Orthogonality and Injectivity Logics for proper classes of morphisms

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Completeness of orthogonality and injectivity logics can be seen as refined versions of two classical problems: the Orthogonal Subcategory Problem and the Small Object Argument. In [1] and [2] we defined these logics on morphisms in appropriate categories, and proved various completeness results of the form

$$\mathrm{H} \models f \iff \mathrm{H} \vdash f \ .$$

There, given an object A and a class of morphisms H, we write $A \models \mathrm{H}$ to mean that A is *H-injective* (resp., *H-orthogonal*), i.e., for all $h \in \mathrm{H}$, every $k\colon dom(h) \to A$ factorizes (resp. uniquely) through h. A morphism f is a (*injectivity*, or resp. *orthogonality*) *consequence of H*, written $\mathrm{H} \models f$, if for all A, we have $A \models \mathrm{H} \Rightarrow A \models f$. We say that f is a *formal consequence of H*, written $\mathrm{H} \vdash f$, if f can be obtained from the members of H by the utilization of some (categorical) "deduction rules" defined there.

Here, we obtain new such completeness results, focusing on proper classes H of morphisms. As a particular consequence, Kelly's result ([4], Theorem 10.2) on the existence of constructive reflectors in orthogonal subcategories with respect to classes "mostly" made of epis, is extended in various ways. Completeness theorems for injectivity logic are also obtained, showing in particular that the Small Object Argument holds for classes mostly made of epis (resp. strong epis) in locally bounded (resp. locally presentable) categories, but proving, rather surprisingly, that this does not hold in locally ranked categories. This adds to the main result in [3]. The wide picture is further clarified through counterexamples and some intriguing open problems, contrasting the situation in the three types of categories mentioned above, as well as between the orthogonality and the injectivity cases.

References

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[4] Kelly, G.M., *A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on*, Bull. Autral. Math. Soc. 22 (1980) 1-83.