Constructively completely distributive lattices in presheaf categories

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An ordered set L is a sup lattice if and only if the down-segment embedding of L into its complete lattice of down-sets, $\downarrow : L \to DL$, has a left adjoint $\bigvee : DL \to L$. Since [3] a complete lattice has been said to be constructively completely distributive (CCD) if $\bigvee : DL \to L$ has itself a left adjoint. In [4] there is a characterization of sup-lattices in terms of functors $L : \mathbf{C}^{op} \to \mathbf{sup}$ ($\mathbf{sup}$ the category of sup-lattices and sup-preserving arrows with respect to the base topos $\mathbf{Set}$). We produce a similar characterization of CCD lattices in terms of fuctors $L : \mathbf{C}^{op} \to \mathbf{ccd}$ ($\mathbf{ccd}$ the category of ccd lattices and arrows that preserve both, suprema and infima). Since L takes values in ccd we have, for every $f : B \to C$ in $\mathbf{C}$ an adjoint string $\bigvee_f \dashv Lf \dashv \bigwedge_f$. This characterization hinges on the (surprising) apperance of a right adjoint to $\bigwedge_f$ and on a condition called complete Frobenius reciprocity.

References

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