$*$-Autonomous functor categories

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Given a mere functor between $*$-autonomous categories, $M\colon \mathcal{J} \to \mathcal{K}$, it is possible to define a second functor $G\colon \mathcal{J} \to \mathcal{K}$ by $G(x) = (M(x^*))^*$ which should be intuitively thought of as the dual of M. [In the non-symmetric case, it is natural to consider two dual functors: $G^+(x) = (M({}^*x))^*$, and $G^-(x) = {}^*(M(x)^*)$. The remarks below assume symmetry for ease of presentation only.]

A particular case of interest is when M is (lax) monoidal with respect to the tensor structures on $\mathcal{J}$ and $\mathcal{K}$. [Recall that a $*$-autonomous category has, in general, two monoidal structures: one closed, the other co-closed; the former is called *tensor* and the latter, *par*.] Then, of course, G is comonoidal (oplax monoidal) with respect to the par structures on $\mathcal{J}$ and $\mathcal{K}$. But, as pointed out by Cockett and Seely [2], this is really only the beginning: M and G are related by four natural transformations (*linear strengths* and *linear costrengths*) which, intuitively, describe a two-sided action of M on G and a two-sided coaction of G on M. These natural transformations are required to satisfy coherence conditions which, again intuitively, say that the action of M on G is G-coequivariant, and that the coaction of G on M is M-equivariant.

Certainly, in the case where $\mathcal{J}$ is the terminal category, these intuitions are borne out, as shown by the author in [3].

Now suppose that $\mathcal{K}$ has colimits of size $\mathcal{J}$; then it is possible to define a *convolution tensor* on the functor category $\mathcal{K}^{\mathcal{J}}$ in such a way that monoidal functors $\mathcal{J} \to \mathcal{K}$ are in bijective correspondence with monoids in $\mathcal{K}^{\mathcal{J}}$, and such that actions of monoidal functors on mere functors are just that: actions of a monoid on an object. Dually, if $\mathcal{K}$ has limits of size $\mathcal{J}$, then we can also define a *co-convolution par* on $\mathcal{K}^{\mathcal{J}}$ such that comonoidal functors $\mathcal{J} \to \mathcal{K}$ correspond to comonoids in $\mathcal{K}^{\mathcal{J}}$.

But in order to fully substantiate our intuitions, it is necessary to first construct linear distributions relating the convolution tensor and the co-convolution par—without them we have, for instance, no means of describing a G-coaction on $M \otimes G$, and therefore cannot make sense of the idea that the left action $M \otimes G \to G$ should be G-coequivariant. The construction of such linear distributions is our first main result. [This is done in greater generality than that presented above: it suffices, for example, that $\mathcal{J}$ and $\mathcal{K}$ be *bilinear*, in the sense of [1].]

Given the existence of a linear distributive structure on $\mathcal{K}^{\mathcal{J}}$, we can revisit original intuition: that (even in the case where M is a mere functor) the functor G should be regarded as the dual of M. Our second main result is that one can construct a

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linear adjoint overlying M and G; this entails that the functor category $\mathcal{K}^{\mathcal{J}}$ is itself $*$-autonomous and that G is indeed (isomorphic to) M^*. [In the non-symmetric case, $G^+ \cong M^*$ and $G^- \cong {}^*M$.]

These theorems were first proven in the case where $\mathcal{J} = \rightarrow$ and $\mathcal{K} = \mathbf{Sup}$ [4]; the concept of *cyclic Frobenius monoid* in the $*$-autonomous category $\mathbf{Sup}^{\rightarrow}$ has proven surprisingly useful, and will be discussed in David's talk.

References

[1] J. R. B. Cockett and R. A. G. Seely, *Proof theory for full intuitionistic linear logic, bilinear logic, and MIX categories*, Theory Appl. Categ. 3 (1997) 85–131 .

[2] J. R. B. Cockett and R. A. G. Seely, *Linearly distributive functors*, J. Pure Appl. Algebra 143 (1999) 155–203.

[3] J.M. Egger, *The Frobenius relations meet linear distributivity*, submitted to *Theory and Applications of Categories.*

[4] J.M. Egger and David Kruml, *Girard couples in quantale theory.*