Join restriction categories: the importance of being adhesive

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The completeness theorem for restriction categories [1] states that every restriction category is a full subcategory of a partial map category. A join restriction category is a restriction category in which every set of compatible maps has a join. A natural question is: for what sort of partial map category are join restriction categories complete in the above sense.

The answer, somewhat to our surprise, is that the partial map category $\mathsf{Par}(\mathbf{X}, \mathcal{M})$ must be $\mathcal{M}$-adhesive, [3], and, furthermore, the stable sytem of monics $\mathcal{M}$ must be closed to $\mathcal{M}$-gaps. Adhesive categories use Van Kampen colimits in their definition and arose from the desire to give a general categorical framework in which double-pushout rewriting was possible. On the other hand, a key construction for join restriction categories is the manifold completion [2]. This implies that this sort of rewriting can be viewed abstractly as surgery on manifolds.

Given a restriction category there is a universal way to complete it to a join restriction category by considering down-closed subsets of compatible maps as the maps. On the other hand, given a representation of a restriction category into a partial map category $\mathsf{Par}(\mathbf{X}, \mathcal{M})$, where $\mathbf{X}$ is $\mathcal{M}$-adhesive, there is a canonical way of enlarging the system of monics $\mathcal{M}$ to include all the $\mathcal{M}$-gaps. This gives, therefore, an alternative method for adding joins as the standard Yoneda representation of restriction categories [1] is of this form.

This second technique for obtaining a join completion underlies, for example, the building of schemes in algebraic geometry: here one starts with $\mathsf{Par}(\mathsf{CRing}^{\mathrm{op}}, \mathcal{L})$ (where $\mathcal{L}$ is the class of localizations) and using the Yoneda representation one builds spaces which are locally ringed. The fact that these two constructions actually coincide, therefore, gives an alternate view of schemes as manifolds over the, possibly more simply described, universal join completion.

References

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[2] M. Grandis, *Cohesive categories and manifolds*, Ann. Mat. Pura Appl. 157 (1990) 199-244.

[3] S. Lack and P. Sobocinski, *Adhesive and quasiadhesive categories*, Theoretical Informatics and Applications 39 (2005) 511-546.

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