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A note on a variational formulation of electrodynamics

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Abstract

We present a variational formulation of electrodynamics using de Rham even and odd differential forms. Relying on a variational principle more complete than the Hamilton principle our formulation leads to field equations with external sources and permits the derivation of the constitutive relations.

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1. Introduction

A general framework for variational formulations of physical theories was presented in [1]. Applications to statics and dynamics of mechanical systems appear in [2, 3]. This note presents an introduction to a more complete variational formulation to be presented in a future publication.

Our formulation of electrodynamics is special relativistic. The use of de Rham even and odd differential forms ([5, 6]) permits a rigorous formulations of electrodynamics and the description of the transformation properties of electromagnetic fields relative to reflections (cf. [4]). Relying on a variational principle more complete than the Hamilton principle our formulation leads to field equations with external sources and permits the derivation of the constitutive relations which are usually postulated separately since the variations normally considered are not general enough to derive them from the variational principle.

We interpret a domain in space-time as an odd de Rham 4-current. This permits a treatment of different types of boundary problems in an unified way. In particular we obtain a smooth transition to the infinitesimal version by using a current with a one point support.

2. Currents

Let M be the affine Minkowski space-time of special relativity with the 4-dimensional model space V and a metric tensor $g : V \to V^*$ of signature $(1, 3)$. The vector space of even differential q-forms in M will be denoted by $\Phi^q_e(M)$ and space of odd differential q-forms will be denoted by $\Phi^q_o(M)$. The symbol $\Phi^q_p(M)$ will be used to denote either of the two spaces when the distinction is not relevant.

An *even* or *odd de Rham current* of dimension q on M is a linear function

$$\boldsymbol{c}\colon \Phi^q_p(M) \to \mathbb{R}\colon A \mapsto \int_{\boldsymbol{c}} A. \tag{1}$$

Domains in space-time will be treated as currents. The boundary $\partial\boldsymbol{c}$ of a current $\boldsymbol{c}$ is defined by assuming that Stokes theorem holds for all currents as it holds for domains.

In addition to domains in space-time odd de Rham currents most frequently used are the *Dirac currents*. A Dirac current $w\delta(x)$ is an odd current of dimension 4 defined in terms of a point $x \in M$ and an odd 4-vector w. If A is an odd 4-form, then

$$\int_{w\delta(x)} A = \langle A(x), w\rangle. \tag{2}$$

3. The space of fields

Let $\mathsf{C}M$ be the space of odd 4-currents with compact supports in M. We consider the set $\mathsf{X}(\Phi^1_e(M); \mathsf{C}M)$ of pairs $(A, \boldsymbol{c})$, where $\boldsymbol{c}$ is an odd current of dimension 4 in M with a compact support $\mathrm{Sup}(\boldsymbol{c})$ and A is an even 1-form

$$A\colon U \to \wedge^1_e V^* \tag{3}$$

defined in an open set $U \subset M$ containing the support of $\boldsymbol{c}$. The symbols $\wedge^q_e V^*$ and $\wedge^q_o V^*$ denote respectively the vector spaces of even and odd q-covectors. The 1-form A will represent the *electromagnetic potential.* Its differential $F = \mathrm{d}A$ is the *electromagnetic field.*

A mapping

$$\kappa\colon M \times \wedge^1_e V^* \times \wedge^2_e V^* \to \wedge^4_o V^* \tag{4}$$

is said to be *quadratic* if for each $x \in M$ there exists a symmetric bilinear mapping

$$\delta^2\kappa_x\colon \left(\wedge^1_e V^* \times \wedge^2_e V^*\right) \times \left(\wedge^1_e V^* \times \wedge^2_e V^*\right) \to \wedge^4_o V^* \tag{5}$$

such that the mappings $\kappa_x = \kappa(x, \cdot, \cdot)$ and $\delta^2\kappa_x$ are in the relation

$$\kappa_x(a, f) = \frac{1}{2}\delta^2\kappa_x((a, f)(a, f)), \tag{6}$$

for each $(a, f) \in \wedge_e^1 V^* \times \wedge_e^2 V^*$. We will use the set of all quadratic mappings (4) to introduce an equivalence relation in the set $\mathsf{X}(\Phi_e^1(M); \mathsf{C}M)$.

Pairs $(A, \boldsymbol{c})$ and $(A', \boldsymbol{c}')$ are equivalent if

$$\int_{\boldsymbol{c}'} \kappa \circ (x, A', \mathrm{d}A') = \int_{\boldsymbol{c}} \kappa \circ (x, A, \mathrm{d}A) \tag{7}$$

for each quadratic mapping (4). Equivalence classes will be called *fields*. Our fields are similar to those used by Freed in [8]. The space of fields will be denoted by $\mathsf{Q}(\Phi_e^1(M); \mathsf{C}M)$ or simply Q. The equivalence class of $(A, \boldsymbol{c})$ will be denoted by $\mathsf{q}(A, \boldsymbol{c})$. The symbol q will denote a generic element of Q. There is a natural projection $\varepsilon\colon Q \to \mathsf{C}M\colon \mathsf{q}(A, \boldsymbol{c}) \mapsto \boldsymbol{c}$ from the space of fields to the space $\mathsf{C}M$ of currents in M which is similar to a vector fibration. Each fibre $\varepsilon^{-1}(\boldsymbol{c})$ of the projection ε is a vector space denoted by $\mathsf{Q}(\Phi_e^1(M); \boldsymbol{c})$ or $Q_{\boldsymbol{c}}$.

4. Functions, vertical vectors and covectors in the space of fields

With each quadratic mapping (4) we associate the function

$$k\colon \mathsf{Q}(\Phi_e^1(M); \mathsf{C}M) \to \mathbb{R}\colon \mathsf{q}(A, \boldsymbol{c}) \mapsto \int_{\boldsymbol{c}} \kappa \circ (x, A, \mathrm{d}A). \tag{8}$$

Functions constructed in this way will be considered differentiable. The space of such functions will be denoted by $\mathsf{K}(\Phi_e^1(M); \mathsf{C}M)$. It is easy to verify that these differentiable functions separate points of Q, i.e. if $k(q') = k(q)$ for each $k \in \mathsf{K}(\Phi_e^1(M); \mathsf{C}M)$, then $q' = q$.

The tangent space to the vector space $Q_{\boldsymbol{c}} = \varepsilon^{-1}(\boldsymbol{c})$ is the space $Q_{\boldsymbol{c}}$ itself. It follows that the *vertical tangent bundle* of the vector fibration ε is the space

$$\mathsf{V}Q = Q \underset{(\varepsilon,\varepsilon)}{\times} Q = \{(q, \delta q) \in Q \times Q;\ \varepsilon(q) = \varepsilon(\delta q)\}. \tag{9}$$

There is no obvious choice of the bundle dual to $\mathsf{V}Q$. Using the fibre derivatives of functions $k \in \mathsf{K}(\Phi_e^1(M); \mathsf{C}M)$ as models of covectors we obtain the following result. A covector p is an equivalence class of triples $(G, J, \boldsymbol{c})$ of an odd 2-form $G\colon U \to \wedge_o^2 V^*$, an odd 3-form $J\colon U \to \wedge_o^3 V^*$, and a current $\boldsymbol{c}$ with support contained in U. The objects G and J are interpreted as the *electromagnetic induction* and the *current* respectively. Elements $(G, J, \boldsymbol{c})$ and $(G', J', \boldsymbol{c}')$ are equivalent if $\boldsymbol{c} = \boldsymbol{c}'$ and

$$\int_{\boldsymbol{c}} \left(\frac{1}{c^2} J' \wedge \delta A - \frac{1}{4\pi c}\mathrm{d}\left(G' \wedge \delta A\right)\right) = \int_{\boldsymbol{c}} \left(\frac{1}{c^2} J \wedge \delta A - \frac{1}{4\pi c}\mathrm{d}\left(G \wedge \delta A\right)\right), \tag{10}$$

for each $\delta A\colon U \to \wedge_e^1 V^*$. The equivalence class of $(G, J, \boldsymbol{c})$ is denoted by $\mathsf{p}(G, J, \boldsymbol{c})$.

The vector space $\Pi_{\boldsymbol{c}}$ of covectors associated with the current $\boldsymbol{c}$ is the dual of the space $Q_{\boldsymbol{c}}$ with the pairing

$$\langle \mathsf{p}(G, J, \boldsymbol{c}), \mathsf{q}(\delta A, \boldsymbol{c}) \rangle_{\boldsymbol{c}} = \int_{\boldsymbol{c}} \left(\frac{1}{c^2} J \wedge \delta A - \frac{1}{4\pi c} \mathrm{d} \left(G \wedge \delta A \right) \right). \tag{11}$$

The space of all covectors is the union $\Pi = \bigcup_{\boldsymbol{c} \in \mathsf{C}\mathbb{R}} \Pi_{\boldsymbol{c}}$. There is a natural projection $\varepsilon' \colon \Pi \to \mathsf{C}M \colon \mathsf{p}(G, J, \boldsymbol{c}) \mapsto \boldsymbol{c}$. The *phase space* is the space

$$\boldsymbol{Ph} = Q \underset{(\varepsilon, \varepsilon')}{\times} \Pi = \{(q, p) \in Q \times \Pi;\ \varepsilon(q) = \varepsilon'(p)\}. \tag{12}$$

The symbol $\boldsymbol{Ph_c}$ will denote the set $Q_{\boldsymbol{c}} \times \Pi_{\boldsymbol{c}} \subset \boldsymbol{Ph}$.

5. A virtual action principle for electrodynamics

In this section a variational principle for electrodynamics similar to the virtual action principle of analytical mechanics (see [3]) will be formulated.

The *action* is the differentiable function

$$W \colon \mathsf{Q}(\Phi^1_e(M); \mathsf{C}M) \to \mathbb{R} \colon \mathsf{q}(A, \boldsymbol{c}) \mapsto \int_{\boldsymbol{c}} L \circ (A, \mathrm{d}A) \tag{13}$$

derived from the quadratic *Lagrangian density*

$$L \colon \wedge^1_e V^* \times \wedge^2_e V^* \to \wedge^4_o V^* \colon (a, f) \mapsto -\frac{1}{8\pi c} \langle f, \wedge^2_e g^{-1}(f) \rangle \sqrt{|g|}. \tag{14}$$

We are using the symbol $\sqrt{|g|}$ to denote the odd 4-covector derived from the metric tensor g (see [4] or [7]), while the symbol $\wedge^2_e g^{-1}$ denotes the inverse mapping of $\wedge^2_e g \colon \wedge^2_e V \to \wedge^2_e V^*$ characterized by the equality $\wedge^2_e g(v_1 \wedge v_2) = g(v_1) \wedge g(v_2)$ for even simple 2-vectors.

A phase $\boldsymbol{ph} = (\mathsf{q}(A, \boldsymbol{c}), \mathsf{p}(G, J, \boldsymbol{c}))$ satisfies the *virtual action principle* if the equality

$$\mathrm{D}W(q, \delta q) - \langle p, \delta q \rangle_{\boldsymbol{c}} = 0 \tag{15}$$

holds for each *virtual displacement* $\delta q = \mathsf{q}(\delta A, \boldsymbol{c}) \in Q_{\boldsymbol{c}}$. For each current $\boldsymbol{c}$ the *dynamics* associated with the current $\boldsymbol{c}$ is the set $\boldsymbol{D_c} \subset \boldsymbol{Ph_c}$ of phases which satisfy the virtual action principle. The *dynamics* is the subset $\boldsymbol{D} = \bigcup_{\boldsymbol{c} \in \mathsf{C}\mathbb{R}} \boldsymbol{D_c}$ of the *phase space* $\boldsymbol{Ph}$ defined above.

A *phase space trajectory* is a triple of differential forms

$$(A, G, J) \colon U \to \wedge^1_e V^* \times \wedge^2_o V^* \times \wedge^3_o V^*. \tag{16}$$

The dynamics of a system can also be represented as a set $\mathbf{D}$ of phase space trajectories (A, G, J) such that for each current $\boldsymbol{c}$ with support included in U the phase $\boldsymbol{ph} = (\mathsf{q}(A, c), \mathsf{p}(G, J, \boldsymbol{c}))$ is in $\boldsymbol{D_c}$.

The equation (15) is too abstract to be used directly. A more concrete expression of the virtual action principle will be given in the Proposition 1.

The *left interior multiplications* are the operations

$$\lrcorner : \wedge^q_p V \times \wedge^{q'}_{p'} V^* \to \wedge^{q'-q}_{pp'} V^*, \tag{17}$$

defined for $q \leqslant q'$ by $\langle w' \lrcorner a, w\rangle = \langle a, w' \wedge w\rangle$. The parity pp' which appears in this definition is constructed by assigning the numerical values $+1$ and -1 to e and p respectively. The parity of the multivector w must match the parity of the multicovector $w' \lrcorner a$.

Proposition 1 *A phase* $\boldsymbol{ph} = (\mathsf{q}(A, c), \mathsf{p}(G, J, \boldsymbol{c}))$ *satisfies the virtual action principle if and only if the equality*

$$\begin{aligned}\frac{1}{4\pi c}\int_{\boldsymbol{c}} &\left(\mathrm{d}\left(\left(\wedge^2_e g^{-1} \circ \mathrm{d}A\right) \lrcorner \sqrt{|g|}\right) \wedge \delta A - \mathrm{d}\left(\left(\left(\wedge^2_e g^{-1} \circ \mathrm{d}A\right) \lrcorner \sqrt{|g|}\right) \wedge \delta A\right)\right)\\ &= \int_{\boldsymbol{c}} \left(\frac{1}{c^2} J \wedge \delta A - \frac{1}{4\pi c}\mathrm{d}\left(G \wedge \delta A\right)\right),\end{aligned} \tag{18}$$

is satisfied for each virtual displacement $\delta q = \mathsf{q}(\delta A, \boldsymbol{c})$.

A phase space trajectory belongs to the dynamics $\mathbf{D}$, if and only if it satisfies the virtual action principle for each current $\boldsymbol{c}$ with support included in its domain of definition. There is a characterization of the dynamics of phase space trajectories in terms of differential equations. This is shown in the following propositions.

Theorem 2 *A phase space trajectory* (A, G, J) *belongs to the dynamics* $\mathbf{D}$ *if and only if it satisfies the Euler-Lagrange equation*

$$\mathrm{d}\left(\left(\wedge^2_e g^{-1} \circ \mathrm{d}A\right) \lrcorner \sqrt{|g|}\right) = \frac{4\pi}{c} J \tag{19}$$

and the constitutive relation

$$G = \left(\wedge^2_e g^{-1} \circ \mathrm{d}A\right) \lrcorner \sqrt{|g|}. \tag{20}$$

The constitutive relation (20) produced by our variational principle corresponds to the momentum-velocity relation of analytical mechanics.

Proposition 3 *A phase space trajectory* (A, G, J) *satisfies the Euler-Lagrange equation and the constitutive relation if and only if it satisfies the Maxwell's equations*

$$\mathrm{d}G = \frac{4\pi}{c} J \tag{21}$$

and the constitutive relation

$$G = \left(\wedge^2_e g^{-1} \circ F\right) \lrcorner \sqrt{|g|}, \tag{22}$$

with $F = \mathrm{d}A$.

6. The Dynamics in a compact domain

Let the current $\boldsymbol{c}$ consist in integrating an odd 4-form on a compact domain $K \subset M$ with smooth boundary ∂K. Field configurations, tangent vectors and covectors are equivalence classes of equivalence relations based on the equalities (7) and (10).

It follows that a field $q = \mathsf{q}(A, K)$ is represented by the restriction

$$A|K \colon K \to \wedge_e^1 V^* \tag{23}$$

of the potential A to the domain K. A tangent vector $\delta q = \mathsf{q}(\delta A, K)$ is represented by the restriction

$$(\delta A)|K \colon K \to \wedge_e^1 V^* \tag{24}$$

of the variation δA to the domain K. A covector $p = \mathsf{p}(G, J, K)$ is represented by the pair of the restrictions

$$G|\partial K \colon \partial K \to \wedge_o^2 V^*, \qquad J|\overset{\circ}{K} \colon \overset{\circ}{K} \to \wedge_o^3 V^* \tag{25}$$

of the electromagnetic induction G to the boundary ∂K of the domain K and of the current J to the interior $\overset{\circ}{K}$ of the domain K.

The *dynamics* in the domain K is the set $\boldsymbol{D}_K \subset \boldsymbol{Ph}$ of phases satisfying the virtual action principle. It is characterized by the following proposition.

Proposition 4 *A phase $\boldsymbol{ph} = (\mathsf{q}(A, K), \mathsf{p}(G, J, K))$, defined in a compact domain K, belongs to the dynamics $\boldsymbol{D}_K$ if and only if the Euler-Lagrange equation*

$$\mathrm{d}\left(\left(\wedge_e^2 g^{-1} \circ \mathrm{d}A\right) \lrcorner \sqrt{|g|}\right) |\overset{\circ}{K} = \frac{4\pi}{c} J|\overset{\circ}{K} \tag{26}$$

and the constitutive relation

$$G|\partial K = \left(\left(\wedge_e^2 g^{-1} \circ \mathrm{d}A\right) \lrcorner \sqrt{|g|}\right) |\partial K \tag{27}$$

are satisfied.

7. The Lagrangian formulation

The Lagrangian formulation of dynamics is the infinitesimal limit of the formulation in a compact domain with the domain shrinking to a point. A formal method which greatly simplifies the passage to the infinitesimal limit is to replace the compact domain — which is used exclusively as domain of integration — with the current $\boldsymbol{c} = \delta(x)w$, where $\delta(x)$ is the Dirac delta function in $x \in M$ and $w \in \wedge_o^4 V$ is an odd 4-vector, with $w \neq 0$. The construction of infinitesimal fields, tangent vectors and covectors is based on the equalities (7) and (10) which in this case reduce to pairings of odd 4-covectors with the odd 4-vector $w \neq 0$.

It follows that an infinitesimal field $q = \mathsf{q}(A, \boldsymbol{c})$ is represented by the pair

$$(A(x), F(x)) \in \wedge_e^1 V^* \times \wedge_e^2 V^*, \tag{28}$$

a tangent vector $\delta q = \mathsf{q}(\delta A, \boldsymbol{c})$ is represented by the pair

$$(\delta A(x), \delta F(x)) \in \wedge_e^1 V^* \times \wedge_e^2 V^*, \tag{29}$$

and a covector $p = \mathsf{p}(G, J, \boldsymbol{c})$ is represented by the pair

$$\left(G(x), \mathrm{d}G(x) - \frac{4\pi}{c} J(x)\right) \in \wedge_o^2 V^* \times \wedge_o^3 V^*. \tag{30}$$

The pairing $\langle\!\langle p, \delta q \rangle\!\rangle_{\boldsymbol{c}}$ defined by the equality (11) assumes the form

$$\langle\!\langle p, \delta q \rangle\!\rangle^{\mathsf{L}} = -\frac{1}{4\pi c} \left\langle \left(\mathrm{d}G(x) - \frac{4\pi}{c} J(x) \right) \wedge \delta A(x) + G(x) \wedge \delta F(x), w \right\rangle. \tag{31}$$

We have constructed the space of infinitesimal fields $Q_\delta = \wedge_e^1 V^* \times \wedge_e^2 V^*$ and the space of infinitesimal covectors $\Pi_\delta = \wedge_o^2 V^* \times \wedge_o^3 V^*$. Hence, the infinitesimal phase space is $\boldsymbol{Ph}_\delta = Q_\delta \times \Pi_\delta = \wedge_e^1 V^* \times \wedge_e^2 V^* \times \wedge_o^2 V^* \times \wedge_o^3 V^*$.

The infinitesimal action is $W(\mathsf{q}(A, \delta(x)w)) = \langle L(A(x), F(x)), w \rangle$ and the infinitesimal dynamics is the set

$$\boldsymbol{D}_\delta = \Big\{ (a, f, g, h) \in \boldsymbol{Ph}_\delta;\ \forall_{(\delta a, \delta f) \in \wedge_e^1 V^* \times \wedge_e^2 V^*} \\ \mathrm{D}L(a, f, \delta a, \delta f) = -\frac{1}{4\pi c} \left(h \wedge \delta a + g \wedge \delta f \right) \Big\}. \tag{32}$$

Applied to a phase $\boldsymbol{ph} = (\mathsf{q}(A, \delta(x)w), \mathsf{p}(G, J, \delta(x)w))$, with $w \neq 0$, the action principle results in the equations

$$G(x) = \left(\wedge_e^2 g^{-1}(F(x))\right) \lrcorner \sqrt{|g|}, \qquad \mathrm{d}G(x) = \frac{4\pi}{c} J(x). \tag{33}$$

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