Fixed points of endomorphisms over special confluent rewriting systems

Pedro V. Silva

University of Porto
(Centre of Mathematics, Faculty of Sciences)

Vienna, 25th November 2010

A tale of two theorems

Theorem FIN

Let G be a group in class $\mathcal{G}$ and let φ be an endomorphism of G with property $\mathcal{P}$. Then $\mathrm{Fix}\,\varphi$ is f.g.

A tale of two theorems

Theorem FIN

Let G be a group in class $\mathcal{G}$ and let φ be an endomorphism of G with property $\mathcal{P}$. Then $\mathrm{Fix}\,\varphi$ is f.g.

Theorem INF

Let G be a hyperbolic group in class $\mathcal{G}$ and let φ be a monomorphism of G with property $\mathcal{P}$. Let Φ denote the continuous extension of φ to the space of ends of G. Then $\mathrm{Fix}\,\Phi$ is "f.g."

A brief history

	Free groups		
	automorphisms	monomorphisms	endomorphisms
Theorem FIN	Gersten 1984	Goldstein and Turner 1985	Goldstein and Turner 1986
Theorem INF	Cooper 1987	Silva 2009	—

A brief history

	Free groups		
	automorphisms	monomorphisms	endomorphisms
Theorem FIN	Gersten 1984	Goldstein and Turner 1985	Goldstein and Turner 1986
Theorem INF	Cooper 1987	Silva 2009	—

	Free products of cyclic groups		
	automorphisms	monomorphisms	endomorphisms
Theorem FIN	Collins and Turner 1994	Sykiotis 2006	Sykiotis 2006 Silva 2010
Theorem INF	Gaboriau, Jaeger, Levitt and Lustig 1998	Silva 2009	—

A general approach

- In 2004, Cassaigne and the speaker initiated an approach to these problems in the context of the theory of rewriting systems...
- ...and so we could cover groups and monoids

A general approach

- In 2004, Cassaigne and the speaker initiated an approach to these problems in the context of the theory of rewriting systems...
- ...and so we could cover groups and monoids
- On doing so, we can distinguish what is specific of groups and automorphisms (often requiring algebraic geometry techniques)...
- ...from what can be studied within a combinatorial automata-theoretic framework (providing more general results)

Special rewriting systems

A – a finite alphabet

$R \subseteq A^+ \times \{1\}$ – a special rewriting system over A

Special rewriting systems

A – a finite alphabet

$R \subseteq A^+ \times \{1\}$ – a special rewriting system over A

$u \longrightarrow v$ if $u = xry,\ v = xy$ for some $x, y \in A^*$, $(r, 1) \in R$

$\xrightarrow{*}$ – the reflexive and transitive closure of $\longrightarrow$

Special rewriting systems

A – a finite alphabet

$R \subseteq A^+ \times \{1\}$ – a special rewriting system over A

$u \longrightarrow v$ if $u = xry,\ v = xy$ for some $x, y \in A^*$, $(r, 1) \in R$

$\xrightarrow{*}$ – the reflexive and transitive closure of $\longrightarrow$

$R^\sharp$ – congruence generated by R

$M = A^*/R^\sharp$ – the monoid defined by R

Confluency

- A rewriting system R over A is said to be confluent if the diagram

$$\begin{tikzcd} u \arrow[r, "*"] \arrow[d, "*"] & v \arrow[d, dashed, "*"] \\ w \arrow[r, dashed, "*"] & z \end{tikzcd}$$

can always be completed

Confluency

- A rewriting system R over A is said to be confluent if the diagram

$$\begin{tikzcd} u \arrow[r, "*"] \arrow[d, "*"'] & v \arrow[d, dashed, "*"] \\ w \arrow[r, dashed, "*"] & z \end{tikzcd}$$

  can always be completed
- Monoids defined by special confluent rewriting systems (SC monoids) provide models for partial reversibility in Computer Science

Normal forms

- $u \in A^*$ is irreducible if $u \xrightarrow{*} v$ implies $v = u$
- $\overline{u}$ – the unique irreducible word such that $u \xrightarrow{*} \overline{u}$

Normal forms

- $u \in A^*$ is irreducible if $u \xrightarrow{*} v$ implies $v = u$
- $\overline{u}$ – the unique irreducible word such that $u \xrightarrow{*} \overline{u}$
- M has a set of normal forms $A_R^* \subseteq A^*$ consisting of all irreducible words
- $M \cong (A_R^*, \circ)$ for $u \circ v = \overline{uv}$

Normal forms

- $u \in A^*$ is irreducible if $u \xrightarrow{*} v$ implies $v = u$
- $\overline{u}$ – the unique irreducible word such that $u \xrightarrow{*} \overline{u}$
- M has a set of normal forms $A_R^* \subseteq A^*$ consisting of all irreducible words
- $M \cong (A_R^*, \circ)$ for $u \circ v = \overline{uv}$
- A_R^ω – the set of all infinite irreducible words over A
- $A_R^\infty = A_R^* \cup A_R^\omega$

The prefix ultrametric

- For all $\alpha = a_1 a_2 \ldots$, $\beta = b_1 b_2 \ldots \in A_R^\infty$ $(a_i, b_j \in A)$, let

$$r(\alpha,\beta) = \begin{cases} \min\{n \in \mathbb{N} \mid a_n \neq b_n\} & \text{if } \alpha \neq \beta \\ \infty & \text{if } \alpha = \beta \end{cases}$$

and $d(\alpha,\beta) = 2^{-r(\alpha,\beta)}$

The prefix ultrametric

- For all $\alpha = a_1 a_2 \ldots$, $\beta = b_1 b_2 \ldots \in A_R^\infty$ $(a_i, b_j \in A)$, let

$$r(\alpha,\beta) = \begin{cases} \min\{n \in \mathbb{N} \mid a_n \neq b_n\} & \text{if } \alpha \neq \beta \\ \infty & \text{if } \alpha = \beta \end{cases}$$

  and $d(\alpha,\beta) = 2^{-r(\alpha,\beta)}$
- Then (A_R^∞, d) is the completion of (A_R^*, d) and it is compact

The geodesic metric

- The Cayley graph $\Gamma_A(M)$ has vertex set A_R^* and edges

$$u \xrightarrow{a} \overline{ua} \qquad (u \in A_R^*, a \in A)$$

The geodesic metric

- The Cayley graph $\Gamma_A(M)$ has vertex set A_R^* and edges

$$u \xrightarrow{a} \overline{ua} \qquad (u \in A_R^*, a \in A)$$

- The geodesic metric s in A_R^* is defined by the length of the shortest path in the undirected Cayley graph
- Then A_R^ω turns out to be the space of ends of A_R^*:

The hyperbolic topology

Theorem (Cassaigne and Silva 2006)

(i) (A_R^*, s) is a hyperbolic metric space and we can therefore define a metric on its space of ends by means of the Gromov product to get the hyperbolic topology $\mathcal{G}$

(ii) The prefix metric d on A_R^ω induces the hyperbolic topology $\mathcal{G}$

(iii) The monoid A_R^* is word hyperbolic in the sense of Gilman

The hyperbolic topology

Theorem (Cassaigne and Silva 2006)

(i) (A_R^*, s) is a hyperbolic metric space and we can therefore define a metric on its space of ends by means of the Gromov product to get the hyperbolic topology $\mathcal{G}$

(ii) The prefix metric d on A_R^ω induces the hyperbolic topology $\mathcal{G}$

(iii) The monoid A_R^* is word hyperbolic in the sense of Gilman

...so we can get away with undirected Cayley graphs (!)

Continuous extensions

Theorem (Cassaigne and Silva 2006)

Let φ be a nontrivial endomorphism of A_R^*. Then the following conditions are equivalent and decidable:

(i) φ can be extended to a continuous mapping $\Phi : A_R^\infty \to A_R^\infty$

(ii) φ is uniformly continuous for the prefix metric

(iii) $w\varphi^{-1}$ is finite for every $w \in A_R^*$

- $\mathrm{Fix}\varphi = \{u \in A_R^* \mid u\varphi = u\} \leq A_R^*$
- $\mathrm{Fix}\Phi = \{\alpha \in A_R^\infty \mid \alpha\Phi = \alpha\}$

The bounded reduction property

Proposition

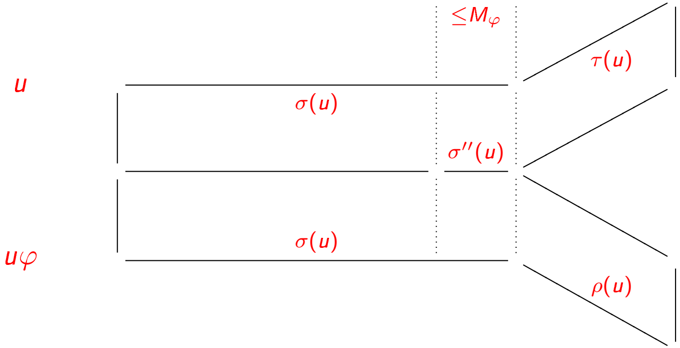
Let $\varphi \in \mathrm{End}A_R^*$ be uniformly continuous for the prefix metric. Then there exists some $M_\varphi > 0$ such that, whenever $uv \in A_R^*$, the reduction of $(u\varphi)(v\varphi)$ involves at most M_φ letters from $u\varphi$ and M_φ letters from $v\varphi$.

$u\varphi$		$v\varphi$	
remains	$\leq M_\varphi$	$\leq M_\varphi$	remains

Proved by Cooper (1987) for free group automorphisms

Defining a quotient

- Fix $\varphi : A_R^* \to A_R^*$ uniformly continuous
- Ladra and Silva 2007:



Defining a quotient

- $\lambda(u)\sigma''(u)\tau(u)$ is the shortest suffix of u satisfying

$$|\lambda(u)\sigma''(u)\tau(u)| \geq t_R - 1 \quad \text{or} \quad \lambda(u)\sigma''(u)\tau(u) = u$$

- ...so that $\lambda(u)$ is a suffix of $\sigma'(u)$ of bounded length

Defining a quotient

- $\lambda(u)\sigma''(u)\tau(u)$ is the shortest suffix of u satisfying

$$|\lambda(u)\sigma''(u)\tau(u)| \geq t_R - 1 \quad \text{or} \quad \lambda(u)\sigma''(u)\tau(u) = u$$

- ...so that $\lambda(u)$ is a suffix of $\sigma'(u)$ of bounded length
- write $C(u) = (\lambda(u), \sigma''(u), \tau(u), \rho(u))$

Defining a quotient

- $\lambda(u)\sigma''(u)\tau(u)$ is the shortest suffix of u satisfying

$$|\lambda(u)\sigma''(u)\tau(u)| \geq t_R - 1 \quad \text{or} \quad \lambda(u)\sigma''(u)\tau(u) = u$$

- ...so that $\lambda(u)$ is a suffix of $\sigma'(u)$ of bounded length
- write $C(u) = (\lambda(u), \sigma''(u), \tau(u), \rho(u))$

Lemma

Let $u, v \in A_R^*$ be such that $C(u) = C(v)$ and $ua \in A_R^*$. Then $va \in A_R^*$ and $C(ua) = C(va)$.

An automaton for FixΦ...

We build a deterministic A-automaton $\mathcal{A}'_\varphi = (Q', q_0, T', E')$ by taking

- $Q' = \{C(u) \mid u \leq \alpha$ for some $\alpha \in \mathrm{Fix}\Phi\}$
- $q_0 = C(1)$
- $T' = \{C(u) \in Q' \mid \tau(u) = \rho(u) = 1\}$
- $E' = \{(C(u), a, C(v)) \in Q' \times A \times Q' \mid v = ua\}$

An automaton for FixΦ...

We build a deterministic A-automaton $\mathcal{A}'_\varphi = (Q', q_0, T', E')$ by taking

- $Q' = \{C(u) \mid u \leq \alpha$ for some $\alpha \in \mathrm{Fix}\Phi\}$
- $q_0 = C(1)$
- $T' = \{C(u) \in Q' \mid \tau(u) = \rho(u) = 1\}$
- $E' = \{(C(u), a, C(v)) \in Q' \times A \times Q' \mid v = ua\}$

Proposition

$\mathrm{Fix}\Phi = L_\infty(\mathcal{A}'_\varphi)$

...where $L_\omega(\mathcal{A}'_\varphi)$ is the set of labels of infinite paths $q_0 \xrightarrow{\alpha} \dots$ and $L_\infty(\mathcal{A}'_\varphi) = L(\mathcal{A}'_\varphi) \cup L_\omega(\mathcal{A}'_\varphi)$

...and another for Fixφ

- Let $S = \{q \in Q' \mid q$ has outdegree $\geq 2\}$
- Let $Q = \{q \in Q' \mid \exists$ a path $q \longrightarrow p \in S \cup T'\}$
- We define $\mathcal{A}_\varphi = (Q, q_0, T, E)$ by taking

$$T = T' \cap Q, \quad E = E' \cap (Q \times A \times Q)$$

...and another for Fixφ

- Let $S = \{q \in Q' \mid q$ has outdegree $\geq 2\}$
- Let $Q = \{q \in Q' \mid \exists$ a path $q \longrightarrow p \in S \cup T'\}$
- We define $\mathcal{A}_\varphi = (Q, q_0, T, E)$ by taking

$$T = T' \cap Q, \quad E = E' \cap (Q \times A \times Q)$$

Proposition

Fix$\varphi = L(\mathcal{A}_\varphi)$

Finite-splitting

We say that φ is finite-splitting if S is finite

Theorem FIN

If φ is a finite-splitting uniformly continuous endomorphism, then $\mathrm{Fix}\,\varphi$ is rational

Finite-splitting

We say that φ is finite-splitting if S is finite

Theorem FIN

If φ is a finite-splitting uniformly continuous endomorphism, then $\mathrm{Fix}\,\varphi$ is rational

Theorem INF

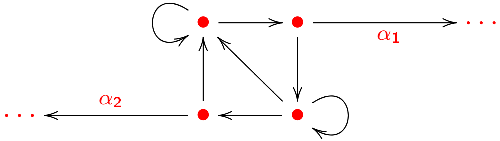
If φ is a finite-splitting uniformly continuous endomorphism, then there exist $L, L_1, \ldots, L_s \in \mathrm{Rat}\, A^*$ and $\alpha_1, \ldots, \alpha_s \in A_R^\omega$ such that

$$\mathrm{Fix}\,\Phi = L^c \;\cup\; L_1\alpha_1 \;\cup\; \ldots \;\cup\; L_s\alpha_s,$$

where L^c denotes the topological closure of L

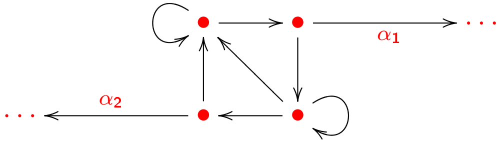
$\mathcal{A}'_\varphi$ and $\mathcal{A}_\varphi$

- $\mathcal{A}'_\varphi$ is a finite automaton with a few infinite hairs adjoined



$\mathcal{A}'_\varphi$ and $\mathcal{A}_\varphi$

- $\mathcal{A}'_\varphi$ is a finite automaton with a few infinite hairs adjoined



- To get $\mathcal{A}_\varphi$, we remove the hairs from the terminal vertices onwards and eventually a few other vertices
- The (infinite) hairs correspond to (Lyapunov stable) attractors or repellers

What is finite-splitting anyway?

- Finite-splitting is a property of infinite automata which are not constructible...
- ...so who can tell when does such a property occur?

What is finite-splitting anyway?

- Finite-splitting is a property of infinite automata which are not constructible...
- ...so who can tell when does such a property occur?
- We can't!
- So the plan is to identify nice subclasses of finite-splitting endomorphisms with good properties

Boundary-injectivity

- We say that a uniformly continuous endomorphism φ of A_R^* is boundary-injective if the extension Φ is injective

Boundary-injectivity

- We say that a uniformly continuous endomorphism φ of A_R^* is boundary-injective if the extension Φ is injective
- A monomorphism needs not to be boundary-injective

Theorem

If φ is boundary-injective, then it is uniformly continuous and finite-splitting

Boundary-injectivity

- We say that a uniformly continuous endomorphism φ of A_R^* is boundary-injective if the extension Φ is injective
- A monomorphism needs not to be boundary-injective

Theorem

If φ is boundary-injective, then it is uniformly continuous and finite-splitting

Theorem

Given a uniformly continuous endomorphism φ of A_R^*, it is decidable whether or not φ is injective or boundary-injective

Bounded length decrease

- An endomorphism φ of A_R^* has bounded length decrease if

$$\exists d_\varphi \in \mathbb{N}\ \forall u \in A_R^* \qquad |u| - |u\varphi| \leq d_\varphi$$

Bounded length decrease

- An endomorphism φ of A_R^* has bounded length decrease if

$$\exists d_\varphi \in \mathbb{N}\ \forall u \in A_R^* \qquad |u| - |u\varphi| \leq d_\varphi$$

- d_φ can be arbitrarily large

Theorem

If φ has bounded length decrease, then it is uniformly continuous and finite-splitting

This subclass is even nicer!

Theorem

Given a uniformly continuous endomorphism φ of A_R^*, it is decidable whether or not φ has bounded length decrease. If this is the case, d_φ can be effectively computed.

This subclass is even nicer!

Theorem

Given a uniformly continuous endomorphism φ of A_R^*, it is decidable whether or not φ has bounded length decrease. If this is the case, d_φ can be effectively computed.

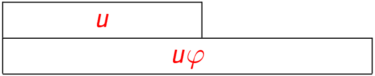
Theorem

Given an endomorphism φ of A_R^* having bounded length decrease, we can effectively construct finite A-automata $\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_s$ and $\alpha_1, \ldots, \alpha_s \in A_R^\omega$ such that

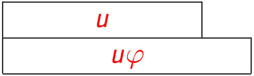
(i) $\mathrm{Fix}\,\varphi = L(\mathcal{A})$

(ii) $\mathrm{Fix}\,\Phi = L_\infty(\mathcal{A}) \cup L(\mathcal{A}_1)\alpha_1 \cup \ldots \cup L(\mathcal{A}_s)\alpha_s$

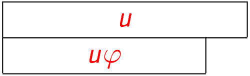
How do we get finite-splitting?



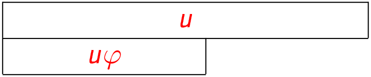
no elements in S



bounded number of cases



bounded number of cases



decider

Which groups are covered by our results?

Not too many...

Proposition

The SC groups are precisely the free products of cyclic groups

Which groups are covered by our results?

Not too many...

Proposition

The SC groups are precisely the free products of cyclic groups

...but still we can get some new group theory results

An alternative proof...

We can get an alternative proof for

Theorem FIN (Sykiotis 2006)

Let φ be a monomorphism of a f.g. SC group. Then $\mathrm{Fix}\,\varphi$ is f.g.

An alternative proof...

We can get an alternative proof for

Theorem FIN (Sykiotis 2006)

Let φ be a monomorphism of a f.g. SC group. Then $\mathrm{Fix}\,\varphi$ is f.g.

We use the following:

- every monomorphism of such a group is boundary-injective
- every rational subgroup of a group is f.g. (Anisimov and Seifert's Theorem)

...and a new result

Theorem INF

Let φ be a monomorphism of a f.g. SC group. Then $\mathrm{Fix}\,\Phi = (\mathrm{Fix}\,\varphi)^c \cup (\mathrm{Fix}\,\varphi)X$ for some finite $X \subseteq \mathrm{Fix}\,\Phi$.

...and a new result

Theorem INF

Let φ be a monomorphism of a f.g. SC group. Then $\mathrm{Fix}\,\Phi = (\mathrm{Fix}\,\varphi)^c \cup (\mathrm{Fix}\,\varphi)X$ for some finite $X \subseteq \mathrm{Fix}\,\Phi$.

- Apparently, this is new even for free group monomorphisms
- The automorphism case (for free products of cyclic groups) is due to Gaboriau, Jaeger, Levitt and Lustig (1998)

Embedding monoids into groups

- A monoid M is directly finite if

$$\forall x, y \in M \quad xy = 1 \Rightarrow yx = 1$$

Embedding monoids into groups

- A monoid M is directly finite if

$$\forall x, y \in M \quad xy = 1 \Rightarrow yx = 1$$

- The bicyclic monoid is defined by $\langle a, b \mid ab \to 1 \rangle$

Embedding monoids into groups

- A monoid M is directly finite if

$$\forall x, y \in M \quad xy = 1 \Rightarrow yx = 1$$

- The bicyclic monoid is defined by $\langle a, b \mid ab \to 1 \rangle$

Proposition

The following conditions are equivalent for an SC monoid M:

(i) M is embeddable into some group
(ii) M contains no bicyclic submonoid
(iii) M is directly finite
(iv) M is a free product of a free monoid and cyclic groups

Introducing elements of finite order

- We can adapt Goldstein and Turner's automata-theoretic proof from free groups to SC groups
- The theorem follows also from the work of Sykiotis, which used different techniques

Introducing elements of finite order

- We can adapt Goldstein and Turner's automata-theoretic proof from free groups to SC groups
- The theorem follows also from the work of Sykiotis, which used different techniques
- We may assume that $A = A_0 \cup A_1 \cup A_1^{-1}$ and there exist $m_a \geq 2$ for every $a \in A_0$ such that

$$R = \{(a^{m_a}, 1) \mid a \in A_0\} \cup \{(aa^{-1}, 1), (a^{-1}a, 1) \mid a \in A_1\}$$

- For every $a \in A_0$, write $a^{-1} = a^{m_a - 1}$

Building automata

- For every $g \in G$, let $Q(g) = \overline{g^{-1}(g\varphi)}$
- Then $g \in \mathrm{Fix}\varphi$ if and only if $Q(g) = 1$

Building automata

- For every $g \in G$, let $Q(g) = \overline{g^{-1}(g\varphi)}$
- Then $g \in \mathrm{Fix}\varphi$ if and only if $Q(g) = 1$
- We define an A-automaton $\mathcal{A}_\varphi = (Q, 1, 1, E)$ by

$$Q = \{Q(g) \mid g \in G\}$$
$$E = \{(Q(g), a, Q(ga)) \mid g \in G, \; a \in A\}$$

- Clearly, $\mathcal{A}_\varphi$ is a complete accessible deterministic automaton and

$$L(\mathcal{A}_\varphi) = (\mathrm{Fix}\varphi)\pi^{-1}$$

Building automata

- For every $g \in G$, let $Q(g) = \overline{g^{-1}(g\varphi)}$
- Then $g \in \mathrm{Fix}\varphi$ if and only if $Q(g) = 1$
- We define an A-automaton $\mathcal{A}_\varphi = (Q, 1, 1, E)$ by

$$Q = \{Q(g) \mid g \in G\}$$
$$E = \{(Q(g), a, Q(ga)) \mid g \in G,\ a \in A\}$$

- Clearly, $\mathcal{A}_\varphi$ is a complete accessible deterministic automaton and

$$L(\mathcal{A}_\varphi) = (\mathrm{Fix}\varphi)\pi^{-1}$$

- We define a subautomaton $\mathcal{A}'_\varphi = (Q, 1, 1, E')$ through

$$E' = \{(p, a, q) \in E \mid aq \text{ is irreducible}\}$$

The results

- Let $d_\varphi = \max\{|a\varphi|;\ a \in A\}$ and $m_R = \max\{m_a \mid a \in A_0\}$

The results

- Let $d_\varphi = \max\{|a\varphi|;\ a \in A\}$ and $m_R = \max\{m_a \mid a \in A_0\}$
- If $|q| > (m_R - 1)d_\varphi$, then q has outdegree ≤ 1 in $\mathcal{A}'_\varphi$

The results

- Let $d_\varphi = \max\{|a\varphi|;\ a \in A\}$ and $m_R = \max\{m_a \mid a \in A_0\}$
- If $|q| > (m_R - 1)d_\varphi$, then q has outdegree ≤ 1 in $\mathcal{A}'_\varphi$
- If $(p, a, q) \in E \setminus E'$ and no path of length $\leq s - m_R$ connects 1 to p or q in $\mathcal{A}_\varphi$, then there exists a path $q \xrightarrow{a^{-1}} p$ in $\mathcal{A}'_\varphi$

The results

- Let $d_\varphi = \max\{|a\varphi|;\ a \in A\}$ and $m_R = \max\{m_a \mid a \in A_0\}$
- If $|q| > (m_R - 1)d_\varphi$, then q has outdegree ≤ 1 in $\mathcal{A}'_\varphi$
- If $(p, a, q) \in E \setminus E'$ and no path of length $\leq s - m_R$ connects 1 to p or q in $\mathcal{A}_\varphi$, then there exists a path $q \xrightarrow{a^{-1}} p$ in $\mathcal{A}'_\varphi$

Theorem

Let φ be an endomorphism of a f.g. SC group. Then Fixφ is f.g.

The results

- Let $d_\varphi = \max\{|a\varphi|;\ a \in A\}$ and $m_R = \max\{m_a \mid a \in A_0\}$
- If $|q| > (m_R - 1)d_\varphi$, then q has outdegree ≤ 1 in $\mathcal{A}'_\varphi$
- If $(p, a, q) \in E \setminus E'$ and no path of length $\leq s - m_R$ connects 1 to p or q in $\mathcal{A}_\varphi$, then there exists a path $q \xrightarrow{a^{-1}} p$ in $\mathcal{A}'_\varphi$

Theorem

Let φ be an endomorphism of a f.g. SC group. Then $\mathrm{Fix}\varphi$ is f.g.

Corollary

Let φ be an endomorphism of a f.g. free product of a free monoid and cyclic groups. Then $\mathrm{Fix}\varphi$ is rational.

Free monoid endomorphisms

- Given $\varphi \in \text{End}A^*$, write $m = |A|$ and define

$$A_2 = \{a \in A \mid a\varphi^n = 1 \text{ for some } n \geq 1\}$$
$$A_3 = A \setminus A_2$$
$$A_4 = \{a \in A_3 \mid a\varphi \in A_2^* a A_2^*\}$$

Free monoid endomorphisms

- Given $\varphi \in \mathrm{End} A^*$, write $m = |A|$ and define
$$A_2 = \{a \in A \mid a\varphi^n = 1 \text{ for some } n \geq 1\}$$
$$A_3 = A \setminus A_2$$
$$A_4 = \{a \in A_3 \mid a\varphi \in A_2^* a A_2^*\}$$
- Let Γ be the directed graph with vertex set A and edges $a \longrightarrow b$ whenever b occurs in $a\varphi$
- Then $a \in A_2$ iff there exists no infinite path $a \longrightarrow \cdots$ in Γ
- This is equivalent to say there is no path $a \longrightarrow \cdots$ in Γ of length m, hence

$$A_2 = \{a \in A \mid a\varphi^m = 1\}$$

Free monoid endomorphisms

- Given $\varphi \in \text{End}A^*$, write $m = |A|$ and define
$$A_2 = \{a \in A \mid a\varphi^n = 1 \text{ for some } n \geq 1\}$$
$$A_3 = A \setminus A_2$$
$$A_4 = \{a \in A_3 \mid a\varphi \in A_2^* a A_2^*\}$$
- Let Γ be the directed graph with vertex set A and edges $a \longrightarrow b$ whenever b occurs in $a\varphi$
- Then $a \in A_2$ iff there exists no infinite path $a \longrightarrow \cdots$ in Γ
- This is equivalent to say there is no path $a \longrightarrow \cdots$ in Γ of length m, hence

$$A_2 = \{a \in A \mid a\varphi^m = 1\}$$

- Therefore A_2 is effectively computable, and so are A_3 and A_4

Free monoid endomorphisms

Given $B \subseteq A$, we denote by $\theta_{A,B}$ the retraction endomorphism $A^* \to B^*$ defined by

$$a\theta = \begin{cases} a & \text{if } a \in B \\ 1 & \text{otherwise} \end{cases}$$

Free monoid endomorphisms

Given $B \subseteq A$, we denote by $\theta_{A,B}$ the retraction endomorphism $A^* \to B^*$ defined by

$$a\theta = \begin{cases} a & \text{if } a \in B \\ 1 & \text{otherwise} \end{cases}$$

Lemma

Let $\varphi \in \mathrm{End}A^*$ and $m = |A|$. Then $\mathrm{Fix}\varphi = (A_4\varphi^m)^*$.

Free monoid endomorphisms

Given $B \subseteq A$, we denote by $\theta_{A,B}$ the retraction endomorphism $A^* \to B^*$ defined by

$$a\theta = \begin{cases} a & \text{if } a \in B \\ 1 & \text{otherwise} \end{cases}$$

Lemma

Let $\varphi \in \mathrm{End}A^*$ and $m = |A|$. Then $\mathrm{Fix}\varphi = (A_4\varphi^m)^*$.

- Note that, given an endomorphism φ of $A^* * G$, where G is a group, the restriction $\varphi|_G$ is an endomorphism of G
- Clearly, G is the (unique) maximal subgroup of $A^* * G$

Computing the fixed points

Theorem

Let $M = A_0^* * G$ be f.g., where G is an SC group. Let $\varphi \in \mathrm{End}\, M$ be such that the equation

$$x = v(x\varphi|_G)w \quad (x \in G)$$

has an effectively constructible rational solution set for all $v, w \in G$. Then $\mathrm{Fix}\varphi$ is an effectively constructible rational submonoid of M.

Computing the fixed points

Theorem

Let $M = A_0^* * G$ be f.g., where G is an SC group. Let $\varphi \in \mathrm{End}\, M$ be such that the equation

$$x = v(x\varphi|_G)w \quad (x \in G)$$

has an effectively constructible rational solution set for all $v, w \in G$. Then $\mathrm{Fix}\varphi$ is an effectively constructible rational submonoid of M.

Corollary

Let $M = A_0^* * G$ be f.g., where G is a free group. Let $\varphi \in \mathrm{End}\, M$ be such that $\varphi|_G$ is an automorphism. Then $\mathrm{Fix}\varphi$ is an effectively constructible rational submonoid of M.

Final notes

- This result generalizes Maslakova's Theorem (2003): if φ is a free group automorphism, then $\mathrm{Fix}\varphi$ is an effectively constructible f.g. subgroup

Final notes

- This result generalizes Maslakova's Theorem (2003): if φ is a free group automorphism, then $\mathrm{Fix}\varphi$ is an effectively constructible f.g. subgroup
- If M is a f.g. free product of a free monoid and a free group, then $\mathrm{Fix}\varphi$ is a rational submonoid but not necessarily a f.g. submonoid of M
- This contrasts the case of both free monoids and free groups, when $\mathrm{Fix}\varphi$ is f.g.