Iteration theories as monadic algebras

Jiří Adámek *

Iteration theories, as introduced by Stephen Bloom and Zoltán Ésik, are Lawvere theories equipped with a function assigning to every morphism $e : n \to n + k$ (representing a system of n recursive equations whose left-hand sides are variables and right-hand sides are terms in those n variables and k parameters) a morphism $e^\dagger : n \to k$ (representing a solution of those equations). This function is required to satisfy a number of equational properties, for example the equation $e^\dagger = [e^\dagger, \mathrm{id}] \cdot e$ telling us that $e^\dagger$ is indeed a solution of e. The motivating idea is "to axiomatize the equational properties of all the computationally interesting structures of this kind" (citation from [2]). We claim that this statement is correct, in fact, it can be sharpened to state that iteration theories precisely axiomatize the equational properties of

$$e^\dagger = \text{the least solution of } e$$

as used in Domain Theory (and being, no doubt, computationally interesting).

In fact, Bloom and Ésik proved in [1] that for every finitary signature Σ a free iteration theory on Σ can be described as the theory R_Σ of rational Σ-trees. More precisely, the obvious forgetful functor U from the category ITh of iteration theories to the category $\mathsf{Sgn} = \mathsf{Set}/\mathbb{N}$ of signatures has a left adjoint given by $\Sigma \mapsto R_\Sigma$, where the theory R_Σ has as morphisms from n to 1 all rational Σ-trees in n variables. We prove more:

Theorem. *The forgetful functor $U : \mathsf{ITh} \to \mathsf{Sgn}$ is monadic.*

This tells us that iteration theories can be understood as the monadic algebras for the monad $\mathbb{R}\mathrm{at}$ on the category Sgn given by the above adjunction. And using the rational algebraic theories of the ADJ group [3], which essentailly are ordered algebraic theories with least solutions of iterative equation morphisms, one obtains the same monad $\mathbb{R}\mathrm{at}$ as the monad of free rational theories. It is easy to see that every rational theory has a free completion to a continuous theory as used in Domain Theory. Therefore, every equational property satisfied by the "least solution" function can be derived from the axioms of iteration theories. (And vice versa — this direction is contained in [1].)

References

[1] S. L. Bloom and Z. Ésik, *Iteration Theories*, Springer Verlag, 1993.

*Joint work with Stefan Milius and Jiří Velebil.

[2] S. L. Bloom and Z. Ésik, Fixed-point operations on ccc's, Part I, *Theoret. Comput. Sci.* 155 (1996), 1–38.

[3] J. B. Wright, J. W. Thatcher, E. G. Wagner and J. A. Goguen, Rational algebraic theories and fixed-point solutions, *Proc. 17th IEEE Symposium on Foundations of Computing*, Houston, Texas, 1976, 147–158.