Flow invariants for irreducible sofic systems

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Based on joint work with Béal, Berstel, Perrin; Boyle, Carlsen

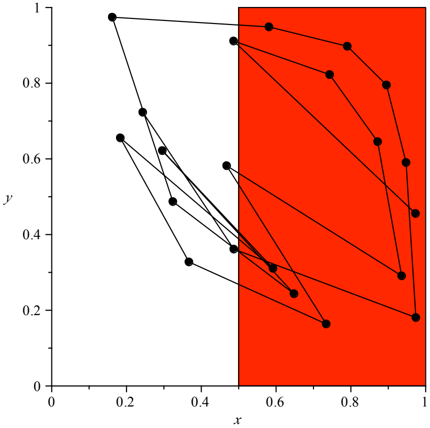
Content

See also

- M.-P. Béal, S. Eilers, J. Berstel, and D. Perrin: *Symbolic Dynamics*. Chapter for "Handbook in Automata Theory". ArXiV 2010.
- M. Boyle, T.M. Carlsen, and S. Eilers: *Flow equivalence of sofic systems*. ArXiV 2011 (sorry!).
- D. Lind, B. Marcus: *Introduction to symbolic dynamics and coding*. Cambridge University Press, 1995.

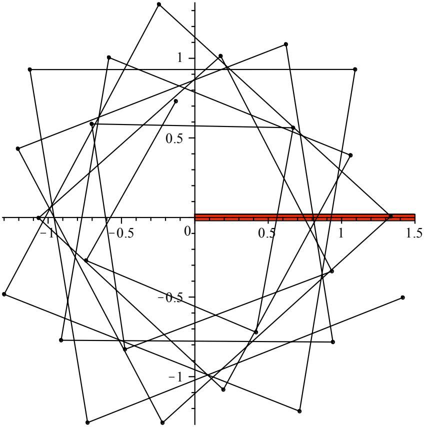
Outline

Baker's map



$$10111010010100111100\cdots$$

Irrational rotation



$$000100010010001001000100\cdots$$

Symbolic dynamics

Let $\mathfrak{a}$ be a finite set and equip $\mathfrak{a}^{\mathbb{Z}}$ with the product topology based on the discrete topology on $\mathfrak{a}$.

Definition

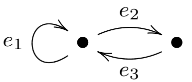
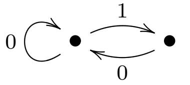
A **shift space** is a subset X of $\mathfrak{a}^{\mathbb{Z}}$ which is closed and closed under the **shift map**

$$\sigma : \mathfrak{a}^{\mathbb{Z}} \to \mathfrak{a}^{\mathbb{Z}} \qquad \sigma((x_i)) = (x_{i+1})$$

Definition

A shift space is **irreducible** if some orbit $\{\sigma^n(x) \mid n \in \mathbb{N}\}$ is dense.

3 constructions

Name	Input	Description	Example
$\mathsf{X}^{(W)}$	List of words W	Sequences not containing words from W	$W = \{11\}$
X_G	Graph G	Infinite paths on G	
$\mathsf{L}_{\mathcal{A}}$	Automaton $\mathcal{A}$	Words recognized by $\mathcal{A}$	

Forbidden word shifts

Let W be a set of finite words on $\mathfrak{a}$.

Definition

$\mathsf{X}^{(W)}$ is the shift space $\{x \in \mathfrak{a}^{\mathbb{Z}} \mid \forall i < j : x_i \cdots x_j \notin W\}$

Example

With $\mathfrak{a} = \{0, 1\}$ and $W = \{11\}$ the shift space $\mathsf{X}^{(W)}$ contains elements such as

$$\cdots 0100001000100010000101010100100100010001 0\cdots$$

Lemma

For any shift space X, $X = \mathsf{X}^{(W)}$ where W is chosen as the complement of the language

$$\mathcal{L}(X) = \{x_i \cdots x_j \mid x \in X, i < j\}$$

Edge shifts

Let a graph $G = (V, E, r, s)$ be given with

- Vertices V
- Edges E enumerated $\{e_1, \dots e_n\}$
- Range and source maps $r, s : E \to V$.

Definition

X_G is the shift space $\mathsf{X}^{(W)}$ with alphabet E and

$$W = \{e_i e_j \mid r(e_i) \neq s(e_j)\}$$

Example

With $G =$
$$\begin{tikzcd} \bullet \arrow[loop left, "e_1"] \arrow[r, bend left, "e_2"] & \bullet \arrow[l, bend left, "e_3"] \end{tikzcd}$$
, X_G contains elements such as

$$\cdots e_1 e_1 e_2 e_3 e_2 e_3 e_2 e_1 e_1 e_2 e_3 e_2 e_3 e_1 e_2 e_3 e_2 e_3 e_1 e_1 e_1 e_1 e_1 e_1 e_2 \cdots$$

Labeled edge shifts

Convention

All automata $\mathcal{A} = (V, E, r, s, \mathfrak{a}, \lambda)$ are finite and all states are both initial and final. Thus, they are given by the underlying graph (V, E, r, s) and a labelling map $\Lambda : E \to \mathfrak{a}$

Definition

We denote by $\mathsf{X}_{\mathcal{A}}$ the edge shift associated to the underlying graph of $\mathcal{A}$ and by

$$\lambda : \mathsf{X}_{\mathcal{A}} \to \mathfrak{a}^{\mathbb{Z}}$$

the *labeling map* induced by Λ. The shift *recognized by* $\mathcal{A}$ is $\mathsf{L}_{\mathcal{A}} = \lambda(\mathsf{X}_{\mathcal{A}})$.

Labeled edge shifts

Example

With $\mathcal{A} = \begin{tikzcd} \bullet \arrow[loop left, "0"] \arrow[r, bend left, "1"] & \bullet \arrow[l, bend left, "0"] \end{tikzcd}$ the shift space $\mathsf{X}_{\mathcal{A}}$ contains elements such as

$$\cdots 01000010001000100001010101001001000100010 \cdots$$

Outline

Definition

Let $X \subseteq \mathfrak{a}^{\mathbb{Z}}$ and $Y \subseteq \mathfrak{b}^{\mathbb{Z}}$. $\phi : X \to Y$ is the (m, n) *sliding block code* given by a map

$$\Phi : \mathfrak{a}^{n+1+m} \to \mathfrak{b}$$

when

$$\phi(x)_i = \Phi(x_{i-m} \cdots x_{i+n})$$

Lemma

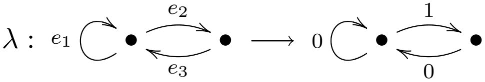
The following are equivalent:

- ϕ *is continuous and shift-commuting*
- ϕ *is a sliding block code*

Definition

X and Y are *conjugate* when there is a bijective sliding block code $\phi : X \to Y$

With $\mathcal{A}$ as above,



becomes a conjugacy. Indeed, the labeling map is always a $(0, 0)$ sliding block code induced by Λ. And in this case it has a $(1, 0)$ block inverse μ given by

$$00 \mapsto e_1 \qquad 01 \mapsto e_2 \qquad 10 \mapsto e_3$$

For instance,

$$\begin{gathered} \mu \circ \lambda(\cdots e_1 e_2 e_3 e_1 e_1 e_1 e_2 e_3 e_1 \ldots) = \\ \mu(\cdots 010000100 \ldots) = \\ \cdots e_2 e_3 e_1 e_1 e_1 e_2 e_3 e_1 \ldots \end{gathered}$$

Multiplicity set

Definition

With a given map $\lambda : \mathsf{X}_{\mathcal{A}} \to \mathsf{L}_{\mathcal{A}}$ we set

$$\begin{array}{rcl} \widetilde{\mathsf{L}_{\mathcal{A}}} & = & \{x \in \mathsf{L}_{\mathcal{A}} \mid |\lambda^{-1}(\{x\})| > 1\} \\ \widetilde{\mathsf{X}_{\mathcal{A}}} & = & \lambda^{-1}(\widetilde{\mathsf{L}_{\mathcal{A}}}) \end{array}$$

and restrict λ to

$$\widetilde{\lambda} : \widetilde{\mathsf{X}_{\mathcal{A}}} \to \widetilde{\mathsf{L}_{\mathcal{A}}}$$

Example

With $\mathcal{A} = \begin{tikzcd} \bullet \arrow[loop left, "0"] \arrow[r, bend left, "1"] & \bullet \arrow[l, bend left, "0"] \end{tikzcd}$ and $\mathcal{B} = \begin{tikzcd} \bullet \arrow[loop left, "1"] \arrow[r, bend left, "0"] & \bullet \arrow[l, bend left, "0"] \end{tikzcd}$ we get $\widetilde{\mathsf{L}_{\mathcal{A}}} = \emptyset$ and $\widetilde{\mathsf{L}_{\mathcal{B}}} = \{0^{\infty}\}$.

Shifts of finite type

Definition

A shift space is a *shift of finite type (SFT)* if is has the form $\mathsf{X}^{(W)}$ with W finite.

Lemma

The following are equivalent:

- X *is an SFT*
- $X \simeq \mathsf{X}_G$ *for some graph* G

Sofic shifts

Definition

A shift space is *sofic* if is has the form $\mathsf{X}^{(W)}$ with W recognizable.

Lemma

The following are equivalent:

- X *is sofic*
- $X \simeq \mathsf{X}_{\mathcal{A}}$ *for some automaton* $\mathcal{A}$

Theorem

When X *is irreducible and sofic, there is a unique deterministic automaton* $\mathcal{A}$ *with fewest possible vertices such that* $X \simeq \mathsf{X}_{\mathcal{A}}$. $\mathcal{A}$ *is called the Fischer cover of* X.

Near Markov and AFT

Definition

We say that an irreducible sofic shift is (right) near Markov if $\widetilde{X_{\mathcal{A}}}$ is finite for its Fischer cover $\mathcal{A}$.

Definition

We say that an irreducible sofic shift is AFT when its Fischer cover has finite left delay: There is a constant ℓ such that when

$$r \xrightarrow{\quad z \quad} q \xrightarrow{a} p \qquad r' \xrightarrow{\quad z \quad} q' \xrightarrow{a} p$$

with $|z| > \ell$, then $q = q'$.

Near Markov and AFT

Theorem

A near Markov shift is AFT.

Theorem

When $\mathsf{L}_{\mathcal{A}}$ is AFT, $\widetilde{\mathsf{X}_{\mathcal{A}}}$ is closed.

The SFT classification problem

Let X and Y be irreducible shifts of finite type given by graphs G and H, respectively. Determine in terms of G and H when X and Y are conjugate.

Theorem (Williams)

Let X_G and X_H be two irreducible SFTs given by graphs with adjacency matrices A and B, respectively. The following conditions are equivalent.

(i) *X_G and X_H are conjugate.*

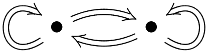
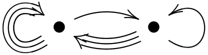
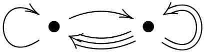
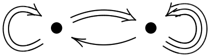
(ii) *There exist nonnegative integral matrices D_i and E_i with*

$$A = D_0E_0, E_0D_0 = D_1E_1, \cdots, E_nD_n = B$$

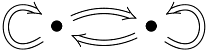
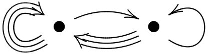
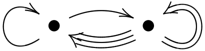
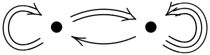
Arsenal of invariants

Real numbers, power series, ordered abelian groups, finitely generated abelian groups, C^*-algebras,...

4 examples

A	G	$h(\mathsf{X}_G)$	$BF(\mathsf{X}_G)$
$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$		4	$(\mathbb{Z}_3, -)$
$\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$		4	$(\mathbb{Z}_3, -)$
$\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$		$\frac{3+\sqrt{13}}{2}$	$(\mathbb{Z}_3, -)$
$\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$		4	$(\mathbb{Z}, 0)$

4 examples

A	G	$h(\mathsf{X}_G)$	$BF(\mathsf{X}_G)$
$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$		4	$(\mathbb{Z}_3, -)$
$\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$		4	$(\mathbb{Z}_3, -)$
$\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$		$\frac{3+\sqrt{13}}{2}$	$(\mathbb{Z}_3, -)$
$\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$		4	$(\mathbb{Z}, 0)$

Theorem (Hamachi-Nasu)

Let X and Y be two irreducible sofic shifts and let $\mathcal{A}, \mathcal{B}$ be their Fischer automata given by alphabetic adjacency matrices A and B. The following conditions are equivalent.

(i) $X \simeq Y$

(ii)
$$\begin{array}{ccc} \mathsf{X}_{\mathcal{A}} & \xrightarrow{\simeq} & \mathsf{X}_{\mathcal{B}} \\ {\scriptstyle\lambda_{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle\lambda_{\mathcal{B}}} \\ \mathsf{L}_{\mathcal{A}} & \xrightarrow{\simeq} & \mathsf{L}_{\mathcal{B}} \end{array}$$

Corollary

When $\mathsf{L}_{\mathcal{A}} \simeq \mathsf{L}_{\mathcal{B}}$, *then* $\mathsf{X}_{\mathcal{A}} \simeq \mathsf{X}_{\mathcal{B}}$.

Example

With $\mathcal{A} = \begin{tikzcd} \bullet \arrow[loop left, "0"] \arrow[r, bend left, "1"] & \bullet \arrow[l, bend left, "0"] \end{tikzcd}$ and $\mathcal{B} = \begin{tikzcd} \bullet \arrow[loop left, "1"] \arrow[r, bend left, "0"] & \bullet \arrow[l, bend left, "0"] \end{tikzcd}$ we get $\widetilde{\mathsf{L}_{\mathcal{A}}} = \emptyset$ and $\widetilde{\mathsf{L}_{\mathcal{B}}} = \{0^\infty\}$.

Hence

$$\begin{tikzcd} \mathsf{X}_{\mathcal{A}} \arrow[r, "\simeq"] \arrow[d, "\lambda_{\mathcal{A}}"'] & \mathsf{X}_{\mathcal{B}} \arrow[d, "\lambda_{\mathcal{B}}"] \\ \mathsf{L}_{\mathcal{A}} \arrow[r, "\simeq"] & \mathsf{L}_{\mathcal{B}} \end{tikzcd}$$

is impossible and $\mathsf{L}_{\mathcal{A}} \not\simeq \mathsf{L}_{\mathcal{B}}$.

Invariant: Fiber product

Definition

The *fiber product* associated to $\lambda : \mathsf{X}_{\mathcal{A}} \to \mathsf{L}_{\mathcal{A}}$ is defined as

$$F[\lambda_{\mathcal{A}}] = \{(x,y) \in \mathsf{X}_{\mathcal{A}}^2 \mid \lambda(x) = \lambda(y)\}$$

Corollary

When $\mathsf{L}_{\mathcal{A}} \simeq \mathsf{L}_{\mathcal{B}}$, *then* $F[\lambda_{\mathcal{A}}] \simeq F[\lambda_{\mathcal{B}}]$.

Lemma

$F[\lambda_{\mathcal{A}}]$ *is a SFT which is reducible unless* $\mathsf{L}_{\mathcal{A}}$ *is an SFT. The diagonal*

$$\Delta = \{(x,x) \mid x \in \mathsf{X}_{\mathcal{A}}\}$$

is an irreducible component of $F[\lambda_{\mathcal{A}}]$. *When* $\mathsf{L}_{\mathcal{A}}$ *is AFT this component communicates with no other irreducible component.*

Outline

Symbol expansion

Fix $a \in \mathfrak{a}$ and $\star \notin \mathfrak{a}$ and define $\eta : \mathfrak{a}^{\mathbb{Z}} \to (\mathfrak{a} \cup \{\star\})^{\mathbb{Z}}$ as the map inserting a $\star$ after each a:

$$\cdots babbbaba \cdots \qquad \mapsto \qquad \cdots ba \star bbba \star ba \star \cdots$$

Definition

The "$a \mapsto a\star$" symbol expansion of a shift space X is the shift space $X_{a \mapsto a\star} = \eta(X)$.

Flow equivalence

Associated to any shift space there is a **suspension flow** given by product topology on

$$SX = \frac{X \times \mathbb{R}}{(x,t) \sim (\sigma(x), t+1)}$$

Definition

X and Y are *flow equivalent* (written $X \simeq_{fe} Y$) when SX and SY are homeomorphic in a way preserving direction in $\mathbb{R}$.

Theorem (Parry-Sullivan)

Flow equivalence is the coarsest equivalence relation containing conjugacy and $X \sim X_{a \to a\star}$

Flow classification

Lemma

If $X \simeq_{fe} Y$ and X is SFT, sofic or irreducible, then so is Y.

The SFT flow classification problem

Let X and Y be irreducible shifts of finite type given by graphs G and H, respectively. Determine in terms of G and H when X and Y are flow equivalent.

The sofic flow classification problem

Let X and Y be irreducible sofic shifts given by Fischer automata $\mathcal{A}$ and $\mathcal{B}$, respectively. Determine in terms of $\mathcal{A}$ and $\mathcal{B}$ when X and Y are flow equivalent.

Flow classifcication of SFTs

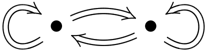
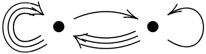
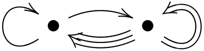
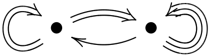
Theorem (Franks)

Let X_G and X_H be two irreducible SFTs given by graphs with adjacency matrices A and B, respectively. The following conditions are equivalent.

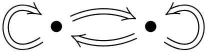
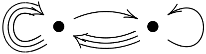
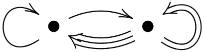
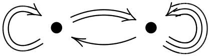
(i) $\mathsf{X}_G \simeq_{fe} \mathsf{X}_H$

(ii)
$$\mathbb{Z}^m/(1-A)\mathbb{Z}^m \simeq \mathbb{Z}^n/(1-B)\mathbb{Z}^n$$
and
$$\operatorname{sgn}\det(1-A) = \operatorname{sgn}\det(1-B)$$

4 examples

A	G	$BF(\mathsf{X}_G)$
$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$		$(\mathbb{Z}_3, -)$
$\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$		$(\mathbb{Z}_3, -)$
$\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$		$(\mathbb{Z}_3, -)$
$\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$		$(\mathbb{Z}, 0)$

4 examples

A	G	$BF(\mathsf{X}_G)$
$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$		$(\mathbb{Z}_3, -)$
$\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$		$(\mathbb{Z}_3, -)$
$\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$		$(\mathbb{Z}_3, -)$
$\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$		$(\mathbb{Z}, 0)$

Flow classifcication of SFTs

Theorem (Boyle-Huang)

The signed K-web is a complete invariant for reducible SFTs.

Theorem (Boyle-Sullivan)

There is a classification theory for equivariant flow equivalence of irreducible SFTs with actions of a finite group G.

Outline

Flow classification of sofics

Theorem

Let X and Y be two irreducible sofic shifts and let $\mathcal{A}, \mathcal{B}$ be their Fischer automata. The following conditions are equivalent.

(i) $X \simeq_{fe} Y$

(ii)
$$\begin{tikzcd} S\mathsf{X}_{\mathcal{A}} \arrow[r, "\sim_+"] \arrow[d, "S\lambda_{\mathcal{A}}"'] & S\mathsf{X}_{\mathcal{B}} \arrow[d, "S\lambda_{\mathcal{B}}"] \\ S\mathsf{L}_{\mathcal{A}} \arrow[r, "\sim_+"] & S\mathsf{L}_{\mathcal{B}} \end{tikzcd}$$

Corollary

If $X \simeq_{fe} Y$ and X is near Markov or AFT, then so is Y.

Flow invariant: n-soficity

Definition

We say that an irreducible sofic shift is n-sofic when for the Fischer cover $\mathcal{A}$, $\lambda : \mathsf{X}_{\mathcal{A}} \to \mathsf{L}_{\mathcal{A}}$ satisfies

$$\max_{x \in \mathsf{L}_{\mathcal{A}}} |\lambda^{-1}(\{x\})| = n$$

Corollary

If $X \simeq_{fe} Y$, and X is n-sofic, then so is Y.

Flow invariant: Fiber product

Corollary

If $\mathsf{L}_{\mathcal{A}} \simeq_{fe} \mathsf{L}_{\mathcal{B}}$, *then* $F[\lambda_{\mathcal{A}}] \simeq_{fe} F[\lambda_{\mathcal{B}}]$

Flow invariant: Multiplicity graph

Collect all periodic words in $\widetilde{\mathsf{X}_{\mathcal{A}}}$ into orbits $\{o_i\}_{i\in I}$ and all periodic words in $\widetilde{\mathsf{L}_{\mathcal{A}}}$ into orbits $\{\omega_j\}_{j\in J}$. Note that a map $\mu : I \to J$ is defined by noting that λ sends o_i to $\omega_{\mu(i)}$. Note also that $|\omega_{\mu(i)}|$ divides $|o_i|$ and set

$$k(i) = \frac{|o_i|}{|\omega_{\mu(i)}|}$$

Definition

The *multiplicity graph* of $\mathcal{A}$ is a bipartite graph $M(\mathcal{A})$ with vertices $I \cup J$ and $k(i)$ edges from i to $\mu(i)$ for each $i \in I$.

Corollary

If $\mathsf{L}_{\mathcal{A}} \simeq_{fe} \mathsf{L}_{\mathcal{B}}$*, then* $M[\lambda_{\mathcal{A}}] \simeq M[\lambda_{\mathcal{B}}]$

Outline

Theorem (Boyle-Carlsen-E)

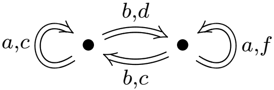
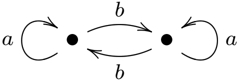
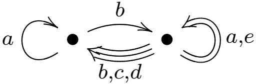
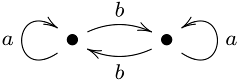
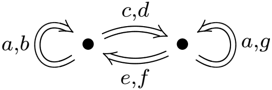
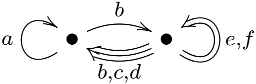
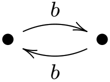
Let X and Y be two irreducible sofic shift spaces with Fischer automata $\mathcal{A}$ and $\mathcal{B}$, respectively, and assume that $\widetilde{\mathsf{X}_\mathcal{A}}$ and $\widetilde{\mathsf{X}_\mathcal{B}}$ are both closed. Then X and Y are flow equivalent exactly when the following conditions hold:

(1) $\mathsf{X}_\mathcal{A} \simeq_{fe} \mathsf{X}_\mathcal{B}$

(2)
$$\begin{tikzcd} S\widetilde{\mathsf{X}_\mathcal{A}} \arrow[r, "\sim+"] \arrow[d, "S\widetilde{\lambda_\mathcal{A}}"'] & S\widetilde{\mathsf{X}_\mathcal{B}} \arrow[d, "S\widetilde{\lambda_\mathcal{B}}"] \\ S\widetilde{\mathsf{L}_\mathcal{A}} \arrow[r, "\sim+"] & S\widetilde{\mathsf{L}_\mathcal{B}} \end{tikzcd}$$

Corollary

Near Markov shifts are classified by the Bowen-Franks invariant of $\mathsf{X}_\mathcal{A}$ and the multiplicity graph $M(\mathcal{A})$.

$\lambda : \mathsf{X}_{\mathcal{A}} \to \mathsf{L}_{\mathcal{A}}$	$\widetilde{\lambda} : \widetilde{\mathsf{X}_{\mathcal{A}}} \to \widetilde{\mathsf{L}_{\mathcal{A}}}$
	
	
	
	

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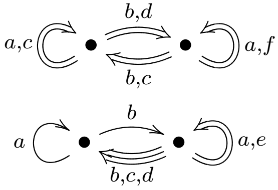	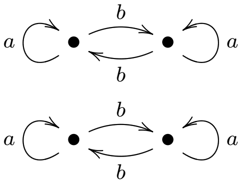
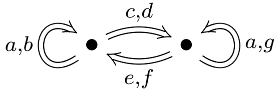	
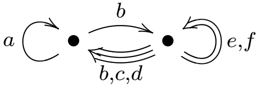	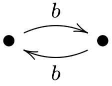

Flow classifcication of SFTs

Classifying 2-sofic AFTs is at least as hard as

Theorem (Boyle-Huang)

The signed K-web is a complete invariant for reducible SFTs.

Theorem (Boyle-Sullivan)

There is a classification theory for equivariant flow equivalence of irreducible SFTs with actions of a finite group G.