Topological spaces, categorically

Dirk Hofmann *

Since F.W. Lawvere's famous 1973 paper [4] it is well-known that (generalised) metric spaces may be seen as examples of V-categories for $\mathsf{V} = [0,\infty]$, "so that enriched category theory can suggest new directions of research in metric space theory and conversely". Thanks to M. Barr [1] we know that topological spaces can be presented as categories as well, by interpreting the convergence relation $\mathfrak{x} \to x$ between ultrafilters and points of a topological space X as arrows in X. Indeed, in this talk we wish to show that general categorical notions and results are meaningful and helpful in topology.

First of all we have to agree on a setting which incorporates both above mentioned sources and which enables us to develop the principal constructions and results. In particular our work on completeness [2] suggested that the concept of a *topological theory* as introduced in [3] fulfils these requirements. Here a topological theory is a triple $\mathcal{T} = (\mathbb{T}, \mathsf{V}, \xi)$ consisting of a monad $\mathbb{T} = (T, e, m)$, a quantale $\mathsf{V} = (\mathsf{V}, \otimes, k)$ and a map $\xi : T\mathsf{V} \to \mathsf{V}$ compatible with the monad and the quantale structure. Given such a theory $\mathcal{T}$, a $\mathcal{T}$-*category* is defined as a pair (X, a) with "hom-objects" $a : TX \times X \to \mathsf{V}$ and with "identity" and "compostion", i.e. such that

$$k \le a(e_X(x), x) \qquad \text{and} \qquad T_\xi a(\mathfrak{X}, \mathfrak{x}) \otimes a(\mathfrak{x}, x) \le a(m_X(\mathfrak{X}), x)$$

where $\mathfrak{X} \in TTX$, $\mathfrak{x} \in TX$ and $x \in X$, and T_ξ is the induced extension of the Set-functor T to V-relations. Accordingly, a $\mathcal{T}$-*functor* $f : (X, a) \to (Y, b)$ is a Set-map $f : X \to Y$ such that

$$a(\mathfrak{x}, x) \le b(Tf(\mathfrak{x}), f(x))$$

for all $\mathfrak{x} \in TX$ and $x \in X$. Clearly, for each quantale V we can consider the identity monad and choose ξ as the identity map, so that we obtain V-categories and V-functors. On the other hand, for the theory $\mathcal{U}$ consisting of the ultrafilter monad, the two-element Boolean algebra $\mathsf{2}$ and ξ (essentially) the identity map, $\mathcal{U}$-categories are precisely topological spaces and $\mathcal{U}$-functors continuous maps.

In this setting, and guided by notions and results of enriched category theory, we study

1. lax Hopf monads and monoidal structures,

2. the dual of a $\mathcal{T}$-category,

*The talk is based on a joint work with M.M. Clementino and W. Tholen.

3. $\mathcal{T}$-modules,

4. the Yoneda Lemma,

5. Lawvere-completeness (resp. Cauchy completeness),

6. Morita equivalence.

References

[1] Michael Barr, *Relational algebras*, In *Reports of the Midwest Category Seminar, IV*, pages 39–55, Lecture Notes in Mathematics, Vol. 137. Springer, Berlin, 1970.

[2] Maria Manuel Clementino and Dirk Hofmann, *Lawvere completeness in Topology*, Technical report, University of Aveiro, 2007; `arXiv:math.CT/0704.3976`.

[3] Dirk Hofmann, *Topological theories and closed objects*, Technical report, University of Aveiro, 2006.

[4] F. William Lawvere, *Metric spaces, generalized logic, and closed categories*, Rend. Sem. Mat. Fis. Milano, 43:135–166 (1974). Also in: Repr. Theory Appl. Categ. 1:1–37 (electronic), 2002.