# Extended real number object in the bornological topos

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A few years ago, F. William Lawvwere [2] pointed out that, over the category $Set$, $\overline{\mathbb{R}}^{+} = [0, \infty]$ is a closed category, and (generalized) metric spaces are $\overline{\mathbb{R}}^{+}$-categories. This means that $\overline{\mathbb{R}}^{+}$ is a good recipient for metrics and norms. Later, Reichman [3] worked out the (Dedekind) extended real line $\overline{\mathbb{R}}_{\mathcal{E}}$, and its positive part $\overline{\mathbb{R}}^{+}_{\mathcal{E}}$, in an elementary topos $\mathcal{E}$ with natural number object.

Our goal is to study these real objects, denoted $\overline{\mathbb{R}}_{b}$ and $\overline{\mathbb{R}}^{+}_{b}$, in the bornological topos $\mathcal{B}$, which is a subtopos of $M$-sets with the monoid $M = Set(\mathbb{N}, \mathbb{N})$ of all sequences of natural numbers. This topos can be presented has the subtopos of all sheaves for the topology generated by the finite epimorphic families in $M$. The study of this topos has been encouraged by Lawvere, and some first steps about are given in [1]. It was proved there that the (Dedekind) real number object in $\mathcal{B}$ is the well known space $\mathbb{R}_b = \ell^{\infty}$.

In order to calculate $\overline{\mathbb{R}}_{b}$ it is useful to consider presheaves at first, that is, the extended real line $\overline{\mathbb{R}}_{m}$ in $MSet$, with $M = Set(\mathbb{N}, \mathbb{N})$. From Reichman's result for arbitrary monoids we calculate

$$\overline{\mathbb{R}}_{m} = \{\mu : \mathcal{P}(\mathbb{N}) \to \overline{\mathbb{R}} \mid \mu \textit{ monotone},\ \mu(\emptyset) = -\infty\},$$

and the same for $\overline{\mathbb{R}}^{+}_{m}$, but with $\overline{\mathbb{R}}^{+}$ and $\mu(\emptyset) = 0$ at the right side. Note that the action is $(\mu \circ f)(A) = \mu(f(A))$. In the bornological topos we obtain $M$-subsets of the above real presheaves, actually:

$$\overline{\mathbb{R}}_{b} = \{\mu : \mathcal{P}(\mathbb{N}) \to \overline{\mathbb{R}} \mid \mu \textit{ preserves finite} \vee\},$$

and similarly $\overline{\mathbb{R}}^{+}_{b}$. Of course, there are the canonical inclusion $\mathbb{R}_b \hookrightarrow \overline{\mathbb{R}}_{b}$, the locale structure given by $inf,\ sup : \mathcal{P}(\overline{\mathbb{R}}_{b}) \to \overline{\mathbb{R}}_{b}$, the closed structure on $\overline{\mathbb{R}}^{+}_{b}$, etc.

## References

[1] L. Español, L. Lambán, *On bornologies, locales and toposes of M-Set*, J. Pure and Appl. Algebra 176 (2002) 113-125.

[2] F. W. Lawvere, *Metric spaces, generalized logic, and closed categories*, Rend. Sem. Mat. e Fisico di Milano 43 (1973) 135-166.

[3] J. Z. Reichman, *Semicontinuous real numbers in a topos*, J. Pure and Appl. Algebra 28 (1983) 81-91.