from the Hodge theory we can obtain

$$H^* (M/G) \cong H^* (M)^G.$$ 

Now, let $M$ and $N$ are two compact manifolds with $G$-action. Then $G$ acts on the product $M \times N$ and we get

$$H^k (M \times N)^G = \bigoplus_{q+p=k} (H^q (M) \otimes H^p (N))^G,$$

since $H^q (M) \otimes H^p (N)$ are $G$-invariant subspaces.
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Example (continued)

Now, take \( M = S^{4n-1} \subset \mathbb{H}^n \) and \( N = \mathbb{T}^4 = \mathbb{H}/\mathbb{Z}^4 \).

Let \( \mathbb{Z}_4 \) (the cyclic group of order 4) act on \( S^{4n-1} \) by

\[ \sigma \cdot (q_1, \ldots, q_n) = (i q_1, \ldots, i q_n), \]

and on \( \mathbb{T}^4 \) by

\[ \sigma \cdot [q] = [i q]. \]

We get

\[ H^k \left( S^{4n-1} \otimes \mathbb{T}^4 \right) \mathbb{Z}_4 = H^k \left( \mathbb{T}^4 \right) \mathbb{Z}_4 \oplus H^{k-4n+1} \left( \mathbb{T}^4 \right) \mathbb{Z}_4. \]

It follows that the Poincaré polynomial of \( (S^{4n-1} \times \mathbb{T}^4) / \mathbb{Z}_4 \) is

\[ (1 + t^{4n-1}) (1 + 4 t^2 + t^4). \]

Thus, \( (S^{4n-1} \times \mathbb{T}^4) / \mathbb{Z}_4 \) cannot be a product of 3-Sasakian and hyper-Kähler manifolds.
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  and on \( T^4 \) by
- \( \sigma \cdot [q] = [iq] \).

We get

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H^k \left( S^{4n-1} \otimes T^4 \right)^{\mathbb{Z}_4} = H^k \left( T^4 \right)^{\mathbb{Z}_4} \oplus H^{k-4n+1} \left( T^4 \right)^{\mathbb{Z}_4}.
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Thus, \((S^{4n-1} \times T^4)/\mathbb{Z}_4\) cannot be a product of 3-Sasakian and hyper-Kähler manifolds.
A contact circle on $M^3$ is a pair of contact forms $(\eta_1, \eta_2)$ such that for any $(\lambda_1, \lambda_2) \in S^1$ the 1-form $\lambda_1 \eta_1 + \lambda_2 \eta_2$ is also a contact form.

A contact p-sphere on $M^{2n+1}$ is given by $(\eta_1, \ldots, \eta_{p+1})$ such that for any $(\lambda_1, \ldots, \lambda_{p+1}) \in S^p$, the 1-form $\lambda_1 \eta_1 + \ldots + \lambda_{p+1} \eta_{p+1}$ is also a contact form.

Theorem (Zessin, 2005)

Any 3-Sasakian manifold $M^{4n+3}$ admits a 2-sphere of contact structures (which is both round and taut).
Contact circles and contact spheres

(Geiges-Gonzalo, 1995)

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Theorem (Zessin, 2005)

Any 3-Sasakian manifold $M^{4n+3}$ admits a 2-sphere of contact structures (which is both round and taut).
A contact sphere is said to be *taut* if all contact forms belonging to the sphere define the same volume form.

A contact sphere is said to be *round* if for any \((\lambda_1, \ldots, \lambda_{p+1}) \in S^p\), the Reeb vector field of

\[
\eta = \sum_{h=1}^{p+1} \lambda_h \eta_h \quad \text{is} \quad \xi = \sum_{h=1}^{p+1} \lambda_h \xi_h.
\]

Zessin showed that: taut \(\iff\) round in dimension 3.
Almost contact spheres

**Definition**

Let \((\phi_1, \xi_1, \eta_1), \ldots, (\phi_{p+1}, \xi_{p+1}, \eta_{p+1})\) be almost contact structures on \(M\). We say that they define an *almost contact sphere* if for any \((\lambda_1, \ldots, \lambda_{p+1}) \in S^p\) the tensors

\[
\phi := \sum_{h=1}^{p+1} \lambda_h \phi_h,
\]

\[
\xi := \sum_{h=1}^{p+1} \lambda_h \xi_h,
\]

\[
\eta := \sum_{h=1}^{p+1} \lambda_h \eta_h,
\]

define an almost contact structure on \(M\).
Almost contact spheres

Theorem

Let \((\phi_\alpha, \xi_\alpha, \eta_\alpha)\) be an almost contact metric 3-structure on \(M\). Then \(M\) carries an almost contact 2-sphere \((\phi, \xi, \eta)\) given by

\[
\begin{align*}
\phi &:= \lambda_1 \phi_1 + \lambda_2 \phi_2 + \lambda_3 \phi_3, \\
\xi &:= \lambda_1 \xi_1 + \lambda_2 \xi_2 + \lambda_3 \xi_3, \\
\eta &:= \lambda_1 \eta_1 + \lambda_2 \eta_2 + \lambda_3 \eta_3,
\end{align*}
\]

where \((\lambda_1, \lambda_2, \lambda_3) \in \mathbb{S}^2\). Furthermore, the Riemannian metric \(g\) is compatible with \((\phi, \xi, \eta)\), and if \((\phi_\alpha, \xi_\alpha, \eta_\alpha)\) is hyper-normal, then \((\phi, \xi, \eta, g)\) is a normal almost contact metric structure on \(M\).
Corollary

A 3-quasi-Sasakian manifold of rank $4l + 3$ $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ defines a 2-sphere of quasi-Sasakian structures $(\phi, \xi, \eta, g)$ of the same rank (which is both round and taut).

In particular:

Corollary

Any 3-Sasakian manifold admits a contact 2-sphere of Sasakian structures (which is both round and taut).
Sasakian spheres

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Topology of 3-quasi-Sasakian manifolds

3-quasi-Sasakian manifolds

\[
\begin{cases}
3\text{-Sasakian manifolds: top rank } 4n+3 \\
3\text{-quasi-Sasakian manifolds of intermediate ranks } 4l + 3, \ 1 \leq l < n \\
3\text{-cosymplectic manifolds: minimum rank } 1
\end{cases}
\]
Main Results on the Betti numbers:

**Theorem (Fujitani, 1966)**

*In any compact Sasakian manifold $M^{2n+1}$, the odd Betti numbers $b_{2k+1}$ are even, for $2k + 1 < n$.***

**Theorem (Galicki-Salamon, 1996)**

*In any compact 3-Sasakian manifold $M^{4n+3}$, the odd Betti numbers $b_{2k+1}$ are zero, for each $k < n$.***
I - Topology of 3-Sasakian manifolds

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In any compact Sasakian manifold $M^{2n+1}$, the odd Betti numbers $b_{2k+1}$ are even, for $2k + 1 < n$.

**Theorem (Galicki-Salamon, 1996)**

In any compact 3-Sasakian manifold $M^{4n+3}$, the odd Betti numbers $b_{2k+1}$ are zero, for each $k < n$.
Theorem (Chinea, de León, Marrero, 1993)

Let $M^{2n+1}$ be a compact cosymplectic manifold. Then,

(i) $b_0 \leq b_1 \leq \ldots \leq b_n$.

(ii) $b_{2p+1} - b_{2p}$ is even, for each $p \leq n$. In particular $b_1$ is odd.

They also proved a version of the strong Lefschetz property.
II - Topology of 3-cosymplectic manifolds

**Definition**

\[ b^h_p := \dim \{ \omega \in \Omega^p(M) \mid \omega \text{ is harmonic, } i_{\xi_\alpha} \omega = 0, \alpha = 1, 2, 3 \} \]

**Theorem**

Let \( M^{4n+3} \) be a compact 3-cosymplectic manifold. Then, for each integer \( p \) such that \( 0 \leq p \leq 2n - 1 \),

(i) \( b^h_{2p+1} \) is divisible by four.

(ii) \( b_p = b^h_p + 3b^h_{p-1} + 3b^h_{p-2} + b^h_{p-3} \).

**Corollary**

For each integer \( p \) such that \( 0 \leq p \leq 2n - 1 \),

\[ b_{2p} + b_{2p+1} = 4k, \text{ for some } k \in \mathbb{N}. \]
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III - Topology of 3-quasi-Sasakian manifolds

We introduce the operators

\[ \theta_{\alpha}X := \begin{cases} 0, & \text{if } X \in \Gamma(E^{4l+3}) \\ \phi_{\alpha}X, & \text{if } X \in \Gamma(E^{4m}) \end{cases} \]

and the associated 2-forms \( \Theta_{\alpha} := g(\cdot, \theta_{\alpha}\cdot) \).

In any 3-quasi-Sasakian manifold each \( \Theta_{\alpha} \) is closed. The fact that the 2-forms \( \Theta_{\alpha} \) are also coclosed follows from the following lemma.

**Lemma**

*In any 3-quasi-Sasakian manifold \( M^{4n+3} \) of non-maximal rank \( 4l + 3 \) one has*

\[ \nabla \Theta_{\alpha} = 0. \]
Then, the following lower bound on the Betti numbers follows.

**Theorem**

In any compact 3-quasi-Sasakian manifold $M^{4n+3}$ of non-maximal rank $4l + 3$, one has the inequality

$$b_{2k} \geq \binom{k + 2}{2} \quad \text{for} \quad 0 \leq k \leq n - l$$

**Corollary**

The sphere $S^{4n+3}$ does not admit any 3-quasi-Sasakian structure of non-maximal rank.
III - Topology of 3-quasi-Sasakian manifolds

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**Corollary**

*The sphere $S^{4n+3}$ does not admit any 3-quasi-Sasakian structure of non-maximal rank.*
III - Topology of 3–quasi-Sasakian manifolds

Stronger bounds on the Betti numbers of compact 3-quasi-Sasakian manifolds are obtained after recognising that there is a decomposition of the space of harmonic forms

\[ \Omega^k_{\Delta}(M) = \bigoplus_{s+t=k} \Omega_{\Delta}^{s,t}(M), \]

where

\[ \Omega_{\Delta}^{s,t}(M) := \{ \omega \in \Omega_{\Delta}^{s+t}(M) \mid i_P \omega = s \omega \}, \]

and \( P \) is the projection on the 3-\( \alpha \)-Sasakian part. Then, an action of \( \text{so}(4,1) \) on \( \bigoplus_{t=0}^{4m} \Omega_{\Delta}^{s,t}(M) \) is found and one can prove the following result.

**Theorem**

In any compact 3-quasi-Sasakian manifold \( M^{4n+3} \) of rank \( 4l + 3 \), the odd Betti numbers \( b_{2k+1} \) are divisible by 4, for each \( k < l \).
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