CONDITIONS FOR CONVERGENCE OF MULTIPOINT HERMITE-PADÉ APPROXIMANTS FOR NIKISHIN SYSTEM OF ANALYTIC FUNCTIONS.

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Abstract

Nikishin type system of analytic functions are considered. For such systems, sufficient conditions for the convergence in capacity of multipoint Hermite Padé approximants is given.

1 Introduction

Let Δ be a set, $\Delta \subset \mathbb{R}$; and μ, a finite positive Borel measure on Δ, whose support contains an infinite set of points. We assume that either 1) Δ is bounded, or 2) the moments $|c_n| = |\int_\Delta x^\nu d\mu| < \infty$, $n = 1, 2, \ldots$ exist. We are going to denote $M(\Delta)$ the set of measures μ that have this property.
Set

$$\hat{\mu}(z) = \int_\Delta \frac{d\mu(x)}{z - x}.$$

The Stieljes function $\hat{\mu}(z)$ is analytic in $D = \mathbb{C} \setminus \Delta$. There exist polynomials Q_n, P_n, such that $Q_n \not\equiv 0$, $deg(Q_n) \leq n$, $deg(P_n) \leq n - 1$, $n \in \mathbb{N}$, and

$$[Q_n \hat{\mu} - P_n](z) = O(z^{-n-1}) \in H(D).$$

Finding Q_n reduces to solving a system of n homogeneus linear equations on the $n + 1$ coeficients of Q_n. Thus, a nontrivial solution always exists. Obviously, P_n is the polynomial part of the expansion of $Q_n \hat{\mu}$. The fraction $R_n = \frac{P_n}{Q_n}$ is known as the diagonal Padé approximant of order n.

An old problem is to find sufficient conditions for the uniform convergence of diagonal Padé approximants for Stieljes Function. Two such conditions are:

- Δ compact, or
- $\sum_{\nu \geq 1} \frac{1}{c_i^{\frac{1}{2i}}} = \infty$

In this work we are going to prove an extension of this result for a certain system of functions.

2 Hermite-Padé Approximants

An extension of Padé approximation for systems of functions is given by the so called Hermite-Padé approximants. This concept was introduced by Hermite in connection whith the proof of the trascendence of number e.

Let $f_1, f_2, \cdots, f_m$ be a set of m formal power series in a neighborhood of infinity ($z = \infty$).

$$f_i(z) = A_{k_i,i} z^{k_i} + A_{k_i-1,i} z^{k_i-1} + \ldots, \qquad i = 1, \cdots, m.$$

Let $r_1, r_2, \cdots, r_m$ be an arbitrary set of nonnegative integers. As in the case of diagonal Padé approximants, it is easy to verify that there exists a polynomial $Q_n \not\equiv 0$, $deg(Q_n) \leq n = r_1 + r_2 + \cdots + r_m$, such that:

$$[Q_n f_i - P_{n,i}](z) = A_i z^{-r_i-1} + \ldots, \qquad i = 1, \cdots, m.$$

The construction of Q_n reduces to finding a nontrivial solution of a homogeneus system of n lineal equations on the $n+1$ coeficients of Q_n. $P_{n,i}$ is the polynomial part of the expansion of $Q_n f_i$. Obviously, $deg(Q_n) \leq n + k_i$.

The set of fractions $\{R_{n,i} = \frac{P_{n,i}}{Q_n}\}$ is called diagonal Hermite-Padé approximants of the system $\{f_i :\}$ $i = 1, 2, \cdots, m$ of order n, associated to the system of indices $r_1, r_2, \cdots, r_m$.

3 Nikishin Systems

Let $\{(\Delta_j, \mu_j)\}$, $j = 1, 2, \cdots, m$ be m pairs formed by an interval, $\Delta_j \subset \mathbb{R}$, and $\mu \in M(\Delta_i)$. Further $\forall j < m$, $\Delta_j \cap \Delta_{j+1} = \emptyset$. We say that the system of functions $\hat{\sigma}(z) = \{\hat{\sigma}_1(z), \hat{\sigma}_2(z), \cdots, \hat{\sigma}_m(z)\}$ is the Nikishin System [N] generated by such pairs on $D = \mathbb{C} \setminus \Delta_1$, if these functions are defined as fallows:

$$\begin{aligned}
\hat{\sigma}_1(z) &= \int_{\Delta_1} \frac{d\mu_1(x_1)}{z-x_1} = \int_{\Delta_1} \frac{d\sigma_1(x_1)}{z-x_1} \\
\hat{\sigma}_2(z) &= \int_{\Delta_1} \frac{d\mu_1(x_1)}{z-x_1} \int_{\Delta_2} \frac{d\mu_2(x_2)}{x_1-x_2} = \int_{\Delta_1} \frac{d\sigma_2(x_1)}{z-x_1} \\
&\vdots \\
\hat{\sigma}_m(z) &= \int_{\Delta_1} \frac{d\mu_1(x_1)}{z-x_1} \int_{\Delta_2} \frac{d\mu_2(x_2)}{x_1-x_2} \int_{\Delta_3} \cdots \int_{\Delta_m} \frac{d\mu_m(x_m)}{x_{m-1}-x_m} = \int_{\Delta_1} \frac{d\sigma_m(x_1)}{z-x_1}
\end{aligned}$$

Now, the question is: which the sufficient conditions are for uniform convergence of Hermite-Padé Aproximants of a Nikishin systems? Uniform convergence is a very strong criteria for Hermite-Padé approximation. We will use a weaker form of convergence. This is going be convergence in capacity.

4 Convergence in Capacity

Let E be a compact set, $E \subset \mathbb{C}$, and $\mu \in M(E)$. We call energy of μ to

$$I_\mu(z) = \int\int log \frac{1}{|z-\beta|} d\mu(\beta) d\mu(z).$$

Robin's Constant is defined as

$$I(E) = inf\{I_\mu : \mu \in M_1(E)\},$$

and the logarithmic capacity of E is given by

$$C(E) = exp(-I(E)).$$

If h is an arbitrary subset of $\mathbb{C}$, it's capacity is given by

$$C(h) = sup\{C(E) : E \subset h\}$$

For each $\epsilon > 0$, the convergence in capacity is defined by:

$$C(\{z \in K : |(\hat{\sigma}_i - R_{n,i})(z)| \geq \epsilon\}) \to 0, \qquad i = 1, \cdots, m, \qquad s \to \infty$$

The question now is: which conditions are sufficient in order to have convergence in capacity.

5 Conditions for Hermite-Padé Approximants

An answer to the question of convergence in capacity for Hermite-Padé approximation is given [BL]. The result may be stated as follows.

Theorem Let c be a constant such that $\forall s \in \mathbb{N}$, we have $r_i \geq \frac{n}{m} - c$, $i = 1, 2, \cdots, m$, $n = n(s) = r_1(s) + r_2(s) + \cdots + r_m(s)$. Assume that either:

- Δ_2 is bound, or.
- $\sum_{\delta \geq 1} \frac{1}{c_\delta^{2\delta}} = \infty$

Then, for all compact $K \subset D = \mathbb{C} \setminus \Delta_1$ and each $\epsilon > 0$,

$$C(\{z \in K : |(\hat{\sigma}_i - R_{n,i})(z)| \geq \epsilon\}) \to 0, \qquad i = 1, \cdots, m, \qquad s \to \infty.$$

6 Conditions for Multipoint Hermite-Padé Aproximants

Let L be a table of points, $L = \cup L_{n,i}$, $L_{n,i} = \{L_{n,i,k} \in \mathbb{R} \setminus \Delta_1\}$, $k = 1, 2, \cdots, n + r_i$. We define the family of polynomials $\{W_{n,i}\}$:

$$W_{n,i} = \prod_{k=1}^{n+r_i} (1 - \frac{x}{L_{n,i,k}}), \qquad i = 1, \cdots, m$$

There exist polynomials $Q_n^*, P_{n,i}^*, P_{n,i}; i = 1, \cdots, m$, such that $Q_n{}^* \not\equiv 0$, $deg(Q_n^*) \leq n$, $deg(P_{n,i}^*) \leq n-1$, $deg(P_{n,i}) \leq n-1$, and

$$\frac{Q_n^*\hat{\sigma}_i - P_{n,i}^*}{w_{n,i}} = Q_n^*\hat{\sigma}_i^n - P_{n,i} = O(z^{-r_i-1}) \in H(D),$$

where

$$\hat{\sigma}_i^n = \int_{\Delta_1} \frac{d\sigma_i}{(z-x)W_{n,i}(x)} = \int_{\Delta_1} \frac{d\sigma_i^{n(s)}}{z-x}.$$

The family of fractions $\{R_{n,i}^* = \frac{P_{n,i}^*}{Q_n^*}\}$, $i = 1 \cdots, m$ is the multipoit Hermite-Padé approximant. From the definition, it follows that

$$[\frac{Q_n^*\hat{\sigma}_i^n - P_{n,i}}{\omega_{n,i}}](z) = [\frac{Q_n^*\hat{\sigma}_i - P_{n,i}^*}{W_{n,i}\omega_{n,i}}](z) = O(z^{-n-l}) \in H(D)$$

Where $\omega_{n,i}$, $i = 1, \cdots, m$ are polynomials whose zeros lie in Δ_2.

In the multipoint case we can add another sufficient condition for convergence.

Theorem Let c be a constant such that $\forall s \in \mathbb{N}$, we have $r_i \geq \frac{n}{m} - c$, $i = 1, 2, \cdots, m$, $n = n(s) = r_1(s) + r_2(s) + \cdots + r_m(s)$. Assume that one the fo llowing conditions is satisfied:

- Δ_2 is bound
- $\sum_{\delta \geq 1} \frac{1}{c_\delta^{2\delta}} = \infty$
- the table of points L is lie on a bounded set and for each i, the number of diferent zeros tends to infinity as $s \to \infty$

Then, for any compact $K \subset D = \mathbb{C} \setminus \Delta_1$, and each $\epsilon > 0$,

$$C(\{z \in K : |(\hat{\sigma}_i - R_{n,i})(z)| \geq \epsilon\}) \to 0, \qquad i = 1, \cdots, m, \qquad s \to \infty.$$

references

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