Reduction of Poisson-Nijenhuis Lie algebroids

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work in progress with Juan Carlos Marrero and Edith Padrón**

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Poisson PT@Porto, 15 May 2010
Aim of our study:

- Given a Poisson-Nijenhuis Lie algebroid \((A, P, N)\) we want to reduce it to a symplectic-Nijenhuis Lie algebroid \((\tilde{A}, \tilde{\Omega}, \tilde{N})\) with \(\tilde{\Omega}\) symplectic and also \(\tilde{N}\) nondegenerate.
Motivation

But first a preliminary question:

- Why study Poisson-Nijenhuis Lie algebroids?
Poisson-Nijenhuis manifolds (briefly)

**Ingredients:** $M$ manifold, $\Lambda$ bivector field and $N$ $(1,1)$-tensor on $M$

- $\Lambda$ is a Poisson structure, i.e., $[\Lambda, \Lambda] = 0$
- $N$ is Nijenhuis operator, i.e., $T_N = 0$
- $\Lambda$ and $N$ satisfy the compatibility conditions

$$N \circ \Lambda^\# = \Lambda^\# \circ N^*, \quad C(\Lambda, N) = 0$$

where $\Lambda^\# : T^*M \to TM, \quad \Lambda^\#(\alpha) = i_\alpha \Lambda$

\[ (M, \Lambda, N) \text{ Poisson-Nijenhuis manifold} \]
Poisson-Nijenhuis manifolds

\((M, \Lambda, N)\) Poisson-Nijenhuis manifold \(\Rightarrow \Lambda^\#_i = N\Lambda^\#_{i-1}\)

Poisson structures \(\Lambda_i, \Lambda_j\) are compatible

Particular case:

Bi-hamiltonian manifold \((M, \Lambda_0, \Lambda_1)\) with \(\Lambda_0\) symplectic structure

\[\downarrow\]

\((M, \Lambda_0, N = \Lambda_1^\# \circ (\Lambda_0^\#)^{-1})\) Poisson-Nijenhuis manifold

\[\downarrow\]

\(X_1 = \Lambda_1^\#(dH_0) = \Lambda_0^\#(dH_1)\) bi-Hamiltonian vector field

\[\downarrow\]

\(X_i = N^{i-1}X_1\) sequence of bi-Hamiltonian v. fields
A simple Example: Toda lattice (for two particles)

$\mathbb{R}^4$ with coordinates $(q^1, q^2, p_1, p_2)$

$$H_1 = \frac{1}{2}(p_1^2 + p_2^2) + e^{q^1 - q^2}$$

**Poisson structures**

$$\Lambda_0 = \frac{\partial}{\partial q^1} \wedge \frac{\partial}{\partial p_1} + \frac{\partial}{\partial q^2} \wedge \frac{\partial}{\partial p_2}$$

$$\Lambda_1 = -\frac{\partial}{\partial q^1} \wedge \frac{\partial}{\partial q^2} + p_1 \frac{\partial}{\partial q^1} \wedge \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial q^2} \wedge \frac{\partial}{\partial p_2} + e^{q^1 - q^2} \frac{\partial}{\partial p_2} \wedge \frac{\partial}{\partial p_1}$$

$$N = \Lambda_1^\# \circ (\Lambda_0^\#)^{-1}, \quad (\mathbb{R}^4, \Lambda_0, N) \text{ PN-manifold}$$

$$X_1 = \Lambda_0^\#(dH_1) = \Lambda_1^\#(dH_0), \quad H_0 = p_1 + p_2$$
Toda lattice in Flaschka coordinates

\[ \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}^4, \quad (t, (q^1, q^2, p_1, p_2)) \to (q^1 + t, q^2 + t, p_1, p_2) \]

\[ \mathbb{R}^4 / \mathbb{R} \cong (\mathbb{R}^+) \times \mathbb{R}^2 \]

\[ [(q^1, q^2, p_1, p_2)] \to (e^{q_1 - q_2}, p_1, p_2) \]

\[ \pi : \mathbb{R}^4 \to \mathbb{R}^+ \times \mathbb{R}^2, \quad (q^1, q^2, p_1, p_2) \to (e^{q_1 - q_2}, p_1, p_2) \]

\((a, b_1, b_2)\) the coordinates on the reduced space \(\mathbb{R}^+ \times \mathbb{R}^2\)
Overview

Example

Short review of Lie alg.

Poisson-Nijenhuis Lie alg.

Reduction of PN Lie alg.

The Reduced nondeg. SN Lie alg.

Toda lattice in Flaschka coordinates

Poisson reduced structures

\[ \tilde{\Lambda}_0 = a \frac{\partial}{\partial a} \wedge \left( \frac{\partial}{\partial b_1} - \frac{\partial}{\partial b_2} \right) \]

\[ \tilde{\Lambda}_1 = a \frac{\partial}{\partial a} \wedge \left( b_1 \frac{\partial}{\partial b_1} - b_2 \frac{\partial}{\partial b_2} \right) - a \frac{\partial}{\partial b_1} \wedge \frac{\partial}{\partial b_2} \]

\[ \tilde{H}_1 = \frac{1}{2}(b_1^2 + b_2^2) + a, \quad \tilde{H}_0 = b_1 + b_2 \]

\[ \tilde{X}_1 = \tilde{\Lambda}_0^\#(d\tilde{H}_1) = \tilde{\Lambda}_1^\#(d\tilde{H}_0) \]

\[ \nexists \tilde{N} \text{ such that } \tilde{\Lambda}_1^\# = \tilde{N} \circ \tilde{\Lambda}_0^\# \]
What happens?

The answer is in the theory of Poisson-Nijenhuis Lie algebroids
Lie algebroids

**Definition (Pradines)**

A **Lie algebroid** is a vector bundle $\tau_A : A \to M$ endowed with

(i) an **anchor**, i.e., a vector bundle morphism $\rho_A : A \to TM$

(ii) a Lie algebra bracket on $\Gamma(A), [ , ]_A$, such that

$$[X, fY]_A = f [X, Y]_A + \rho_A(X)(f)Y,$$

for all $X, Y \in \Gamma(A), f \in C^\infty(M)$. 
Lie algebroids

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It follows that $\rho_A([X,Y]_A) = [\rho_A(X), \rho_A(Y)]$. 

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Examples of Lie algebroids

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<th>Example</th>
<th>Description</th>
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<td><strong>The tangent bundle</strong></td>
<td>((A = TM, \rho_A = id_{TM}, [ , ]))</td>
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<td><strong>An involutive distribution</strong></td>
<td>((A = D \subset TM, \rho_A = \iota_D, [ , ]))</td>
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<td><strong>A Lie algebra</strong></td>
<td>((A = \mathfrak{g}, \rho_A = 0, [ , ]_{\mathfrak{g}}))</td>
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Examples of Lie algebroids

The Atiyah algebroid

\[ \pi : M \to M/G \text{ principal } G\text{-bundle} \]

- \[ A = TM/G \to M/G \text{ sections are } G\text{-invariant vector fields} \]
- \[ \rho_A([v]) = T\pi(v) \text{ induced projection map} \]
- \[ [, ]_A = \text{ bracket of } G\text{-invariant vector fields} \]
Cartan calculus

Associated to a given Lie algebroid \((A, [ , ]_A, \rho_A)\) there is a \textit{Lie algebroid differential} \(d^A : \Gamma(\wedge^\bullet A^*) \rightarrow \Gamma(\wedge^{\bullet+1} A^*)\) defined by

\[
(d^A\omega)(X_0, \ldots, X_k) := \sum_{i=0}^k (-1)^i \rho_A(X_i) \left(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_k)\right)
+ \sum_{0 \leq i < j \leq k} (-1)^{i+j} \omega([X_i, X_j]_A, X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k),
\]

for \(\omega \in \Gamma(\wedge^k A^*), X_0, \ldots, X_k \in \Gamma(A)\).

For \(X \in \Gamma(A)\),

\[
\mathcal{L}_X^A := i_X d^A + d^A i_X
\]
Cartan calculus

Associated to a given Lie algebroid \((A, [\ , \ ]_A, \rho_A)\) there is a *Lie algebroid differential* \(d^A : \Gamma(\bigwedge \bullet A^*) \to \Gamma(\bigwedge \bullet A^* + 1)\) defined by

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- For \(X \in \Gamma(A)\),

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\]
Properties of the Lie algebroid differential

- $d^A$ is a graded derivation of degree 1, i.e.,
  
  $d^A(\theta \wedge \omega) = d^A \theta \wedge \omega + (-1)^{\deg(\theta)} \theta \wedge d^A \omega$,

- $d^A \circ d^A = 0$. 
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- $d^A \circ d^A = 0$. 
Morphisms of Lie algebroids

Let \((A, [ , ]_A, \rho_A)\) and \((A', [ , ]_{A'}, \rho_{A'})\) be Lie algebroids. A bundle map

\[
\begin{array}{ccc}
A & \xrightarrow{F} & A' \\
\downarrow \tau_A & & \downarrow \tau_{A'} \\
M & \xrightarrow{f} & M'
\end{array}
\]

is called a **morphism of Lie algebroids** from \(A\) to \(A'\), if

\[
d^A(F^*\alpha') = F^*(d^{A'}\alpha') \quad \text{for all } \alpha' \in \Gamma(\wedge^k A'^*).
\]
Schouten-Gerstenhaber algebra

The Lie algebra bracket on $\Gamma(A)$ can be extended to the exterior algebra $(\Gamma(\wedge \cdot A), \wedge)$. For $X \in \Gamma(A)$ and $P \in \Gamma(\wedge^p A)$,

$$[X, P]_A (\alpha_1, \ldots, \alpha_p) = \rho_A(X)(P(\alpha_1, \ldots, \alpha_p))$$

$$- \sum_{i=1}^p P(\alpha_1, \ldots, \mathcal{L}_X^A \alpha_i, \ldots \alpha_p),$$

If $P \in \Gamma(\wedge^p A)$, $Q \in \Gamma(\wedge^q A)$ and $R \in \Gamma(\wedge^r A)$, then $[P, Q]_A \in \Gamma(\wedge^{p+q-1} A)$ and

- $[P, Q]_A = -(-1)^{(p-1)(q-1)} [Q, P]_A$

- $[P, Q \wedge R]_A = [P, Q]_A \wedge R + (-1)^{(p-1)q} Q \wedge [P, R]_A$

- $(-1)^{(p-1)(r-1)} [P, [Q, R]_A]_A + \text{cyclic perm.} = 0$
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Poisson structures on Lie algebroids

Let $A$ be a Lie algebroid and $P$ a section of the vector bundle $\wedge^2 A \to M$. We denote by $P^\#$ the usual bundle map

$$P^\# : A^* \longrightarrow A : \alpha \longmapsto P^\#(\alpha) = i_\alpha P.$$ 

**Definition**

A **Poisson structure** on $A$ is a section $P \in \Gamma(\wedge^2 A)$, such that

$$[P, P]_A = 0.$$ 

In this case, the bracket

$$[\alpha, \beta]_P := \mathcal{L}^A_{P^\#\alpha} \beta - \mathcal{L}^A_{P^\#\beta} \alpha - d^A(P(\alpha, \beta)), \quad \alpha, \beta \in \Gamma(A^*),$$

is a Lie bracket and $A^*_P = (A^*, [\ , \ ]_P, \rho_A \circ P^\#)$ is a Lie algebroid.
Let $A$ be a Lie algebroid and $P$ a section of the vector bundle $\wedge^2 A \to M$. We denote by $P^\sharp$ the usual bundle map

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Nijenhuis operators

Let $(A, [\ , \ ], \rho_A)$ be a Lie algebroid and $N : A \to A$ a bundle map. The torsion of $N$ is defined by

$$\mathcal{T}_N(X, Y) := [NX, NY]_A - N[X, Y]_N, \quad X, Y \in \Gamma(A),$$

where

$$[X, Y]_N := [NX, Y]_A + [X, NY]_A - N[X, Y]_A, \quad X, Y \in \Gamma(A).$$

When $\mathcal{T}_N = 0$, $N$ is called a Nijenhuis operator, $A_N = (A, [\ , \ ]_N, \rho_N = \rho_A \circ N)$ is a new Lie algebroid and $N : A_N \to A$ is a Lie algebroid morphism.
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\[
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is a Lie algebroid morphism.
On a Lie algebroid $A$ with a Poisson structure $P \in \Gamma(\wedge^2 A)$, we say that a bundle map $N : A \to A$ is compatible with $P$ if

(i) $NP^\sharp = P^\sharp N^*$,

(ii) $[\alpha, \beta]_{NP} - [\alpha, \beta]_P^{N^*} = 0$.

**Definition (Grabowski-Urbanski)**

A Poisson-Nijenhuis Lie algebroid $(A, P, N)$ is a Lie algebroid $A$ equipped with a Poisson structure $P$ and a Nijenhuis operator $N : A \to A$ compatible with $P$. 
Poisson-Nijenhuis Lie algebroids

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Toda lattice in Flaschka coordinates

\[ \mathbb{R}^+ \times \mathbb{R}^2 \text{ with coordinates } (a, b_1, b_2) \]

\[ \pi : \mathbb{R}^4 \rightarrow \mathbb{R}^+ \times \mathbb{R}^2 \]

Poisson reduced structures on \( \mathbb{R}^+ \times \mathbb{R}^2 \)

\[ \bar{\Lambda}_0 = a \frac{\partial}{\partial a} \wedge \left( \frac{\partial}{\partial b_1} - \frac{\partial}{\partial b_2} \right) \]

\[ \bar{\Lambda}_1 = a \frac{\partial}{\partial a} \wedge \left( b_1 \frac{\partial}{\partial b_1} - b_2 \frac{\partial}{\partial b_2} \right) - a \frac{\partial}{\partial b_1} \wedge \frac{\partial}{\partial b_2} \]
Toda lattice in Flaschka coordinates

\[ T(\mathbb{R}^+ \times \mathbb{R}^2) \to \mathbb{R}^+ \times \mathbb{R}^2, \ (\cdot, \cdot, Id) \]

The Lie algebroid

\[ A = \mathbb{R} \times T(\mathbb{R}^+ \times \mathbb{R}^2) \to \mathbb{R}^+ \times \mathbb{R}^2 \]

\{e_0 = (1, 0), e_1 = (0, \frac{\partial}{\partial a}), e_2 = (0, \frac{\partial}{\partial b_1}), e_3 = (0, \frac{\partial}{\partial b_2})\} \]

\[[e_i, e_j]_A = 0, \ \rho(e_0) = 0, \ \rho(e_1) = \frac{\partial}{\partial a}, \ \rho(e_2) = \frac{\partial}{\partial b_1}, \ \rho(e_3) = \frac{\partial}{\partial b_2} \]

Two Poisson structures on the Lie algebroid \( A = \mathbb{R} \times T(\mathbb{R}^+ \times \mathbb{R}^2) \)

\[ P_0 = ae_1 \wedge (e_2 - e_3) + e_0 \wedge e_3 \]

\[ P_1 = ae_0 \wedge e_1 + ae_1(b_1 e_2 - b_2 e_3) + ae_2 \wedge e_3 + b_2 e_0 \wedge e_3 \]

These Poisson structures induce \( \tilde{\lambda}_0, \tilde{\lambda}_1 \) on \( \mathbb{R}^+ \times \mathbb{R}^2 \).
Toda lattice in Flaschka coordinates

The Nijenhuis operator $N$

$$N = P_1^{\#} \circ (P_0^{\#})^{-1} : A \to A$$

The Poisson-Nijenhuis Lie algebroid

$$(A = \mathbb{R} \times T(\mathbb{R}^+ \times \mathbb{R}^2) \to \mathbb{R}^+ \times \mathbb{R}^2, P_0, N)$$

$$P_0^{\#} d^A \bar{H}_1 = P_1^{\#} d^A \bar{H}_0$$

$$\bar{\Lambda}_0^{\#} d \bar{H}_1 = \rho_A(P_0^{\#} d^A \bar{H}_1) = \rho_A(P_1^{\#} d^A \bar{H}_0) = \bar{\Lambda}_1^{\#} d \bar{H}_0$$

Poisson-Nijenhuis Lie algebroids

General case

\( \pi : M \to M/G \) principal bundle

\((M, P, N)\) PN-manifold

\( P, N \) \( G \)-invariants

\( \tilde{\pi} : TM/G \to M/G \)

Atiyah algebroid

\((\tilde{P}, \tilde{N})\) \( PN \)-Lie algebroid

\( M/G \) is not, in general,

PN-manifold!!
Reduction of PN Lie algebroids

\[(A, P, N)\]

Poisson-Nijenhuis Lie algebroid

\[\Downarrow \text{Reduction by restriction}\]

\[(\tilde{A}, \tilde{P}, \tilde{N})\]

symplectic-Nijenhuis Lie algebroid

\[\Downarrow \text{Reduction by projection}\]

\[(\tilde{A}, \tilde{P}, \tilde{N})\]

symplectic-Nijenhuis Lie algebroid with \(\tilde{N}\) nondegenerate

i.e. \(\tilde{P}^\# : \tilde{A}^* \to \tilde{A}\), \(\tilde{N} : \tilde{A} \to \tilde{A}\) isomorphisms
1\textsuperscript{st} step: Reduction by restriction

\[(A, [ , ]_A, \rho_A, P, N)\] Poisson-Nijenhuis Lie algebroid on \(M\).

Distribution \(D \subset TM, D(x) := \rho_A(P^\#(A_x^*)) \subset T_xM\) for \(x \in M\)

\[
\left[\rho_A(P^\#\alpha), \rho_A(P^\#\beta)\right] = \rho_A(P^\#[\alpha, \beta]_P) + D
\]

\(D\) is locally finitely generated

\[
\downarrow
\]

\(D\) is a generalized foliation of \(M\) in the sense of Sussmann.
1st step: Reduction by restriction

\((A, [\ , \ ]_A, \rho_A, P, N)\)  Poisson-Nijenhuis Lie algebroid on \(M\).

Distribution \(D \subset TM\), \(D(x) := \rho_A(P^\#(A^*_x)) \subset T_x M\) for \(x \in M\)

\[
\left[ \rho_A(P^\#\alpha), \rho_A(P^\#\beta) \right] = \rho_A(P^\# [\alpha, \beta]_P) + D
\]

\(D\) is locally finitely generated

\(\downarrow\)

\(D\) is a generalized foliation of \(M\) in the sense of Sussmann.
1\textsuperscript{st} step: Reduction by restriction

\[(A, [\ , \ ]_A, \rho_A, P, N)\] Poisson-Nijenhuis Lie algebroid on \(M\).

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\]

\(D\) is locally finitely generated

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\(D\) is a generalized foliation of \(M\) in the sense of Sussmann.
$1^{st}$ step: Reduction by restriction

- Let $L \subset M$ be a leaf of the foliation $D = \rho_A(P^\#(A^*)) \subset TM$
- Assume: $P^\# : A^* \to A$ has constant rank on each leaf $L$.

\[ A_L := P^\#(A^*)|_L \subset A \to L \text{ is a Lie algebroid} \]

- \[ [P^\#\alpha|_L, P^\#\beta|_L]_{A_L} = P^\# [\alpha, \beta]|_L \in \Gamma(A_L) \]
- \[ \rho_{A_L} = (\rho_A)|_{A_L} : A_L \to TL \]
Furthermore, the inclusion maps

\[ A_L \rightarrow A \]

\[ \downarrow (\tau_A)|_{A_L} \quad \downarrow \tau_A \]

\[ L \rightarrow M \]

\[ \downarrow \downarrow \]

give a morphism of Lie algebroids, i.e.

\[ A_L \rightarrow L \] is a Lie subalgebroid of \( A \rightarrow M \).
1\textsuperscript{st} step: Reduction by restriction

\[
\begin{array}{c}
L \xrightarrow{X_L} A_L \\
\downarrow \iota \downarrow \downarrow \\
M \xrightarrow{P^\# \alpha} A
\end{array}
\]

\[\alpha \in \Gamma(A^*)\]

The symplectic structure \(\Omega_L : L \to \wedge^2 A_L^*\)

\[\Omega(X_L, Y_L) = P(\alpha, \beta) \circ \iota\]

Nijenhuis tensor \(N_L : A_L \to A_L\)

\[I \circ N_L(X_L) = N(P^\# \alpha) \circ \iota\]
1st step: Reduction by restriction

**Theorem 1**

Let \( (A, P, N) \) be a Poisson-Nijenhuis Lie algebroid such that the Poisson structure has constant rank in the leaves of the foliation \( D = \rho_A(P^\#(A^*)) \). Then, we have a symplectic-Nijenhuis Lie algebroid \( (A_L, \Omega_L, N_L) \) on each leaf \( L \) of \( D \).
2\textsuperscript{nd} step: Reduction by projection

**Lie algebroid epimorphism**

Let $\tau_A : A \to M$ and $\tau_{\tilde{A}} : \tilde{A} \to \tilde{M}$ be Lie algebroids

\[
\begin{array}{ccc}
A & \xrightarrow{\Pi} & \tilde{A} \\
\downarrow{\tau_A} & & \downarrow{\tau_{\tilde{A}}} \\
M & \xrightarrow{\pi} & \tilde{M}
\end{array}
\]

- $(\Pi, \pi)$ epimorphism of vector bundles
- $d^A(\Pi^*\tilde{\alpha}) = \Pi^*(d^{\tilde{A}}\tilde{\alpha})$ for all $\tilde{\alpha} \in \Gamma(\wedge^k\tilde{A}^*)$ and all $k$
Projectability

- **Π-projectable 1-section:** \( X \in \Gamma(A) \) such that there exists \( \tilde{X} \in \Gamma(\tilde{A}) \) with \( \Pi \circ X = \tilde{X} \circ \pi \).

- **Π-projectable 2-section:** \( P \in \Gamma(\wedge^2 A) \) such that for all \( \tilde{\alpha} \in \Gamma(\tilde{A}^*) \) the 1-section \( P^\#(\Pi^*\tilde{\alpha}) \in \Gamma(A) \) is Π-projectable

\[ \downarrow \]

\( \tilde{P} \in \Gamma(\wedge^2 \tilde{A}), \quad (\tilde{P}^\#\tilde{\alpha}) \circ \pi = \Pi(P^\#(\Pi^*\tilde{\alpha})). \)

- **Π-projectable (1,1)-section:** \( N: A \to A \) vector bundle morphism such that

\[ N(\Gamma_p(A)) \subseteq \Gamma_p(A) \quad \text{and} \quad N(\Gamma(\text{Ker} \Pi)) \subseteq \Gamma(\text{Ker} \Pi) \]

\[ \downarrow \]

\( \tilde{N} : \tilde{A} \to \tilde{A}, \quad (\tilde{N}\tilde{X}) \circ \pi = \Pi(NX). \)
Projectability

- **Π-projectable 1-section**: \( X \in \Gamma(A) \) such that there exists \( \tilde{X} \in \Gamma(\tilde{A}) \) with \( \Pi \circ X = \tilde{X} \circ \pi \).

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  \[
  \tilde{P} \in \Gamma(\wedge^2 \tilde{A}), \quad (\tilde{P}^\# \tilde{\alpha}) \circ \pi = \Pi(P^\#(\Pi^*\tilde{\alpha})).
  \]

- **Π-projectable (1, 1)-section**: \( N: A \rightarrow A \) vector bundle morphism such that
  
  \[
  N(\Gamma_p(A)) \subseteq \Gamma_p(A) \quad \text{and} \quad N(\Gamma(\text{Ker}\Pi)) \subseteq \Gamma(\text{Ker}\Pi)
  \]
  
  \[
  \tilde{N}: \tilde{A} \rightarrow \tilde{A}, \quad (\tilde{N}\tilde{X}) \circ \pi = \Pi(NX).
  \]
Projectability

- **Π-projectable 1-section**: $X \in \Gamma(A)$ such that there exists $\tilde{X} \in \Gamma(\tilde{A})$ with $\Pi \circ X = \tilde{X} \circ \pi$.

- **Π-projectable 2-section**: $P \in \Gamma(\wedge^2 A)$ such that for all $\tilde{\alpha} \in \Gamma(\tilde{A}^\ast)$ the 1-section $P^\#(\Pi^* \tilde{\alpha}) \in \Gamma(A)$ is Π-projectable.

- **Π-projectable (1,1)-section**: $N: A \to A$ vector bundle morphism such that

  \[
  N(\Gamma_p(A)) \subseteq \Gamma_p(A) \quad \text{and} \quad N(\Gamma(Ker\Pi)) \subseteq \Gamma(Ker\Pi)
  \]

  \[
  \tilde{N} : \tilde{A} \to \tilde{A}, \quad (\tilde{N}\tilde{X}) \circ \pi = \Pi(NX).
  \]
**Theorem**

*Let* $(\Pi, \pi) : A \to \tilde{A}$ *be a Lie algebroid epimorphism. Assume that* $(P, N)$ *is a Poisson-Nijenhuis structure on* $A$ *such that* $P$ *and* $N$ *are* $\Pi$*-projectable. Then, $(\tilde{P}, \tilde{N})$ *is a Poisson-Nijenhuis structure on* $\tilde{A}$. *
Complete and vertical lifts

- \((A, [, ]_A, \rho_A)\) a Lie algebroid
- \(X \in \Gamma(A)\)

The vertical lift of \(X\): \(X^v \in \mathfrak{X}(A)\)

\((i)\) \(X^v(f \circ \tau_A) = 0,\; f \in C^\infty(M)\),
\((ii)\) \(X^v(\hat{\alpha}) = \alpha(X) \circ \tau_A,\; \alpha \in \Gamma(A^*)\).

Here, if \(\alpha \in \Gamma(A^*)\) then \(\hat{\alpha}: A \rightarrow \mathbb{R}\) is defined by

\[\hat{\alpha}(a) = \alpha(\tau_A(a))(a),\; \text{for all } a \in A.\]
Complete and vertical lifts

- $(A, [ , ]_A, \rho_A)$ a Lie algebroid
- $X \in \Gamma(A)$

**The vertical lift of $X$: $X^v \in \mathfrak{x}(A)$**

1. $X^v(f \circ \tau_A) = 0$, $f \in C^\infty(M)$,
2. $X^v(\hat{\alpha}) = \alpha(X) \circ \tau_A$, $\alpha \in \Gamma(A^*)$.

**The complete lift of $X$: $X^c \in \mathfrak{x}(A)$**

1. $X^c(f \circ \tau_A) = \rho_A(X)(f) \circ \tau_A$, $f \in C^\infty(M)$,
2. $X^c(\hat{\alpha}) = \mathcal{L}^A_X\alpha$, $\alpha \in \Gamma(A^*)$.

Here, if $\alpha \in \Gamma(A^*)$ then $\hat{\alpha}: A \to \mathbb{R}$ is defined by

$$\hat{\alpha}(a) = \alpha(\tau_A(a))(a), \quad \text{for all } a \in A.$$
Let $\tau_A : A \to M$ a vector bundle and $(A, [\cdot, \cdot]_A, \rho_A)$ a Lie algebroid. Consider a Lie subalgebroid $\tau_B : B \to M$ of $A$.

**Key Fact**

The distributions $\rho_A(B)$ and $\mathcal{F}$ defined by

$$\mathcal{F}_a := \{X^c(a) + Y^v(a) \mid X, Y \in \Gamma(B)\} \subseteq T_aA, \quad \text{for all } a \in A$$

are generalized foliations.

Now assume that

(i) $\rho_A(B)$ and $\mathcal{F}$ are regular foliations;
(ii) For all $x \in M$, $a_x, a'_x \in L_\mathcal{F} \implies a_x - a'_x \in B_x$. 
Reduction by lifts of sections of a Lie subalgebroid

Let \( \tau_A : A \to M \) a vector bundle and \((A, [, ]_A, \rho_A)\) a Lie algebroid. Consider a Lie subalgebroid \( \tau_B : B \to M \) of \( A \).

**Key Fact**

The distributions \( \rho_A(B) \) and \( F \) defined by

\[
F_a := \{ X^c(a) + Y^v(a) \mid X, Y \in \Gamma(B) \} \subseteq T_aA, \quad \text{for all } a \in A
\]

are generalized foliations.

Now assume that

(i) \( \rho_A(B) \) and \( F \) are regular foliations;
(ii) For all \( x \in M, a_x, a'_x \in L_F \implies a_x - a'_x \in B_x \).
Reduction by lifts of sections of a Lie subalgebroid

Let $\tau_A : A \to M$ a vector bundle and $(A, [\cdot, \cdot]_A, \rho_A)$ a Lie algebroid. Consider a Lie subalgebroid $\tau_B : B \to M$ of $A$.

**Key Fact**

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(i) $\rho_A(B)$ and $\mathcal{F}$ are regular foliations;

(ii) For all $x \in M$, $a_x, a'_x \in L_{\mathcal{F}} \implies a_x - a'_x \in B_x$. 
We define \( \tilde{\tau}_A : \tilde{A} = A/F \to \tilde{M} = M/\rho_A(B) \) such that the following diagram is commutative

\[
\begin{align*}
A & \xrightarrow{\Pi} \tilde{A} = A/F \\
\downarrow \tau_A & \quad \downarrow \tilde{\tau}_A \\
M & \xrightarrow{\pi} \tilde{M} = M/\rho_A(B)
\end{align*}
\]

**Proposition**

In the above conditions we can define a Lie algebroid structure on \( \tilde{\tau}_A : \tilde{A} = A/F \to \tilde{M} = M/\rho_A(B) \) such that the above diagram is an epimorphism of Lie algebroids.
Reduction by lifts of sections of a Lie subalgebroid

We define $\tau_{\tilde{A}}: \tilde{A} = A/F \rightarrow \tilde{M} = M/\rho_A(B)$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
A & \xrightarrow{\Pi} & \tilde{A} = A/F \\
\downarrow{\tau_A} & & \downarrow{\tau_{\tilde{A}}} \\
M & \xrightarrow{\pi} & \tilde{M} = M/\rho_A(B)
\end{array}
\]

Proposition

In the above conditions we can define a Lie algebroid structure on $\tau_{\tilde{A}}: \tilde{A} = A/F \rightarrow \tilde{M} = M/\rho_A(B)$ such that the above diagram is an epimorphism of Lie algebroids.
The Riesz index

Let \((A, P, N)\) a Poisson-Nijenhuis Lie algebroid. For any \(x \in M\) consider the map \(N_x: A_x \to A_x\). Recall that there exists a smallest integer \(k > 0\) such that the sequences

\[
\text{Im } N_x \supseteq \text{Im } N_x^2 \supseteq \ldots
\]

and

\[
\text{ker } N_x \subseteq \text{ker } N_x^2 \subseteq \ldots
\]

both stabilize at rank \(k\). That is,

\[
\text{Im } N_x^k = \text{Im } N_x^{k+1} = \ldots, \quad \text{while } \text{Im } N_x^{k-1} \neq \text{Im } N_x^k,
\]

and

\[
\text{ker } N_x^k = \text{ker } N_x^{k+1} = \ldots, \quad \text{while } \text{ker } N_x^{k-1} \neq \text{ker } N_x^k.
\]

The integer \(k\) is called the Riesz index of \(N\) at \(x\).
The Reduced nondegenerate SN Lie algebroid

**Theorem 2**

Let $(A \to M, [, ]_A, \rho_A, \Omega, N)$ be a symplectic-Nijenhuis Lie algebroid such that

1) $N$ has constant Riesz index $k$.

2) $\rho_A(B)$ and $\mathcal{F}$ are regular foliations for $B = \ker N^k$.

3) For all $x \in M$, $a_x, a'_x \in L_{\mathcal{F}} \implies a_x - a'_x \in \ker(N^k_x)$.

Then, we can induce a symplectic-Nijenhuis Lie algebroid structure $([, ]_{\tilde{A}}, \rho_{\tilde{A}}, \tilde{\Omega}, \tilde{N})$ on $\tilde{A} = A/\mathcal{F} \to \tilde{M} = M/\rho_A(\ker N^k)$ with $\tilde{N}$ nondegenerate.
The Reduced nondegenerate SN Lie algebroid

Theorem 2

Let \((A \to M, [\cdot, \cdot]_A, \rho_A, \Omega, N)\) be a symplectic-Nijenhuis Lie algebroid such that

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The Reduced nondegenerate SN Lie algebroid

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Let \((A \to M, [\ , \ ]_A, \rho_A, \Omega, N)\) be a symplectic-Nijenhuis Lie algebroid such that

1) \(N\) has constant Riesz index \(k\).

2) \(\rho_A(B)\) and \(F\) are regular foliations for \(B = \ker N^k\).

3) For all \(x \in M\), \(a_x, a'_x \in L_F \implies a_x - a'_x \in \ker(N^k_x)\).

Then, we can induce a symplectic-Nijenhuis Lie algebroid structure 
\([\ , \ ]_{\tilde{A}}, \rho_{\tilde{A}}, \tilde{\Omega}, \tilde{N}\) on \(\tilde{A} = A/F \to \tilde{M} = M/\rho_A(\ker N^k)\) with \(\tilde{N}\) nondegenerate.
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3) For all \(x \in M\), \(a_x, a'_x \in L\mathcal{F} \implies a_x - a'_x \in \ker(N_x^k)\).

Then, we can induce a symplectic-Nijenhuis Lie algebroid structure \(([\cdot, \cdot]_{\tilde{A}}, \rho_{\tilde{A}}, \tilde{\Omega}, \tilde{N})\) on \(\tilde{A} = A/\mathcal{F} \rightarrow \tilde{M} = M/\rho_A(\ker N^k)\) with \(\tilde{N}\) nondegenerate.
Reduction: Summary

\[(A \rightarrow M, [ , ]_A, \rho_A, P, N)\]

Poisson-Nijenhuis Lie algebroid

\[\Downarrow \quad D = \rho_A(P^\#(A^*))\]

\[(A_L = P^\#(A^*)|_L \rightarrow L, [ , ]_{A_L}, \rho_{A_L}, \Omega_L, N_L)\]

symplectic-Nijenhuis Lie algebroid

\[\Downarrow \quad \mathcal{F} = \{X^c + Y^\vee / X, Y \in \Gamma(\ker N_{L}^k)\}\]

\[(\tilde{A} = A_L / \mathcal{F} \rightarrow \tilde{L} = L / \rho_{A_L}(\ker N_{L}^k), [ , ]_{\tilde{A}}, \rho_{\tilde{A}}, \Omega_{\tilde{A}}, N_{\tilde{A}})\]

symplectic-Nijenhuis Lie algebroid with \(N_{\tilde{A}}\) nondegenerate
Reduction: Summary

\((A \to M, [, ]_A, \rho_A, P, N)\)

Poisson-Nijenhuis Lie algebroid

\[ \downarrow \quad D = \rho_A(P^*(A^*)) \]

\((A_L = P^*(A^*)|_L \to L, [, ]_{A_L}, \rho_{A_L}, \Omega_L, N_L)\)

symplectic-Nijenhuis Lie algebroid

\[ \downarrow \quad \mathcal{F} = \{ X^c + Y^\vee / X, Y \in \Gamma(\ker N_L^k) \} \]

\((\tilde{A} = A_L / \mathcal{F} \to \tilde{L} = L / \rho_{A_L}(\ker N_L^k), [, ]_{\tilde{A}}, \rho_{\tilde{A}}, \Omega_{\tilde{A}}, N_{\tilde{A}})\)

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References