Lecture 5

Exercises

Computational Mathematics

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Exercises

- Exercise 1: Consider $K_n = [b \mid Ab \mid \dots \mid A^{n-1}b]$ (Krylov matriz) for a given matriz $A \in \mathbb{R}^{n\times n}$ and a vector $b \in \mathbb{R}^n$.
  In Matlab: `Kn = gallery('krylov',A,b,n)`
  1. Varying n between 2 and 20, choose a random matriz A and b and determine the condition number
     $$\text{cond}(K_n) = \|K_n\|_2 \|K_n^{-1}\|_2$$
     of matrix K_n and plot the data in logarithmic scale.
  2. Let us suppose that we want to solve the system $K_n x = c$, where c is determined such that $x = [1, 1, \dots, 1]^T$ is the exact solution of the system. Plot the evolution of the relative error with respect the calculated approximation $\bar{x}$,
     $$r(\bar{x}) = \|x - \bar{x}\|_2 / \|x\|_2,$$
     as a function of n (make the range from 2 to 20).
  3. Prove that, if $K_n \bar{x} = \bar{c}$,
     $$r(\bar{x}) \leqslant \text{cond}(K_n) r(\bar{c})$$
     and plot the evolution of this upper bond for the relative error as a function of n (make the range from 2 to 20).

Exercises

- Exercise 2: Consider the SOR method for $Ax = b$.

  1. Consider the tridiagonal matrix A with 2 on the diagonal and -1 above and below the diagonal. Construct the right-hand side vector so that $x = [1, 1, ..., 1]^T$ is the true solution.
  2. For each value of $\omega = 1, 1.01, 1.02, ..., 1.99, 2.0$, apply 100 iterations of SOR starting with $x^{(0)} = 0$. Do this for A of order 10, 20, and 50. Measure the error at the end of 100 iterations, call it e, and set $p = \sqrt[100]{e}$. The value of p is the "average" rate of convergence of the iteration; the error was reduced by this much on each iteration. Note that the error of $x^{(0)}$ is 1.
  3. For each of the three cases make a performance plot of p versus ω and estimate the optimum value of ω. If the plot is too coarse, make additional runs to fill in the gaps.
  4. Discuss the behavior of the performance profiles and their implications for the difficulty of finding optimum SOR factors.

Exercises

- Exercise 3: In the last lectures we considered the Conjugate Gradient (CG) method, the Krylov subspace methods and Richardson iteration method.

  1. Explain the relationship between these three methods.
  2. How does the CG method fit into the broader framework of Krylov subspace methods? Furthermore, discuss how Richardson iteration can be viewed as a special case of both the CG method and Krylov methods.
  3. Provide insights into the similarities and differences between these iterative techniques, and discuss the implications of understanding this relationship for solving linear systems efficiently.
  4. Prove that, for SPD matrices, solving $Ax = b$ is equivalent to finding the minimizer $x \in \mathbb{R}^n$ of the quadratic form

     $$\phi(y) = \frac{1}{2} y^T A y - y^T b$$

     and explain how this can be used to develop an interactive method for solving a system of linear equations.