New aspects of the Schreier-Mac Lane's extension theorem

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It is classical that any extension of groups:

$$1 \longrightarrow K \rightarrowtail^{k} X \overset{f}{\twoheadrightarrow} Y \longrightarrow 1$$

determines, via conjugation in the group X, a group homomorphism $\phi : Y \rightarrow AutK/IntK$, called the *abstract kernel* of this extension, which allows to recover the set $Ext_\phi(Y,K)$ of all isomorphic classes of extensions with abstract kernel ϕ. This comes from the fact that, with the combinatorial notion of *obstruction*, it was possible to show that there is on $Ext_\phi(Y,K)$ an action of the abelian group $Ext_{\bar{\phi}}(Y,ZK)$ of extensions with abelian kernel the centre ZK of the group K and with module structure given by the restriction $\bar{\phi}(y)$ to ZK of the automorphisms $\phi(y)$; $y \in Y$.

The recent introduction of the notion of *split extension classifier* (spec) (Borceux, Janelidze, Kelly) and *action groupoid* (Borceux, Bourn) in the context of protomodular categories will allow us to show that the previous result on extensions fully holds in any pointed protomodular Barr exact category $\mathbb{C}$ with specs. It is not only a simple generalization of the result in Gp, but also an enlightenment about the structure of extensions, since they will appear as morphisms in a certain groupoid $Tors\mathbb{C}$, which will give another meaning to the group action we recalled above. The main point is that the *action groupoid* $\underline{D}_1X$ of an object X retains all what remains of the abelianity of X and that the kernel of its normalization $j_X : X \rightarrow DX$ is the centre ZX of the object X. On the other hand, with any abstract kernel $\phi : Y \rightarrow Q_K$, where Q_K is the π_0 of the groupoid $\underline{D}_1K$, is associated a groupoid $\underline{D}_1\phi$ which will appear to be the *categorical expression of the obstruction.*