```
In[1]:= Clear["Global`*"]; z = 1 + I;
        {Re[z], Im[z], Abs[z], Arg[z], Conjugate[z]}
```

Out[2]= $\{1, 1, \sqrt{2}, \frac{\pi}{4}, 1 - i\}$

```
In[3]:= expression = Expand[(x + I * y)^3]
```

Out[3]= $x^3 + 3\, i\, x^2\, y - 3\, x\, y^2 - i\, y^3$

```
In[4]:= Re[expression]
```

Out[4]= $-3\, \mathrm{Im}[x^2\, y] + \mathrm{Im}[y^3] + \mathrm{Re}[x^3 - 3\, x\, y^2]$

```
In[5]:= Im[expression]
```

Out[5]= $\mathrm{Im}[x^3 - 3\, x\, y^2] + 3\, \mathrm{Re}[x^2\, y] - \mathrm{Re}[y^3]$

```
In[6]:= re[w_] := ComplexExpand[Re[w]];
        im[w_] := ComplexExpand[Im[w]];
        {re[expression], im[expression]}
```

Out[8]= $\{x^3 - 3\, x\, y^2, 3\, x^2\, y - y^3\}$

```
In[9]:= ComplexExpand[(Cos[θ] + I * Sin[θ])^5]
```

Out[9]= $\mathrm{Cos}[\theta]^5 - 10\, \mathrm{Cos}[\theta]^3\, \mathrm{Sin}[\theta]^2 + 5\, \mathrm{Cos}[\theta]\, \mathrm{Sin}[\theta]^4 + i\, (5\, \mathrm{Cos}[\theta]^4\, \mathrm{Sin}[\theta] - 10\, \mathrm{Cos}[\theta]^2\, \mathrm{Sin}[\theta]^3 + \mathrm{Sin}[\theta]^5)$

```
In[10]:= Expand[(Cos[θ]^5 - 10 Cos[θ]^3 Sin[θ]^2 + 5 Cos[θ] Sin[θ]^4) /. {Sin[θ]^2 → (1 - Cos[θ]^2), Sin[θ]^4 → (1 - Cos[θ]^2)^2}]
```

Out[10]= $5\, \mathrm{Cos}[\theta] - 20\, \mathrm{Cos}[\theta]^3 + 16\, \mathrm{Cos}[\theta]^5$

```
In[11]:= TrigExpand[Cos[5 θ]]
```

Out[11]= $\mathrm{Cos}[\theta]^5 - 10\, \mathrm{Cos}[\theta]^3\, \mathrm{Sin}[\theta]^2 + 5\, \mathrm{Cos}[\theta]\, \mathrm{Sin}[\theta]^4$

```
In[12]:= Expand[Cos[5 θ], Trig → True]
```

Out[12]= $\mathrm{Cos}[\theta]^5 - 10\, \mathrm{Cos}[\theta]^3\, \mathrm{Sin}[\theta]^2 + 5\, \mathrm{Cos}[\theta]\, \mathrm{Sin}[\theta]^4$

In[13]:=
```
Expand[% /. Sin[θ]^k_ → (1 - Cos[θ]^2)^(k/2)]
```

Out[13]= $5\,\mathrm{Cos}[\theta] - 20\,\mathrm{Cos}[\theta]^3 + 16\,\mathrm{Cos}[\theta]^5$

In[14]:=
```
TrigReduce[5 Cos[θ] - 20 Cos[θ]^3 + 16 Cos[θ]^5]
```

Out[14]= $\mathrm{Cos}[5\,\theta]$

In[15]:=
```
TrigFactor[Cos[5 θ]]
```

Out[15]= $\mathrm{Cos}[\theta]\,(1 - 2\,\mathrm{Cos}[2\,\theta] + 2\,\mathrm{Cos}[4\,\theta])$

In[16]:=
```
Solve[w^5 - 1 == 0, w]
```

Out[16]= $\{\{w \to 1\}, \{w \to -(-1)^{1/5}\}, \{w \to (-1)^{2/5}\}, \{w \to -(-1)^{3/5}\}, \{w \to (-1)^{4/5}\}\}$

In[17]:=
```
Table[Cos[2*k*π/5] + I*Sin[2*k*π/5], {k, 0, 4, 1}]
```

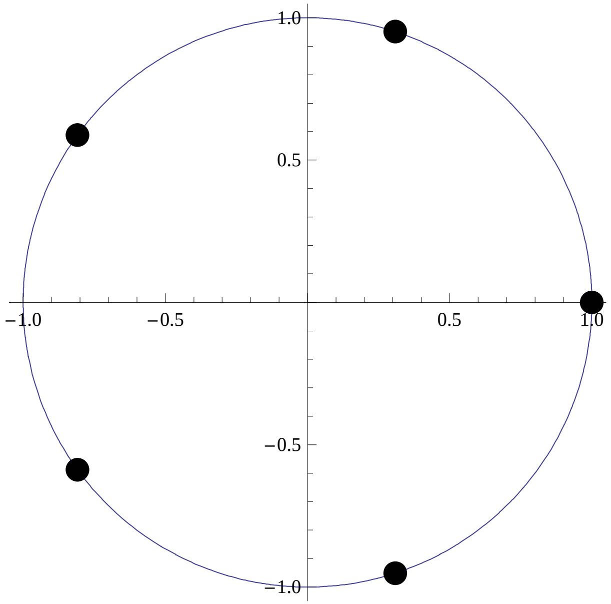
Out[17]= $\left\{1,\ i\sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}}+\frac{1}{4}\left(-1+\sqrt{5}\right),\ \frac{1}{4}\left(-1-\sqrt{5}\right)+i\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}},\ \frac{1}{4}\left(-1-\sqrt{5}\right)-i\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}},\ -i\sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}}+\frac{1}{4}\left(-1+\sqrt{5}\right)\right\}$

In[18]:=
```
complexPlot[z_List] := Module[{points}, points = Map[{Re[#], Im[#]} &, z];
   ParametricPlot[{Cos[θ], Sin[θ]}, {θ, -π, π}, AspectRatio -> 1,
    PlotRange → {{-1.05, 1.05}, {-1.05, 1.05}},
    PlotRegion → {{0.03, 0.97}, {0.03, 0.97}},
    Epilog → {PointSize[0.04], Map[Point, points]}]];
```

$$w = \frac{1}{4}\left(-1+\sqrt{5}\right)+\frac{1}{2}\, i \sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)};$$

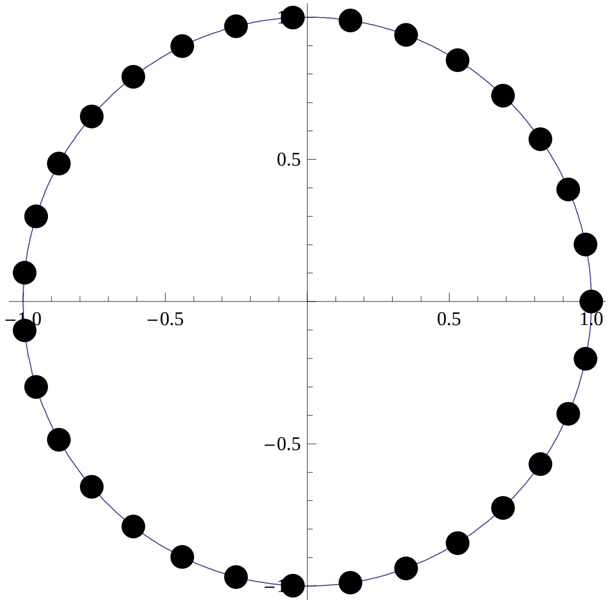
```
complexPlot[{1, w, w^2, w^3, w^4}]
```

Out[20]=



In[21]:=
```
showNthRoots[n_] := Module[{w = Cos[2 * π / n] + I * Sin[2 * π / n]},
   complexPlot[Table[w^m, {m, 1, n}]]];
showNthRoots[31]
```

Out[22]=



In[23]:=
```
Collect[Expand[x^3 + a x^2 + b x + c /. x → X + A], X]
```

Out[23]= $a A^2 + A^3 + A b + c + (2 a A + 3 A^2 + b) X + (a + 3 A) X^2 + X^3$

In[24]:=
```
w = Exp[ⅈ * 2 * π / 3];
Map[{Re[#] + I * Im[#]} &, {w, w + w^2 + w^3}]
```

Out[25]= $\left\{\left\{-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right\}, \{0\}\right\}$

In[26]:=
```
z1 = x - α - β * w^2;
z2 = x - α * w - β * w;
z3 = x - α * w^2 - β;
Simplify[Collect[Expand[z1 * z2 * z3], x]]
```

Out[29]= $x^3 - \alpha^3 + \frac{3}{2}\left(1 + i\sqrt{3}\right) x\,\alpha\,\beta - \beta^3$

In[30]:=
```
expression1 = Expand[Simplify[-α^3 - β^3 - b /. β → -2 a / ((1 + i √3) α)]]
```

Out[30]= $-b - \frac{a^3}{\alpha^3} - \alpha^3$

In[31]:=
```
expression2 = expression1 /. {α^3 → λ, 1 / α^3 → 1 / λ}
```

Out[31]= $-b - \frac{a^3}{\lambda} - \lambda$

In[32]:=
```
Solve[expression2 == 0, λ]
```

Out[32]= $\left\{\left\{\lambda \to \frac{1}{2}\left(-b - \sqrt{-4 a^3 + b^2}\right)\right\}, \left\{\lambda \to \frac{1}{2}\left(-b + \sqrt{-4 a^3 + b^2}\right)\right\}\right\}$

In[33]:=
```
solα = α /. Solve[expression1 == 0, α]
```

Out[33]= $\left\{-\left(-\frac{1}{2}\right)^{1/3}\left(-b - \sqrt{-4 a^3 + b^2}\right)^{1/3}, \frac{\left(-b - \sqrt{-4 a^3 + b^2}\right)^{1/3}}{2^{1/3}}, \frac{(-1)^{2/3}\left(-b - \sqrt{-4 a^3 + b^2}\right)^{1/3}}{2^{1/3}}, \left(-\frac{b}{2} + \frac{1}{2}\sqrt{-4 a^3 + b^2}\right)^{1/3}, -(-1)^{1/3}\left(-\frac{b}{2} + \frac{1}{2}\sqrt{-4 a^3 + b^2}\right)^{1/3}, (-1)^{2/3}\left(-\frac{b}{2} + \frac{1}{2}\sqrt{-4 a^3 + b^2}\right)^{1/3}\right\}$

In[34]:=
```
solβ = Map[-2 a / ((1 + i Sqrt[3]) #) &, solα]
```

Out[34]= $\left\{-\frac{2\,(-1)^{2/3}\,2^{1/3}\,a}{\left(1+i\sqrt{3}\right)\left(-b-\sqrt{-4a^3+b^2}\right)^{1/3}},\ -\frac{2\,2^{1/3}\,a}{\left(1+i\sqrt{3}\right)\left(-b-\sqrt{-4a^3+b^2}\right)^{1/3}},\ \frac{2\,(-2)^{1/3}\,a}{\left(1+i\sqrt{3}\right)\left(-b-\sqrt{-4a^3+b^2}\right)^{1/3}},\ -\frac{2\,a}{\left(1+i\sqrt{3}\right)\left(-\frac{b}{2}+\frac{1}{2}\sqrt{-4a^3+b^2}\right)^{1/3}},\ -\frac{2\,(-1)^{2/3}\,a}{\left(1+i\sqrt{3}\right)\left(-\frac{b}{2}+\frac{1}{2}\sqrt{-4a^3+b^2}\right)^{1/3}},\ \frac{2\,(-1)^{1/3}\,a}{\left(1+i\sqrt{3}\right)\left(-\frac{b}{2}+\frac{1}{2}\sqrt{-4a^3+b^2}\right)^{1/3}}\right\}$

In[35]:=
```
Clear["Global`*"];
quartic = x^4 + p * x^2 + q * x + r;
lhs = x^4 + p * x^2; rhs = -q * x - r;
```

In[38]:=
```
lhs == rhs
```

Out[38]= $p\,x^2 + x^4 == -r - q\,x$

In[39]:=
```
lhs1 = lhs + p * x^2 + p^2;
rhs1 = rhs + p * x^2 + p^2;
Factor[lhs1]
```

Out[41]= $\left(p + x^2\right)^2$

In[42]:=
```
rhs1
```

Out[42]= $p^2 - r - q\,x + p\,x^2$

In[43]:=
```
lhs2 = lhs1 + 2 * z (p + x^2) + z^2;
rhs2 = rhs1 + 2 * z (p + x^2) + z^2;
Factor[lhs2]
```

Out[45]= $\left(p + x^2 + z\right)^2$

In[46]:=
```
a = Coefficient[rhs2, x^2]
```

Out[46]= $p + 2\,z$

In[47]:= b = Coefficient[rhs2, x]

Out[47]= $-q$

In[48]:= c = Expand[rhs2 - a * x^2 - b * x]

Out[48]= $p^2 - r + 2\,p\,z + z^2$

In[49]:= Collect[b^2 - 4 * a * c, z]

Out[49]= $-4\,p^3 + q^2 + 4\,p\,r + \left(-16\,p^2 + 8\,r\right)\,z - 20\,p\,z^2 - 8\,z^3$

In[50]:=
```
lhs = x^4 - 10 x^2; rhs = -8 * x - 5;
lhs1 = lhs - 10 * x^2 + 10^2; rhs1 = rhs - 10 * x^2 + 10^2;
Factor[lhs1]
```

Out[52]= $\left(-10 + x^2\right)^2$

In[53]:=
```
lhs2 = lhs1 + 2 * z (-10 + x^2) + z^2;
rhs2 = rhs1 + 2 * z (-10 + x^2) + z^2;
Factor[lhs2]
```

Out[55]= $\left(-10 + x^2 + z\right)^2$

In[56]:= a = Coefficient[rhs2, x^2]

Out[56]= $-10 + 2\,z$

In[57]:= b = Coefficient[rhs2, x]

Out[57]= -8

In[58]:= c = Expand[rhs2 - a * x^2 - b * x]

Out[58]= $95 - 20\,z + z^2$

In[59]:= cubic = Collect[b^2 - 4 * a * c, z]

Out[59]= $3864 - 1560\,z + 200\,z^2 - 8\,z^3$

In[60]:=
```
reducedcubic = Collect[cubic /. z → Z + 200 / 24, Z]
```

Out[60]= $\frac{3328}{27} + \frac{320\,Z}{3} - 8\,Z^3$

In[61]:=
```
solutionZ = Z /. Solve[reducedcubic == 0, Z]
```

Out[61]= $\left\{-\frac{4}{3},\ \frac{2}{3}\left(1 - 3\sqrt{3}\right),\ \frac{2}{3}\left(1 + 3\sqrt{3}\right)\right\}$

In[62]:=
```
z = solutionZ[[1]] + 200 / 24
```

Out[62]= 7

In[63]:=
```
lhs3 = Factor[lhs2]
```

Out[63]= $\left(-3 + x^2\right)^2$

In[64]:=
```
rhs3 = Factor[rhs2]
```

Out[64]= $4\,(-1 + x)^2$

In[65]:=
```
quad1 = PowerExpand[Sqrt[lhs3] == Sqrt[rhs3]]
```

Out[65]= $-3 + x^2 == 2\,(-1 + x)$

In[66]:=
```
quad2 = PowerExpand[Sqrt[lhs3] == -Sqrt[rhs3]]
```

Out[66]= $-3 + x^2 == -2\,(-1 + x)$

In[67]:=
```
solutionx1 = x /. Solve[quad1, x]
```

Out[67]= $\left\{1 - \sqrt{2},\ 1 + \sqrt{2}\right\}$

In[68]:=
```
solutionx2 = x /. Solve[quad2, x]
```

Out[68]= $\left\{-1 - \sqrt{6},\ -1 + \sqrt{6}\right\}$

In[69]:=
```
Simplify[lhs - rhs /. {x → solutionx1}]
```

Out[69]= {0, 0}

In[70]:=
```
Simplify[lhs - rhs /. {x → solutionx2}]
```

Out[70]= {0, 0}

In[71]:=
```
Solve[lhs == rhs, x]
```

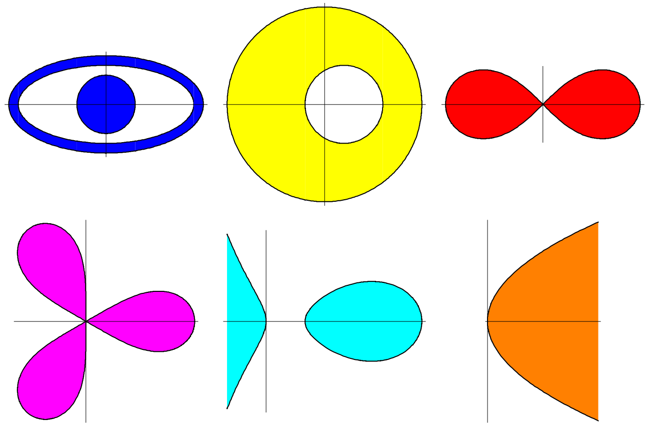
Out[71]= $\{\{x \to 1 - \sqrt{2}\}, \{x \to 1 + \sqrt{2}\}, \{x \to -1 - \sqrt{6}\}, \{x \to -1 + \sqrt{6}\}\}$

In[72]:=
```
Clear["Global`*"];
Off[General::obspkg]; Off[General::newpkg];
Needs["Graphics`InequalityGraphics`"];
? ComplexInequalityPlot
```

ComplexInequalityPlot[ineqs, {z, zmin, zmax}] plots the the region defined by ineqs within the box bounded by {Re[zmin], Im[zmin]} and {Re[zmax], Im[zmax]}. The functions that occur within the inequality need to be real valued functions of a complex argument, e.g. Abs, Re and Im.

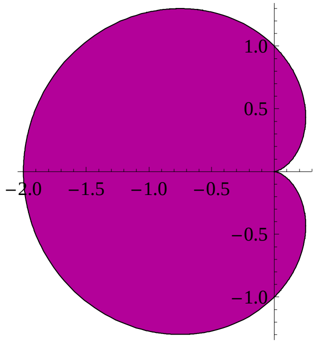
In[76]:=
```
Block[{$DisplayFunction = Identity}, p1 =
   ComplexInequalityPlot[Abs[z] ≤ 0.3 || (Re[z]^2 + 4 Im[z]^2 ≤ 1 && 1.23 Re[z]^2 + 6.25 Im[z]^2 ≥ 1), {z}, Filling → {1 → {{2}, Blue}}, Ticks → None];
  p2 = ComplexInequalityPlot[Abs[z - 0.2] ≥ 0.4 && Abs[z] ≤ 1, {z}, Filling → {1 → {{2}, Yellow}}, Ticks → None];
  p3 = ComplexInequalityPlot[Abs[1 - z^2] ≤ 1, {z}, Filling → {1 → {{2}, Red}}, Ticks → None];
  p4 = ComplexInequalityPlot[Abs[1 - z^3] ≤ 1, {z},
    Filling → {1 → {{2}, Magenta}}, Ticks → None]; p5 = ComplexInequalityPlot[Abs[1 - z^2] ≤ Abs[1 - z + z^2], {z, -1/2 - 2 ⅈ, 2 + 2 ⅈ}, Ticks → None];
  p6 = ComplexInequalityPlot[Abs[z - 1] ≤ Re[z], {z, -1/2 - 4 ⅈ, 3 + 4 ⅈ},
    Filling → {1 → {{2}, Orange}}, Ticks → None];];
Show[GraphicsGrid[{{p1, p2, p3}, {p4, p5, p6}}]]
```

Out[77]=



In[78]:=
```
ComplexInequalityPlot[(Re[z]^2 + Im[z]^2 + Re[z])^2 ≤ Re[z]^2 + Im[z]^2,
  {z, -2 - 2 ⅈ, 1/2 + 2 ⅈ}, Filling → {1 → {{2}, RGBColor[0.7, 0, 0.6]}},
  ImageSize → 72 × 2]
```

Out[78]=



In[79]:=
```
ParametricPlot[Evaluate[Table[{a (1 - Cos[θ]) Cos[θ], a (1 - Cos[θ]) Sin[θ]},
    {a, 0.25, 1, 0.25}]], {θ, 0, 2 π}, AspectRatio → 1,
 PlotStyle → {{Hue[0]}, {Hue[0.3]}, {Hue[0.6]}, {Hue[0.9]}}]
```

Out[79]=