A Geometry for Diagrammatic Computads

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This talk is a preliminary report on a project suggested in [3]. The goal is to give a (combinatorially) geometric description of a rather large class of computads (= free ∞-categories with distinguished generators; all higher-dimensional categories mentioned here are strict), much as [3] does for the class of many-to-one computads. Sources of inspiration were the articles [1], [6] and [5], in which computads are constructed from certain "complexes". In each of them a condition of global acyclicity is imposed (allowing composition to be represented by plain set-theoretic union), which the present work will avoid.

The central notion here is that of a *hypotopic set*. It differs from the notion of an oriented polytopic set in that facets of a cell are allowed (and by further constraints in fact demanded) to occur in arbitrary lower dimensions (hence the prefix 'hypo'). Simplicial and cubical sets as well as dendrotopic sets all can be interpreted as instances of hypotopic sets.

Legitimate cell configurations in a hypotopic set, the *tissues*, form a category, which the expected pasting operations render *ω-categorial*. (So the ω-category laws, including the boundary ones, are satisfied in a weakened form.) Attention is restricted to those hypotopic sets that are *well formed* in that all boundary tissues of cells are well formed, that is, generated from (lower-dimensional) cells by the pasting operations. The problem now is to find "nice" conditions that, while being reasonably inclusive, ensure that (a) the (∞-categorial) full subcategory of well-formed tissues is essentially discrete — then its components form an ∞-category — and (b) this ∞-category is free.

A certainly nice condition is *semiregularity*: the absence of "singular" faces in the source tissues of cells. It can be subdivided according to the *ranks* of these faces. (Facets have rank 0, ridges have rank 1, and so on.) Rank-1 semiregularity is sufficient for (a) alone. As for the stated problem (regarding both (a) and (b)), the presenter's best proven solution to date is rank-1 and -2 semiregularity in conjunction with "translationality" of "enhanced" dual hemigraphs. It includes all dendrotopic sets, as well as simplicial and cubical sets up to dimension 3 (but, unfortunately, not beyond). The presenter's best conjectural (and as yet unrefuted) solution is full semiregularity on its own. It would include all dendrotopic, simplicial and cubical sets. It would also include, for instance, A. J. Power's 4-dimensional (counter-)example.

The category of well-formed hypotopic sets satisfying (a) and (b) is in fact concretely embedded into the category of computads. The full subcategory described by either solution is a concrete presheaf category, as can be shown using the methods of [4]. The composite result is a counterpart to the fact, expounded in [2], that the concrete category of all computads is not a presheaf category.

References

[1] Michael Johnson, *The combinatorics of n-categorical pasting*, Journal of Pure and Applied Algebra 62 (1989), 211–225.

[2] M. Makkai, *The word problem for computads*, `www.math.mcgill.ca/makkai`.

[3] Thorsten Palm, *Dendrotopic Sets*, Fields Institute Communications 43 (2004), 393–443.

[4] Thorsten Palm, *Categories with Slicing*, `www.ics.mq.edu.au/~palm`.

[5] Richard Steiner, *The Algebra of Directed Complexes*, Applied Categorical Structures 1 (1993), 247–284.

[6] Ross Street, *Parity complexes*, Cahiers de Topologie et Géométrie Différentielle Catégoriques 32 (1991), 315–343. *corrigenda*: 35 (1994), 359–361.