The connection between equivalence relations and subgroups

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It is well-known that the lattice of equivalence relations on a set embeds in the lattice of subgroups of the permutation group of that set, while the lattice of subgroups of a group embeds in the lattice of relations on its underlying set [1].

In fact, both these lattices are sublattices of involutive quantales – the quantale of all relations on X, and the quantale of all subsets of G. In their respective quantales, these lattices are both the sublattice of reflexive, symmetric idempotent elements. Therefore, a quantale embedding between these quantales would induce a $\vee$-semilattice embedding between the lattices, and a $\wedge$-preserving quantale embedding would induce a lattice embedding.

These quantales can both be constructed in the following way: take a category $\mathcal{C}$, and take the subsets of its set of morphisms. This is a quantale, with meet and join given by intersection and union respectively, and multiplication given by the set of morphisms that are formed as the composite of a morphism in the first set with a morphism in the second set. The quantale of relations on X is obtained by applying this construction to the indiscrete category on X, while the quantale of subsets of G is obtained by applying this construction to G considered as a 1-object category. The embedding of the quantale of subsets of G into the quantale of equivalence relations of its underlying set is then induced by the functor from the coslice $*/G$ to G, where $*$ is the object of G.

References

[1] G. Birkhoff, *On the Structure of abstract algebras*, Proc. Cambridge Phil. Soc. 31 (1935) 433-454.

[*] Joint work with Geoff Cruttwell, Robert Paré and Richard Wood.